Positroids, knots, and

q,t-Catalan numbers.

Joint work with Thomas Lam

[GL19] P. Galashin and T. Lam. Positroid varieties and cluster algebras. *Ann. Sci. Éc. Norm. Supér., to appear.* arXiv:1906.03501, 2019.

[GL20] P. Galashin and T. Lam. Positroids, knots, and q,t-Catalan numbers. arXiv:2012.09745.

[GL21] P. Galashin and T. Lam. Positroid Catalan numbers. arXiv:2104.05701.

Positroid varieties and cluster algebras.

$\mathrm{Gr}(k,n;\mathbb{C}) := \{V \subseteq \mathbb{C}^n \mid \dim V = k\}$;

$$\mathrm{Gr}(k,n;\mathbb{C}) \cong \frac{\{M \in \mathrm{Mat}_{k\times n}(\mathbb{C}) \mid \mathrm{rk}(M) = k\}}{\text{row operations}}.$$

Theorem ([Sco06]). *The coordinate ring*

$$\mathbb{C}[\mathrm{Gr}(k,n)] = \frac{\mathbb{C}[\Delta_J \mid J \in \binom{[n]}{k}]}{(\textit{Plücker relations})}.$$

is a cluster algebra.

[Sco06] J. S. Scott. Grassmannians and cluster algebras. *Proc. Lond. Math. Soc. (3)*, 92(2):345–380, 2006.

For $I = \{i_1 < \cdots < i_k\}$, $J = \{j_1 < \cdots < j_k\}$, write $I \preceq J$ if $i_1 \le j_1, \ldots, i_k \le j_k$.

Definition. *Schubert cell*:

$$\Omega_I := \{V \in \mathrm{Gr}(k,n) \mid I \text{ is } \preceq\text{-minimal satisfying } \Delta_I \neq 0\}.$$

$$= \{V \in \mathrm{Gr}(k,n) \mid I = \mathrm{pivots}(V)\}.$$

Definition. A *Grassmann necklace* is a sequence $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ of k-element subsets of $[n]$ such that for each $i \in [n]$, we have

$$I_{i+1} = I_i \setminus \{i\} \cup \{j\} \quad \text{for some } j \in [n].$$

Positroid:

$$\mathcal{M}_{\mathcal{I}} := \left\{ J \in \binom{[n]}{k} \,\middle|\, I_i \preceq_i J \text{ for all } i \in [n] \right\},$$

where $\preceq_i$ is the i-th cyclic shift of $\preceq$:

$$i \le_i i+1 \le_i \cdots \le_i i-1.$$

Open positroid variety:

$$\Pi^\circ_{\mathcal{I}} := \left\{ V \in \mathrm{Gr}(k,n) \,\middle|\, \begin{array}{l} \Delta_I(V) \neq 0 \text{ for } I \in \mathcal{I}, \\ \Delta_J(V) = 0 \text{ for } J \notin \mathcal{M}_{\mathcal{I}} \end{array} \right\}$$

Theorem ([GL19]). *The coordinate ring*

$$\mathbb{C}[\Pi^\circ_{\mathcal{I}}] = \frac{\mathbb{C}[\Delta_I^{\pm 1}, \Delta_J \mid I \in \mathcal{I},\ J \in \mathcal{M}_{\mathcal{I}}]}{(\textit{Plücker relations})}.$$

is a cluster algebra.

• For *open Schubert varieties*, this was done in [SSBW19] using results of [Lec16].

[Lec16] B. Leclerc. Cluster structures on strata of flag varieties. *Adv. Math.*, 300:190–228, 2016.

[SSBW19] K. Serhiyenko, M. Sherman-Bennett, and L. Williams. Cluster structures in Schubert varieties in the Grassmannian. *Proc. Lond. Math. Soc. (3)*, 119(6):1694–1744, 2019.

[GL19] P. Galashin and T. Lam. Positroid varieties and cluster algebras. *Ann. Sci. Éc. Norm. Supér., to appear.* arXiv:1906.03501, 2019.

Open Problem. Do this for open Richardson varieties.

Top open positroid variety.

$$\mathcal{I}_{k,n} = (I_1, I_2, \dots, I_n);$$

$$I_i = \{i, i+1, \dots, i+k-1\};$$

$$\Pi^\circ_{k,n} := \{V \in \mathrm{Gr}(k,n) \mid \Delta_{I_i}(V) \neq 0 \text{ for all } i \in [n]\}.$$

Example ($k = 2, n = 4$).

$$\Pi^\circ_{2,4} \cong \left\{ \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \middle| \begin{array}{c} a,b,c,d \in \mathbb{C} : \\ a,d \neq 0, ad - bc \neq 0 \end{array} \right\}.$$

Question. Point count? Poincaré polynomial?

$$\#\Pi^\circ_{2,4}(\mathbb{F}_q) = (q-1)\cdot(q-1)\cdot(q^2-q+1).$$

$\Pi^\circ_{k,n}(\mathbb{C})$ is homotopy equivalent to

$$S^1 \times S^1 \times \text{pinched torus}, \quad \text{so}$$

$$\mathcal{P}(\Pi^\circ_{2,4}; q) = (q+1)\cdot(q+1)\cdot(q^2+q+1)$$

• For $\mathrm{Gr}(k,n)$, point count and Poincaré polynomial coincide, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}, \quad \text{where}$$

$[n]_q := 1+q+\dots+q^{n-1}$, $\quad [n]_q! := [1]_q \cdot [2]_q \cdots [n]_q$.

Reason: *Schubert decomposition*

$$\mathrm{Gr}(k,n) = \bigsqcup_{I \in \binom{[n]}{k}} \Omega_I$$

▷ Over $\mathbb{C}$, this is a CW decomposition into cells of even (real) dimension;

▷ Over $\mathbb{F}_q$, we have $\#\Omega_I(\mathbb{F}_q) = q^{|\lambda(I)|}$.

Problem. No such decomposition is available for $\Pi^\circ_{k,n}$.

Definition (*Rational Catalan numbers*).

• For $a, b \geq 1$ with $\gcd(a,b) = 1$, let

$$C_{a,b} := \frac{1}{a+b}\binom{a+b}{a} = \#\,\mathrm{Dyck}_{a,b}, \quad \text{where}$$

$\mathrm{Dyck}_{a,b} := \{\text{Dyck paths inside an } a \times b \text{ rectangle}\}$

Theorem ([GL20]). *Let* $\gcd(k,n) = 1$.

• Point count *of* $\Pi^\circ_{k,n}$*:*

$$\#\Pi^\circ_{k,n}(\mathbb{F}_q) = (q-1)^{n-1}\cdot C'_{k,n-k}(q), \quad \textit{where}$$

$$C'_{a,b}(q) = \frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q.$$

• Poincaré polynomial *of* $\Pi^\circ_{k,n}$*:*

$$\mathcal{P}(\Pi^\circ_{k,n}; q) = (q+1)^{n-1}\cdot C''_{k,n-k}(q), \quad \textit{where}$$

$$C''_{a,b}(q) = \sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)}.$$

▷ area(P)= *number of squares strictly between P and diagonal;*

Example ($a = 3,\ b = 5$).

$$C'_{3,5}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1.$$

$$C''_{3,5}(q) = q^4 + q^3 + 2q^2 + 2q + 1.$$

Corollary.

$$\mathrm{Prob}(V \in \Pi^\circ_{k,n}(\mathbb{F}_q)) = \frac{(q-1)^n}{q^n - 1}.$$

• Somehow the RHS does not depend on k.

Question. Common generalization?

Left hand side (cohomology).

• Suppose X is an algebraic variety satisfying certain conditions.

• $H^*(X)$ admits a second grading coming from the *Deligne splitting*:

$$H^i(X) = \bigoplus_{p \in \mathbb{Z}} H^{i,(p,p)}(X).$$

• The corresponding *mixed Hodge polynomial* $\mathcal{P}(X; q, t) \in \mathbb{N}[q^{\frac{1}{2}}, t^{\frac{1}{2}}]$ specializes to both $\#X(\mathbb{F}_q)$ and $\mathcal{P}(X; q)$.

Example. $H^*(X)$ could be given by

H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
1	0	1	0	2	0	2	0	1

• $\mathcal{P}(X; q) = q^4 + q^3 + 2q^2 + 2q + 1$.

• Deligne splitting $H^{i,(p,p)}(X)$ could look like

$H^{i,(p,p)}(X)$	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$p = i$	1	0	1	0	1	0	1	0	1
$p = i-1$					1	0	1		

• $\mathcal{P}(X; q, t) = q^4 + q^3 t + q^2 t^2 + q t^3 + t^4 + q^2 t + q t^2$

Right hand side (combinatorics).

• *Rational q,t-Catalan numbers*:

$$C_{a,b}(q,t) := \sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)}.$$

• area(P)= number of squares strictly between P and diagonal;

• dinv(P)= number of pairs (h, v) such that:

▷ h is a horizontal step, v is a vertical step;

▷ h appears to the left of v;

▷ there is a line of slope a/b (parallel to the diagonal) intersecting both h and v.

Example.

$$C_{3,5}(q,t) = q^4 + q^3 t + q^2 t^2 + q^2 t + q t^3 + q t^2 + t^4.$$

• $C_{a,b}(q,t)$ specializes to $C'_{a,b}(q)$ and $C''_{a,b}(q)$:

$$q^{\frac{(a-1)(b-1)}{2}} \cdot C_{a,b}(q, 1/q) = C'_{a,b}(q)$$

$$C_{a,b}(q, 1) = C''_{a,b}(q)$$

Theorem (Follows from [Mel16]).

$$C_{a,b}(q,t) = C_{a,b}(t,q).$$

[Mel16] Anton Mellit. Toric braids and (m,n)-parking functions. arXiv:1604.07456, 2016.

Open Problem. Bijective proof?

Theorem ([GL20]). *Let* $\gcd(k,n) = 1$. *Then the mixed Hodge polynomial of* $\Pi^\circ_{k,n}$ *is given by*

$$\mathcal{P}(\Pi^\circ_{k,n}; q, t) = \left(q^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^{n-1} C_{k,n-k}(q,t).$$

Moreover, the torus T of diagonal matrices acts freely on $\Pi^\circ_{k,n}$*, and we have*

$$\mathcal{P}(\Pi^\circ_{k,n}/T; q, t) = C_{k,n-k}(q,t).$$

[GL20] P. Galashin and T. Lam. Positroids, knots, and q,t-Catalan numbers. arXiv:2012.09745.

Corollary. $C_{a,b}(q,t)$ are q,t-symmetric and q,t-unimodal.

q,t-unimodal means coefficients of $C_{a,b}(q,t)$ at

$$q^d t^0, q^{d-1} t^1, \dots, q^0 t^d$$

form a unimodal sequence, for each d.

• Symmetry is known, unimodality is new.

Explanation for the corollary.

[GSV10] M. Gekhtman, M. Shapiro, and A. Vainshtein. *Cluster algebras and Poisson geometry.*

[LS16] T. Lam and D. E. Speyer. Cohomology of cluster varieties. I. Locally acyclic case.

[MS16] G. Muller and D. E. Speyer. Cluster algebras of Grassmannians are locally acyclic.

▷ On any cluster variety $\mathcal{A}(Q)$, we have the mutation-invariant *GSV form*

$$\gamma_Q = \sum_{u \to v} \frac{dx_u}{x_u} \wedge \frac{dx_v}{x_v}.$$

▷ [LS16]: Q is locally acyclic $\Longrightarrow$ multiplication by $\gamma_Q \in H^{2,(2,2)}(\mathcal{A}_Q)$ acts as a *curious Lefschetz operator* on $H^*(\mathcal{A}_Q)$.

▷ $\Pi^\circ_{k,n}/T$ is a cluster variety, locally acyclic by [MS16].

Example. The quiver of $X := \Pi^\circ_{3,8}/T$ is of type E_8, with $H^{i,(p,p)}(X)$ indeed given by

$H^{i,(p,p)}(X)$	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$p = i$	1	0	1	0	1	0	1	0	1
$p = i-1$					1	0	1		