Positroids, knots, and q,t-Catalan numbers.

joint w/ Thomas Lam

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Positroids and cluster algebras.

$$Gr(k,n) = \{ V \subset \mathbb{C}^n \mid \dim V = k \}$$

$$= \left\{ \begin{matrix} \text{Full rank} \\ k \times n \text{ matrices} \end{matrix} \right\} \Big/ \left(\begin{matrix}\text{row} \\ \text{operations}\end{matrix} \right)$$

Th'm (Scott'06)

$\mathbb{C}[Gr(k,n)]$ is a cluster algebra.

$$\mathbb{C}[Gr(k,n)] = \frac{\mathbb{C}\left[\Delta_J \mid J \in \binom{[n]}{k}\right]}{(\text{Plücker relations})}$$

Gale order on subsets

$I = \{ i_1 < i_2 < \cdots < i_k \}$

$J = \{ j_1 < j_2 < \cdots < j_k \}$

Say $I \preceq J$ iff

$i_1 \le j_1, \quad i_2 \le j_2, \quad \dots, \quad i_k \le j_k.$

Def. Schubert cells:

$$\Omega_I := \left\{ V \in Gr(k,n) \;\middle|\; \begin{matrix} \Delta_I \neq 0 \text{ and } I \\ \text{is } \preceq\text{-minimal} \\ \text{such set} \end{matrix} \right\}$$

$$V = \text{RowSpan}\; k \begin{bmatrix} 1 & * & 0 & * & * & * & 0 & * \\ 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

$\quad I \quad\quad I \quad I \quad n \quad I \qquad\qquad \lambda(I)$

Row-echelon form

$I =$ set of pivot columns

Open positroid variety "=" an intersection of n cyclically shifted Schubert cells.

<u>Def.</u> A <u>Grassmann necklace</u> is a sequence

k-elt subsets of $[n] = \{1,2,\dots,n\}$

$\mathcal{I} = (I_1, I_2, \dots, I_n)$: for each $i \in [n]$

$I_{i+1} = I_i \setminus \{i\} \cup \{j\}$ for some $j \in [n]$.

<u>Positroid</u> $\mathcal{M}_{\mathcal{I}}$:

$$\mathcal{M}_{\mathcal{I}} = \left\{ J \in \binom{[n]}{k} \,\middle|\, I_i \leq_i J \text{ for all } i \in [n] \right\}$$

where $\leq_i$ means

$$i <_i i+1 <_i i+2 <_i \dots <_i i-1$$

<u>Open positroid variety</u>:

$\mathcal{I}$ - Grassmann necklace

$$\Pi^{\circ}_{\mathcal{I}} := \left\{ V \in Gr(k,n) \,\middle|\, \begin{array}{ll} \Delta_{I_i} \neq 0 & I_i \in \mathcal{I} \\ \Delta_J = 0 & \text{for } J \notin \mathcal{M}_{\mathcal{I}} \end{array} \right\}$$

<u>Th'm</u> $\mathbb{C}[\Pi^{\circ}_{\mathcal{I}}]$ is a cluster algebra.

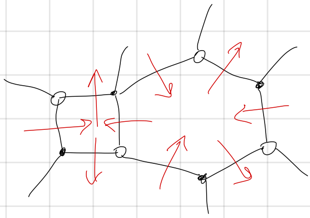
where $\mathbb{C}[\Pi^{\circ}_{\mathcal{I}}] = \dfrac{\mathbb{C}\left[\Delta^{\pm}_{I_i}, \Delta_J \,\middle|\, \begin{array}{l} I_i \in \mathcal{I}, \\ J \in \mathcal{M}_{\mathcal{I}} \end{array}\right]}{(\text{Plücker relations})}$

— For "open Schubert varieties" this was shown by [SSBW'19] using [Lec'16]

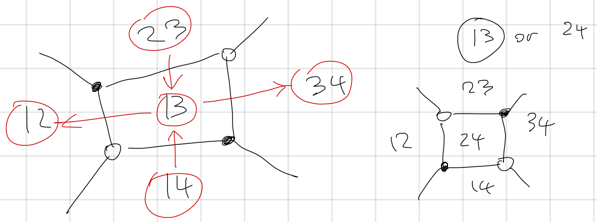
Open problem: do the same for open Richardson varieties.

<u>Q:</u> Why prove such theorems?

Plabic graph



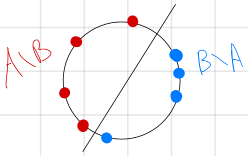
$\mathcal{I} = \{12, 23, 34, 41\}$



Choose a maximal weakly sep. collection from $\mathcal{M}_{\mathcal{I}}$, containing $\mathcal{I}$.

$A, B \in \binom{[n]}{k}$ are weakly sep. if there's no $a<b<c<d$: $a,c \in A\setminus B$, $b,d \in B\setminus A$

or $a,c \in B\setminus A$, $b,d \in A\setminus B$



Oh–Postnikov–Speyer '15

Top open positroid variety

$$\mathcal{I}_{k,n} = \left\{ \begin{array}{l} 1, 2, \dots k \\ \quad 2 \;\; 3 \; \dots \; k+1 \\ \qquad 3 \; \dots \dots \; k+2 \quad \mathcal{I}_3 \\ \qquad\quad \vdots \\ \qquad\qquad n, 1, 2, \dots k-1 \end{array} \right\}$$

$$Gr(k,n) = \bigsqcup_{\mathcal{I}} \Pi^\circ_{\mathcal{I}}$$

$$\mathcal{M}_{\mathcal{I}_{k,n}} = \binom{[n]}{k}$$

$$\Pi^\circ_{k,n} = \left\{ V \in Gr(k,n) \;\middle|\; \Delta_{I_i}(V) \neq 0 \text{ for } i \in [n] \right\}$$

Ex $k=2$, $n=4$. $\mathcal{I} = (\{12\}, \{23\}, \{34\}, \{14\})$

$$\Pi^\circ_{2,4} = \left\{ \boxed{\begin{matrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{matrix}} \;\middle|\; \begin{array}{l} a \neq 0 \\ ad - bc \neq 0 \\ d \neq 0 \end{array} \right\}$$

$\Pi^\circ_{k,n}$ is an alg. variety defined $/\mathbb{Z}$ and over $\mathbb{C}$

Q1: Point count over $\mathbb{F}_q$?

$$\# \Pi^\circ_{2,4}(\mathbb{F}_q) = (q-1)\cdot(q-1)\,(q^2-q+1).$$

$a \neq 0 \Rightarrow q-1$ options for a
$d \neq 0 \Rightarrow q-1$ opts for d

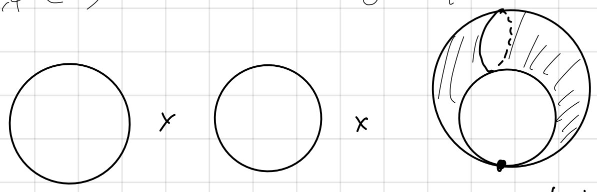
$bc \neq ad$
$b=0$, c - arb. $\Rightarrow q$ opts
$b \neq 0$, $c \neq \frac{ad}{b}$ $\quad q-1$
$q-1$ opts

Q2: Betti numbers of $\Pi^\circ_{k,n}(\mathbb{C})$?

$$P(\Pi^\circ_{k,n}(\mathbb{C}), q) = \sum_{i=0}^{2\dim_{\mathbb{C}}} q^i \beta_i$$

β_i ← i-th Betti number.

$\Pi^\circ_{2,4}(\mathbb{C})$ is homotopy equivalent to



$S^1 \times S^1 \times$ pinched torus

$1+q \qquad 1+q \qquad 1+q+q^2$

$$P(\Pi^\circ_{2,4}(\mathbb{C}), q) = (q+1)(q+1)(q^2+q+1).$$

What happens for $Gr(k,n)$?

Both point count & Poincaré poly are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}$$

$[n]_q = 1+q+\cdots+q^{n-1}$, $\quad [n]_q! = [1]_q \cdot [2]_q \cdots [n]_q$

Reason: Schubert decomp.:

$$Gr(k,n) = \bigsqcup_{I \in \binom{[n]}{k}} \Omega_I.$$

each $\Omega_I \simeq \mathbb{C}^{|\lambda(I)|}$

$$\# \Omega_I(\mathbb{F}_q) = q^{|\lambda(I)|}$$

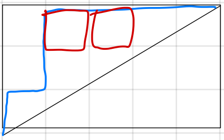
Problem: No such decomposition for $\Pi^\circ_{k,n}$?

Rational Catalan numbers.

Start with $a, b \geq 1$ $\gcd(a,b)=1$.

$$C_{a,b} = \frac{1}{a+b}\binom{a+b}{a} = \# \underbrace{\mathrm{Dyck}_{a,b}}$$

Dyck paths in $a \times b$ rect.



area(P) = 2.

Th'm [GL 20] Let $\gcd(k,n)=1$. Then

Point count:

$$\# \Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^{n-1} \cdot C'_{k,n-k}(q)$$

$$C'_{a,b}(q) = \frac{1}{[a+b]_q} \cdot \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

Poincaré poly:

$$\mathcal{P}(\Pi^{\circ}_{k,n}; q) = (q+1)^{n-1} \cdot C''_{k,n-k}(q),$$

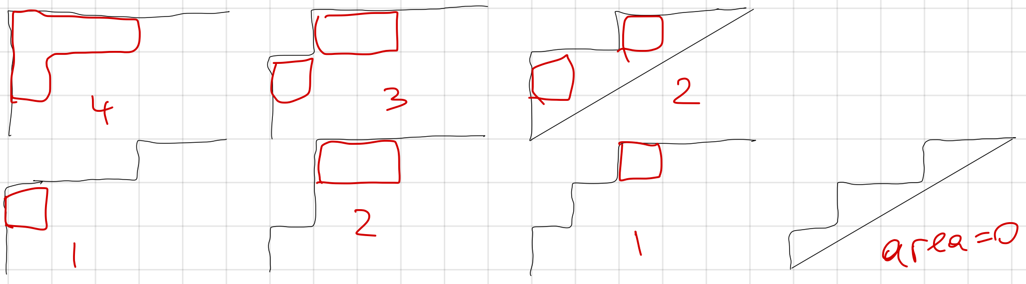
$$C''_{a,b}(q) = \sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)}.$$

where area(P) = # boxes strictly between P & the diagonal

Ex. $a=3$, $b=5$ $(k=3, n=8)$

$$C'_{3,5}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1.$$

$$C''_{3,5}(q) = q^4 + q^3 + 2q^2 + 2q + 1.$$



Cor. Let $\gcd(k,n)=1$

$$\mathrm{Prob}\left(V \in \Pi^{\circ}_{k,n}(\mathbb{F}_q)\right) = \boxed{\frac{(q-1)^n}{q^n-1}}.$$

does not depend on k ???

($Gr(k,n;\mathbb{F}_q)$ is a finite set)

Common generalization.

LHS (cohomology): mixed Hodge structure on $H^*(\Pi^{\circ}_{k,n})$.

Let X be some alg. variety.

Then $H^*(X)$ ← single graded vect space admits a second grading called the Deligne splitting:

$$H^i(X) = \bigoplus_{p \in \mathbb{Z}} H^{i,(p,p)}(X)$$

So instead of $\mathcal{P}(X,q)$, can consider "mixed Hodge poly" $\mathcal{P}(X;q,t) \in \mathbb{N}[q^{1/2},t^{1/2}]$.

Ex Let $X = \boxed{???}$. Then $H^*(X)$ can look like

H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
1	0	1	0	2	0	2	0	1

$$\mathcal{P}(X,q) = q^4 + q^3 + 2q^2 + 2q + 1$$

The $H^{i,(p,p)}(X)$ can look like

$H^{i,(p,p)}(X)$	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$p=i$	1	0	1	0	1	0	1	0	1
$p=i-1$					1	0	1		

$$P(X; q,t) = q^4 + q^3 t + q^2 t^2 + q t^3 + t^4$$
$$+ q^2 t + q t^2$$

each row has terms $q^d, \; q^{d-1/2} t^{1/2}, \; q^{d-1} t, \; \dots$

RHS (combinatorics): $\gcd(a,b) = 1$.

Rat'l q,t - Catalan number

$$C_{a,b}(q,t) = \sum_{P \in Dyck_{a,b}} q^{area(P)} t^{dinv(P)}.$$

$dinv(P)$ = # diagonal inversions of P, i.e.

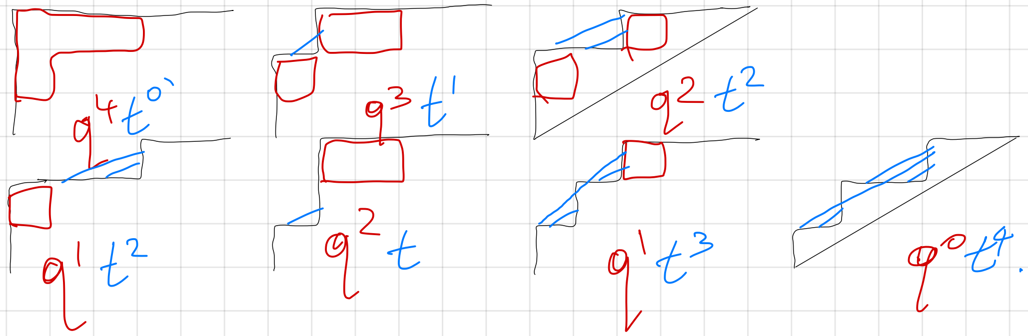
pairs h, v:

h - hor. step, v - vert. step

h is southwest from v

exists a line of slope a/b intersecting both h and v.

Ex $a=3$, $b=5$



$C_{a,b}(q,t)$ specializes to

$$q^{\frac{(a-1)(b-1)}{2}} \cdot C_{a,b}(q, 1/q) = C'_{a,b}(q)$$

$$C_{a,b}(q, 1) = C''_{a,b}(q).$$

Striking property of $C_{a,b}(q,t)$: ([Mel 16])

$$\boxed{C_{a,b}(q,t) = C_{a,b}(t,q)}$$

Open problem: prove this bijectively.

<u>Thm</u> [GL 20] Let $\gcd(k,n)=1$. Then

(1) $\mathcal{P}(\Pi^{\circ}_{k,n}; q,t) = (q^{1/2}+t^{1/2})^{n-1} \, C_{k,n-k}(q,t)$.

(2) Torus T of diag. matrices in SL_n acts on $\Pi^{\circ}_{k,n}$ freely, and

$$\mathcal{P}(\Pi^{\circ}_{k,n}/T; q,t) = C_{k,n-k}(q,t).$$

Corollary $C_{a,b}(q,t)$ are q,t-symmetric and q,t-unimodal.

<u>unimodal</u> means for each d, sequence of coefs at $q^d, q^{d-1}t, \dots, t^d$ is unimodal.

unimodality is new.

Q: how is this a corollary of this?

Reason: cluster algebras.

(1) On any cluster variety $\mathcal{A}(Q)$,

we have a mut-invariant GSV form γ_Q

$$\gamma_Q = \sum_{u \to v} \frac{dx_u}{x_u} \wedge \frac{dx_v}{x_v}$$

(2) if Q is loc. acyclic $\Rightarrow$ $\gamma_Q \in H^{2,(2,2)}(\mathcal{A}(Q))$

mult. by γ_Q acts as a <u>curious</u> <u>Lefschetz operator</u> on $H^*(\mathcal{A}(Q))$.

$H^{i,(p,p)}(X)$	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$p=i$	1	0	1	0	1	0	1	0	1
$p=i-1$					1	0	1		

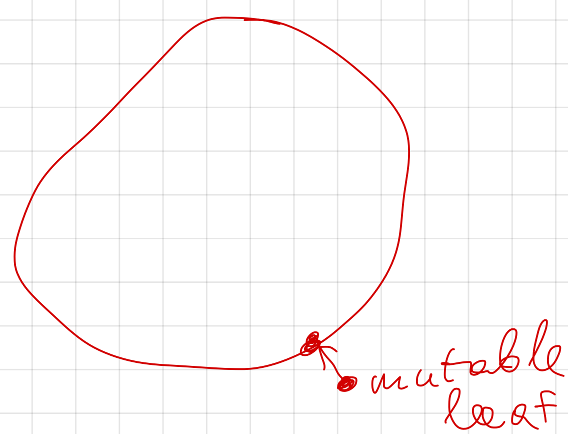
γ_Q

(3) $\Pi^\circ_{k,n}/T$ is a cluster variety (loc. acyclic by [MS'16]).

<u>Ex.</u> $k=3$, $n=8$ $\Pi^\circ_{3,8}/T$ is a cluster variety, $Q = E_8$ without frozen vars.

Thanks!

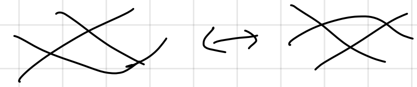
amplituhedron form



Richardson quivers: $v, w \in S_n$ $v \leq w$

– Ingermansson

– Leclerc



✗ – not planar

✓ – Braid moves preserve quiver (up to mut)

✗ – weak sep

✓ – quiver point count appears to $= \# R^{\circ}_{v,w}(\mathbb{F}_q)$

✓ – all quivers appear ~~to~~ be loc. acyclic

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