Braid variety cluster structures.

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Motivation #1.

- Positroid varieties.
  - Cluster structure: GL '19, SSBW '19, Leclerc '14, Muller-Speyer '14, Scott '06
  - Plabic graphs Postnikov '06
- Richardson varieties R°_{uw} ($u \le w$ in Bruhat order)
  - Include positroid var. when w is "Grassmannian"
  - Cluster structure: conj. by Leclerc '14
  - ??? graphs

Motivation #2.

of cluster varieties

- LS '16: Cohomology of positroid/Richardson var.
- GL '20: Cohomology
  - q,t-Catalan numbers
  - knot homology.

$G = SL_n = \begin{bmatrix} * \end{bmatrix}$ $\quad B = \begin{bmatrix} * & \ddots & * \\ 0 & & * \end{bmatrix}$ $\quad B_- = \begin{bmatrix} * & \ddots & 0 \\ * & & * \end{bmatrix}$ $\quad H = \begin{bmatrix} * & \ddots & 0 \\ 0 & & * \end{bmatrix} = B \cap B_-$

$U = \begin{bmatrix} 1 & \ddots & * \\ 0 & & 1 \end{bmatrix}$ $\quad U_- = \begin{bmatrix} 1 & \ddots & 0 \\ * & & 1 \end{bmatrix}$

G/B = flag var.

$R^\circ_{u,w} = (B_- uB \cap BwB)/B \subset G/B$ – Richardson var. (Deodhar '85)

$G^{v,w} = B_- vB_- \cap BwB \subset G$ – double Bruhat cells (Fomin-Zelevinsky '99)

$(W = S_n)$

$G/B = \bigsqcup_{u \le w} R^\circ_{u,w}$ $\qquad G = \bigsqcup_{v,w \in W} G^{v,w}$

Double Braid varieties: $R^\circ_{u,\beta}$ $\qquad I = \{1,2,\dots,n-1\}$

$\beta = i_1 i_2 \cdots i_m \in (\pm I)^m$ – double braid word

$u \in W$, $u \le \pi(\beta)$ $\qquad \pi(\beta)$ = "Demazure product" of β

I and $-I$ commute

- β = reduced word for $w \in W \Rightarrow R^\circ_{u,\beta} \cong R^\circ_{u,w}$
- β = shuffle of (red. word for w on + indices) and (red. word for v on − indices) $\Rightarrow R^\circ_{id,\beta} \cong G^{v,w}$.

Combinatorics

$\beta = i_1 \dots i_m \in (\pm I)^m$, $u \in W$

Def. Positive distinguished subexpr.

Set $u_m := u$. For $c = m, \dots, 2, 1$, set $u_{c-1} = \min(u_c, s_{i_c}^- u_c s_{i_c}^+)$

Write $u \leq \beta$ if $u_0 = id$

Let $J := \{c : u_c = u_{c-1}\}$ – solid crossings

not in J – hollow crossings $\boxed{i_c}$

Def. Almost positive subexpr. Fix $d \in J$.

Set $u_m^{\langle d \rangle} = u$, $u_{c-1}^{\langle d \rangle} = \begin{cases} \min(u_c^{\langle d \rangle}, s_{i_c}^- u_c^{\langle d \rangle} s_{i_c}^+), & c \neq d \\ \max(\text{———} // \text{———}), & c = d. \end{cases}$ "mistake"

Call $d \in J$ mutable if $u_0^{\langle d \rangle} = id$, frozen otherwise.

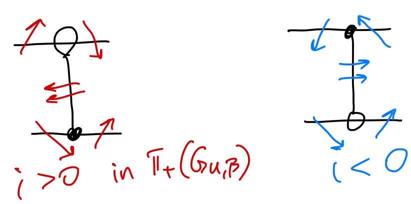
[BFZ '05]

[Ingermanson '19]

"Grid minors": $\Delta_{c,i} = \prod_{d \in J:\, u_c \omega_i \neq u_c^{\langle d \rangle} \omega_i} x_d$ $\qquad \Delta_{c,-i} = \prod_{d \in J:\, u_c^{-1} \omega_i \neq (u_c^{\langle d \rangle})^{-1} \omega_i} x_d$

(Type A only!)

$\omega_i = \{1, \dots, i\}$



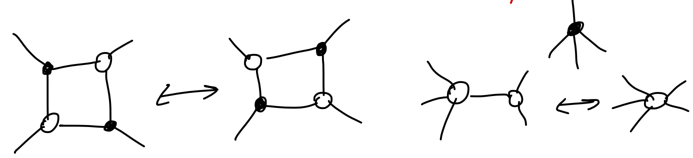
Quiver $Q_{u,\beta}$ For each $c \in J$, sum up half-arrows.

Def (Postnikov '06) A plabic graph G is a planar bicolored graph

vertices of Q_G $\leftrightarrow$ faces of G

Planar dual of G is a quiver Q_G: arrows of Q_G $\leftrightarrow$ edges of G.

Moves:



Def (Goncharov–Kenyon '13) Conjugate surface S_G: G – ribbon graph, clockwise/cclw edge order around black/white vertices.

Faces of G $\to$ some cycles inside S_G

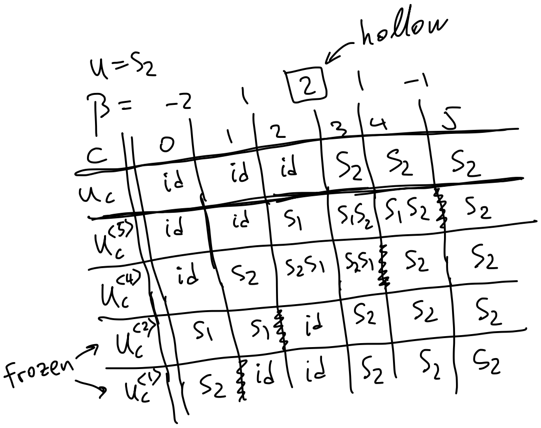
moves preserve S_G. $\boxed{Q_G \Leftrightarrow \text{intersection form of } S_G}$

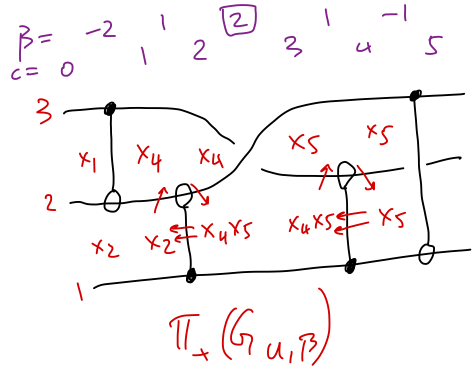
Our construction:

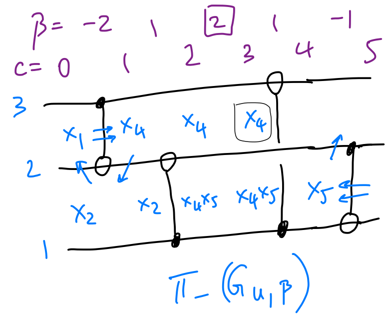
$\Pi_+(G_{u,\beta})$, $\Pi_-(G_{u,\beta})$ are projections of a 3D plabic graph $G_{u,\beta}$.

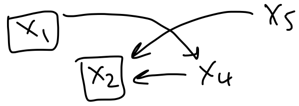
Regions where x_d appears: projections of a 2-disk D_d, ∂D_d – cycle in $G_{u,\beta}$.

Quiver $Q_{u,\beta}$ – intersection form of conjugate surface of $G_{u,\beta}$


hollow
u=s2
β= -2 1 2 1 -1
c 0 1 2 3 4 5
u_c id id id s2 s2 s2
u_c^<3> id id s1 s1s2 s1s2 s2
u_c^<4> id s2 s2s1 s2s1 s2 s2
u_c^<2> s1 s1 id s2 s2 s2
u_c^<1> s2 id id s2 s2 s2
frozen


β= -2 1 2 3 1 -1
c= 0 1 2 3 4 5
3
2
1
x1 x4 x4 x5 x5
x2 x2 x4x5 x4x5 x5
Π+(G u,β)


β= -2 1 2 1 -1
c= 0 1 2 3 4 5
3
2
1
x1 x4 x4 x4
x2 x2 x4x5 x4x5 x5
Π−(G u,β)


x1
x5
x2
x4