Link homology vs usual homology

joint w/ T. Lam

Richardson varieties $\subset$ Braid varieties

(1) Positroid varieties $\subset$ Richardson varieties $\subset$ Braid varieties

$0 \le k \le n$.

Def. $Gr(k,n) = \{ M \in Mat(k \times n, \mathbb{C}) \mid rank(M) = k \} / (\text{row operations})$

Let M_i = i-th column of M. Extend to $(M_i)_{i \in \mathbb{Z}}$ n-periodically: $M_{i+n} = M_i$.

Each <u>open positroid variety</u> is of the form

$$\Pi^\circ_R := \{ M \in Gr(k,n) \mid rank(M_i, M_{i+1}, \dots, M_{j-1}) = r_{ij} \text{ for all } i \le j \}$$

for some fixed $R = (r_{ij})_{i,j \in \mathbb{Z}}$ – "cyclic rank matrix".

Prop. [KLS 13]

Nonempty open. pos. varieties are in bijection* with permutations $f \in S_n$: $k = \# \{ 1 \le i \le n \mid f(i) < i \}$

(* false if $f(i) = i$ for some i.)

$$Gr(k,n) = \bigsqcup_{\substack{f \in S_n \\ k = \#\{\dots\}}} \Pi^\circ_f$$ – stratification into smooth loc. closed affine subvarieties.

Ex. (Top cell) $r_{ij} = \min(k, j-i)$ (maximal possible)

Let $f_{k,n} \in S_n$: $f_{k,n}(i) \equiv i+k \pmod n$ for $i = 1, \dots, n$.

$$\Pi^\circ_{f_{k,n}} = \left\{ M \in Gr(k,n) \;\middle|\; \begin{matrix} \Delta_{1,\dots,k}(M) \neq 0 \\ \Delta_{2,\dots,k+1}(M) \neq 0 \\ \dots \\ \Delta_{n,1,\dots,k-1}(M) \neq 0 \end{matrix} \right\}.$$

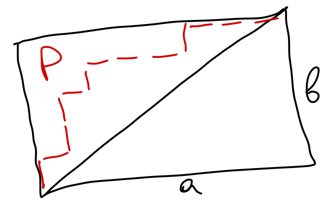
$$\Delta_{i_1 \dots i_k}(M) := \det \begin{pmatrix} | & & | \\ M_{i_1} & \dots & M_{i_k} \\ | & & | \end{pmatrix}$$

Ex. $\Pi^\circ_{f_{2,5}} = \left\{ \begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \middle| \ a, ae-bd, bf-ce, f \neq 0 \right\}$

<u>Thm.</u> [GL20] For $\gcd(k,n)=1$, cohomology of $\Pi^\circ_{f_{k,n}}$ is given by

$$\mathcal{P}(\Pi^\circ_{f_{k,n}}; q,t) = (q^{1/2}+t^{1/2})^{n-1} C_{k,n-k}(q,t), \quad \text{where}$$

- $C_{a,b}(q,t) = \sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)}$



rational q,t-Catalan number

- $\mathcal{P}(X;q,t) \in \mathbb{Z}_{\geq 0}[q^{1/2}, t^{1/2}]$ - mixed Hodge polynomial encoding bigraded dimensions of $H^*(X)$, where X - variety of "mixed Tate type"

Specializations:

$t^{1/2} = -q^{-1/2}$: Poincaré poly $\mathcal{P}(X(\mathbb{C}); t^{1/2})$

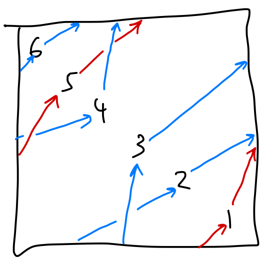
$q^{1/2} = 1$: point count $\# X(\mathbb{F}_q)$.

(2) <u>Positroid links</u>

$f \in S_n \longrightarrow$ link L_f on a torus (or in S^3)

<u>Rule:</u> connect $i \to f(i)$ in NE direction
draw higher slope above lower slope.

<u>Ex.</u> $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix}$



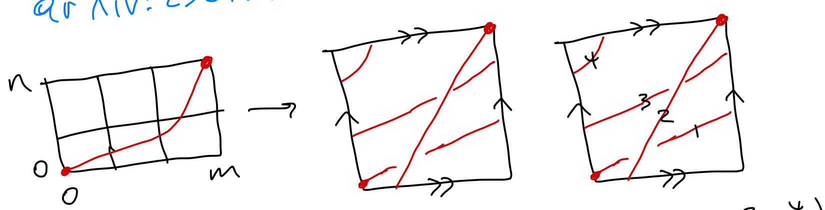
$\#$ cpts of $L_f = c(f) := \#\{\text{cycles of } f\}$

<u>Thm</u> [GL20] $c(f)=1 \Rightarrow \mathcal{P}(\Pi^\circ_f; q,t) = HHH^{q=0}(L_f) \cdot (q^{1/2}+t^{1/2})^{n-1}$

Arbitrary f: RHS $\to$ T-equivariant cohom, where $T=$ diag. matrices in PGL_n

(3) Monotone links vs EHA [GL23] arXiv:2307.16794

Def. Let $h : [0,m] \to [0,n]$
- continuous
- increasing
- $h(0)=0$, $h(m)=n$.



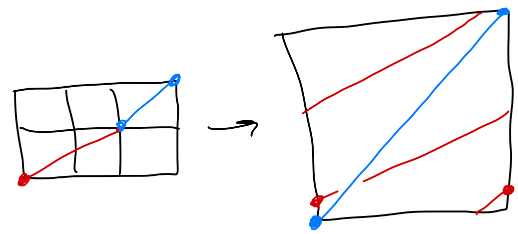
$\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ $\quad f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

Define monotone link L_h

Rules: draw later segments above earlier segments

Q: When is L_h a positroid link?

A: when h is a convex function



[Morton–Samuelson '21]:

$$Sk_t(T^2 - D^2) \xrightarrow{\downarrow} \text{DAHA}/\text{EHA} \xrightarrow{\text{act on } 1} \mathbb{C}[x_1, \dots, x_n]^{S_n}$$

$$L_h \longrightarrow F_h \text{ - symmetric function}$$

Conj [GL23] h – convex function $\Rightarrow$ F_h is Schur-positive.

Mellit: false for some convex h

See also: [BHMPS 21]

Conj. L_h is algebraic $\Rightarrow$ F_h computes $HHH(L_h)$.

Thm Let h be piecewise linear.
L_h is algebraic $\Longleftrightarrow$ h is convex

Thm. Viewed as links in $\mathbb{R}^3$, monotone links are precisely Coxeter links:

introduced by [Oblomkov–Rozansky '20]

closures of $JM_1^{b_1} \cdots JM_k^{b_k} \, \sigma_1^{\varepsilon_1} \cdots \sigma_{k-1}^{\varepsilon_{k-1}}$,

where $\varepsilon_i \in \{0,1\}$, $b_i \in \mathbb{Z}_{\geq 0}$, $JM_i =$