# Sorting probability for Young diagrams

Swee Hong Chan
University of California, Los Angeles
Joint work with Igor Pak and Greta Panova

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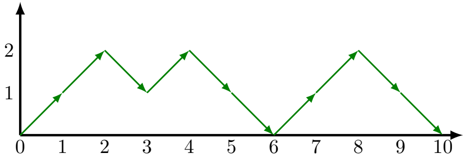
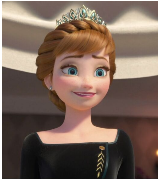
|---|---|---|---|---|
| 1 | 2 | 4 | 7 | 8 |
| 3 | 5 | 6 | 9 | 10 |





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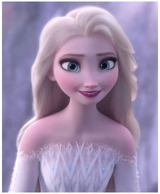

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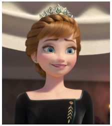
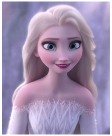
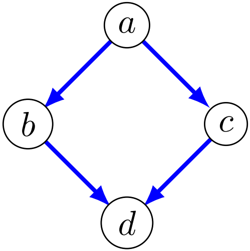
 $<$  $<$  $<$  $<$  $<$ 

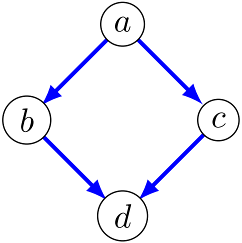
# Partially ordered set

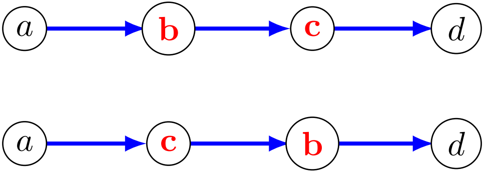
A poset $P$ is a set $X$ with a partial order $\preccurlyeq$ on $X$.



# Linear extension

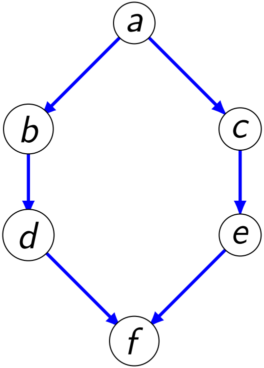
A linear extension $L$ is a complete order of $\preccurlyeq$.



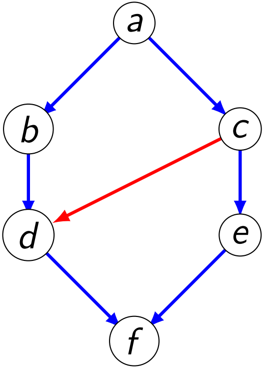

We write $e(P)$ for number of linear extensions of $P$.

# How many steps needed to complete a partial order?



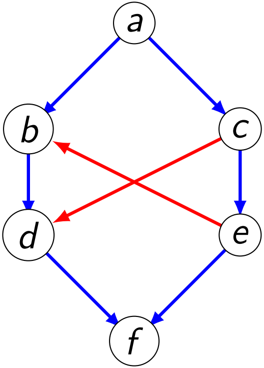
How many steps needed to complete a partial order?

We first compare $c$ and $d$, and get $c \preccurlyeq d$.



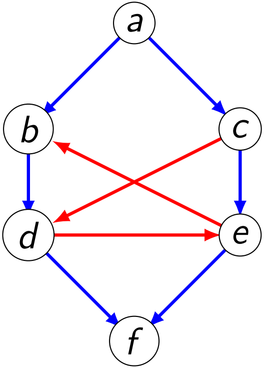
How many steps needed to complete a partial order?

We then compare $b$ and $e$, and get $e \preccurlyeq b$.



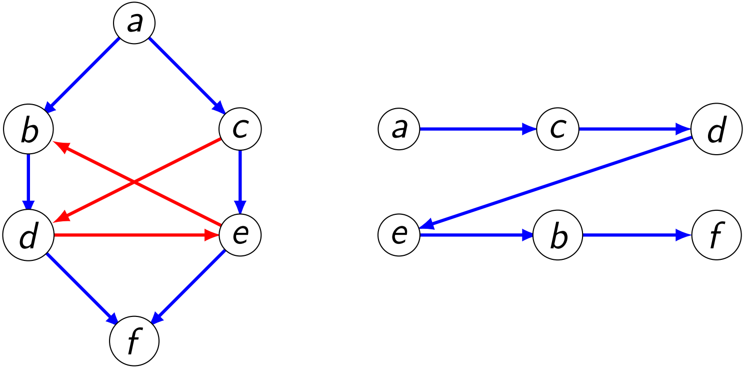
How many steps needed to complete a partial order?

We continue with $d$ and $e$, and get $d \preccurlyeq e$.



How many steps needed to complete a partial order?

Completing the partial order took 3 steps.



# Strategy to complete the partial order

At each step, compare $x$ and $y$ that satisfies

$$\frac{1}{2} - c \quad \leq \quad \mathbf{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{1}{2} + c\,,$$

where $\mathbf{P}$ is uniform on linear extensions of $P$.

# Strategy to complete the partial order

At each step, compare $x$ and $y$ that satisfies

$$\frac{1}{2} - c \quad \leq \quad \mathbf{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{1}{2} + c\,,$$

where $\mathbf{P}$ is uniform on linear extensions of $P$.

Runtime is at most $\big|\log_{\frac{1}{2}+c} e(P)\big|$ steps, optimal up to a multiplicative constant.

# $\frac{1}{3} - \frac{2}{3}$ Conjecture

**Conjecture** (Kislitsyn '68, Fredman '75, Linial '84)

*For every finite poset that is not completely ordered, there exists $x, y$:*

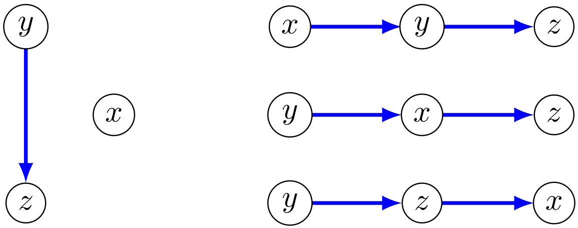
$$\frac{1}{3} \quad \leq \quad \mathbf{P}\left[x \preccurlyeq y\right] \quad \leq \quad \frac{2}{3}.$$

**Quote** (Brightwell-Felsner-Trotter '95)

*"This problem remains one of the most intriguing problems in the combinatorial theory of posets "*

# Why $\frac{1}{3}$ and $\frac{2}{3}$?

The upper,lower bound are achieved by this poset:




$$\mathbf{P}\left[x \preccurlyeq y\right] \;=\; \frac{1}{3}; \qquad \mathbf{P}\left[y \preccurlyeq x\right] \;=\; \frac{2}{3}.$$

# What is known so far

**Theorem (Kahn-Saks '84)**

*For every finite poset, there always exists $x, y$:*

$$\frac{3}{11} \quad \leq \quad \mathbf{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{8}{11},$$

*roughly between* 0.273 *and* 0.727*.*

Proof is based on a geometric approach, using mixed-volume inequalities.

# What is known so far

**Theorem (Brightwell-Felsner-Trotter '95)**

*For every finite poset, there always exists $x, y$:*

$$\frac{5-\sqrt{5}}{10} \quad \leq \quad \mathbf{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{5+\sqrt{5}}{10},$$

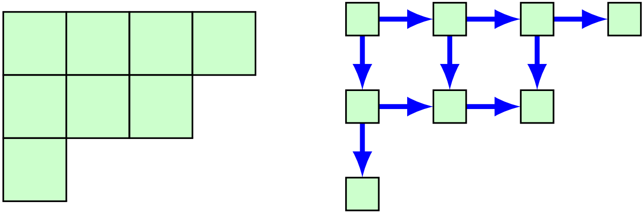
*roughly between* 0.276 *and* 0.724.

Upper-lower bound is tight for infinite posets.

# Young diagrams

Elements of $P_\lambda$ are cells of Young diagram of shape $\lambda$.

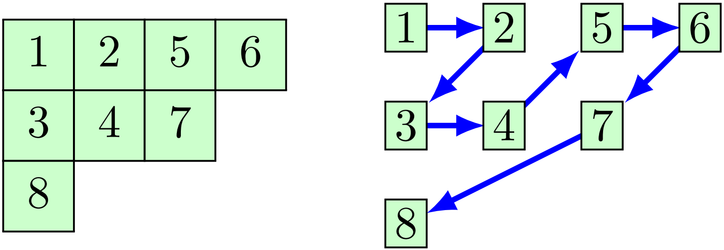
$x \preccurlyeq y \quad$ if $y$ lies to the Southeast of $x$.



Young diagram of shape $\lambda = (4, 3, 1)$

# Young diagrams

Linear extensions of $P_\lambda$ correspond to standard Young tableau of the Young diagram.




Linear extensions are counted by hook-length formulas.

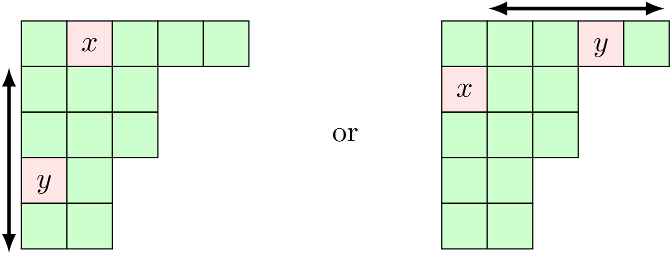
# What is known for Young diagrams

**Theorem (Olson–Sagan '18)**

*There always exists $x, y$:*

$$\frac{1}{3} \quad \leq \quad \mathbf{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{2}{3},$$

*for posets of Young diagrams.*



# What is known for Young diagrams

## Theorem (Olson–Sagan '18)

*There always exists* $x, y$*:*

$$\frac{1}{3} \quad \leq \quad \mathbf{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{2}{3},$$

*for posets of Young diagrams.*

We sketch an alternative proof for Young diagrams using Naruse hook-length formulas.

# Hook-length formulas

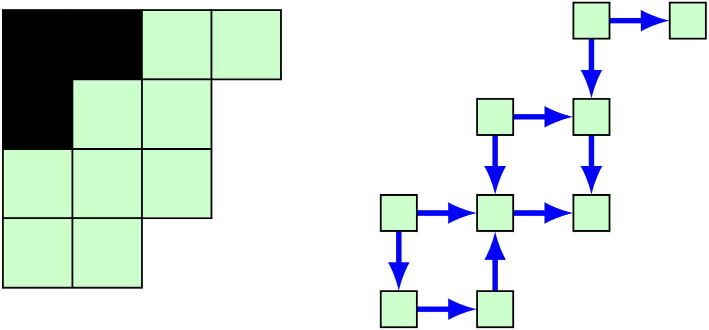
Number of linear extensions of $P_\lambda$ is equal to

$$\mathsf{HLF}(\lambda) \;=\; \frac{(|\lambda|)!}{\prod h_\lambda(x)}\,.$$

| | | | |
|---|---|---|---|
| 7 | 6 | 4 | 1 |
| 5 | 4 | 2 | |
| 4 | 3 | 1 | |
| 2 | 1 | | |

$$\frac{12!}{(7)(6)(4)(1)\,(5)(4)(2)\,(4)(3)(1)\,(2)(1)}$$

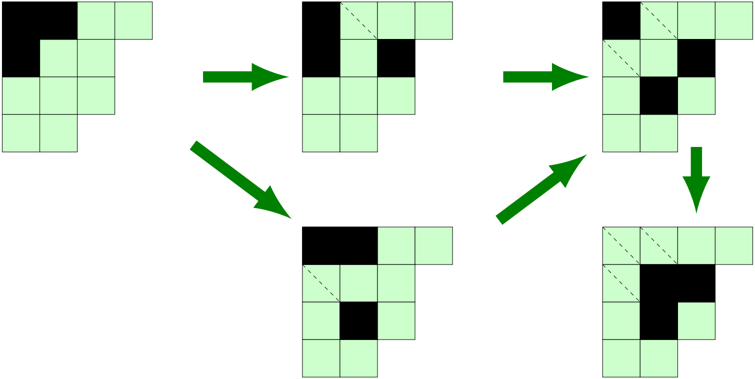
# Skew Young diagrams



Skew Young diagram of shape $\lambda/\mu$,
$\lambda = (5, 3, 3, 1)$ and $\mu = (2, 1)$.

# Excited diagrams

Black boxes can move on the Southeast direction.



# Naruse hook-length formulas

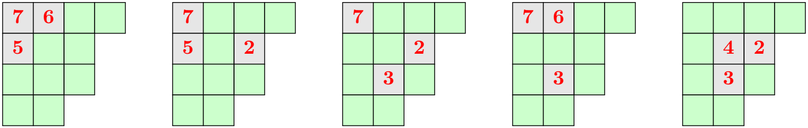
**Theorem**

**(Naruse '14, Morales-Pak-Panova '17)**

*Number of linear extensions of $P_{\lambda/\mu}$ is equal to*

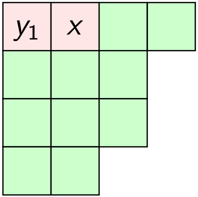
$$\mathsf{HLF}(\lambda)\,\frac{(|\lambda|-|\mu|)!}{(|\lambda|)!}\underbrace{\sum}_{\text{excited diagrams}}\underbrace{\prod_{x\in B} h_\lambda(x)}_{\text{black cells}}\,.$$

# Naruse hook-length formulas




$$\mathsf{HLF}(\lambda)\,\frac{(12-3)!}{(12)!}\Big[(7)(6)(5)\;+\;(7)(5)(2)\;+\;(7)(2)(3)\;+\;(7)(6)(3)\;+\;(4)(2)(3)\Big].$$

# Proof of Olson–Sagan result

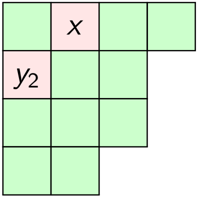



$\mathbf{P}\left[\, x \preccurlyeq y_1 \,\right] \;=$ 

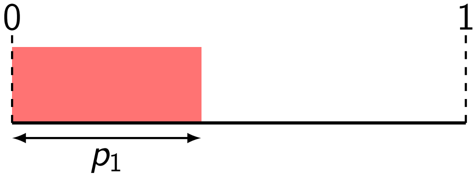

The jump probabilities are

$$p_i \;:=\; \mathbf{P}\left[\, y_i \preccurlyeq x \preccurlyeq y_{i+1} \,\right]$$

# Proof of Olson–Sagan result




$$\mathbf{P}\left[\, x \preccurlyeq y_2 \,\right] \;=$$

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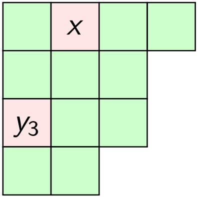
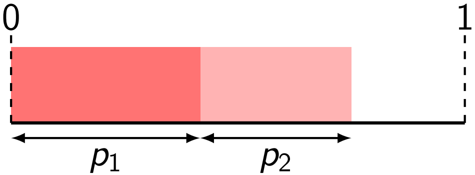



$$\mathbf{P}\big[\, x \preccurlyeq y_3 \,\big] \;=$$

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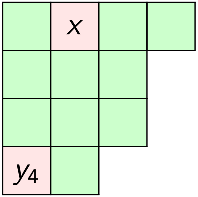
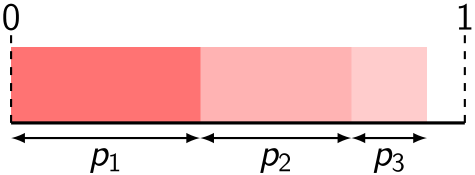



$$\mathbf{P}\left[\, x \preccurlyeq y_4 \,\right] \;=$$

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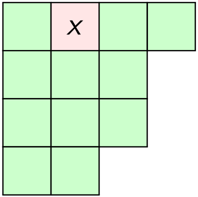
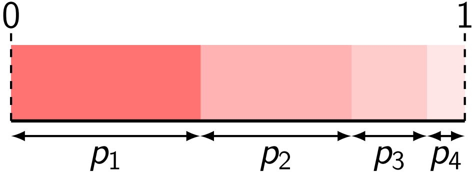
# Proof of Olson–Sagan result




$$\mathbf{P}\left[\, x \preccurlyeq y_5 \,\right] \;=$$



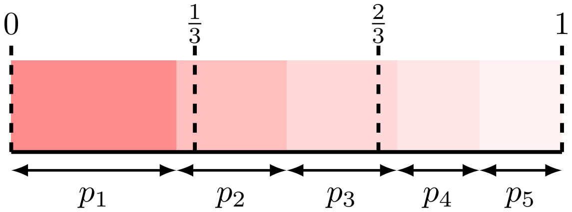

The jump probabilities are

$$p_i \;:=\; \mathbf{P}\left[\, y_i \preccurlyeq x \preccurlyeq y_{i+1} \,\right]$$

# Linial-type argument

Supppose that $p_1, p_2, \ldots, p_\ell$ are all $\leq \frac{1}{3}$.

$\mathbf{P}\left[\, x \preccurlyeq y_i \,\right] \;=$



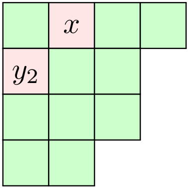

Look at when the probability exceeds $\frac{1}{3}$. Then

$$\frac{1}{3} \quad \leq \quad \mathbf{P}\left[\, x \preccurlyeq y_{i+1} \,\right] \quad \leq \quad \frac{2}{3}\,.$$

# First jump is less than $\frac{1}{3}$

We have $p_1 < \frac{1}{3}$ or $p_1 > \frac{2}{3}$, as otherwise

$$\frac{1}{3} \quad \leq \quad p_1 \ = \ \mathbf{P}\big[x \preccurlyeq y_2\big] \quad \leq \quad \frac{2}{3}\,.$$




By symmetry, $p_1 < \frac{1}{3}$.

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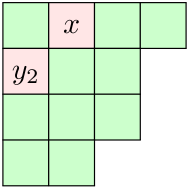


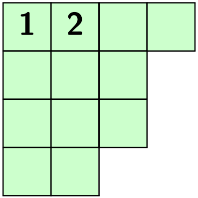

By symmetry, $p_1 < \frac{1}{3}$.

Thus suffices to show $p_1 \geq p_2 \geq \ldots \geq p_\ell$.

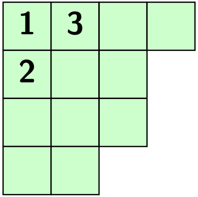
# Skew diagrams enter the scene

Probabilities $p_1$ and $p_2$ are equal to

$$p_1 \;=\; \mathbf{P}\big[\, y_1 \preccurlyeq x \preccurlyeq y_2 \,\big] \;=$$




$$p_2 \;=\; \mathbf{P}\big[\, y_2 \preccurlyeq x \preccurlyeq y_3 \,\big] \;=$$



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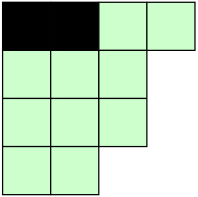

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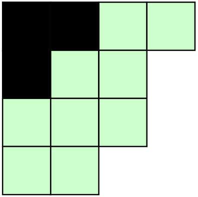
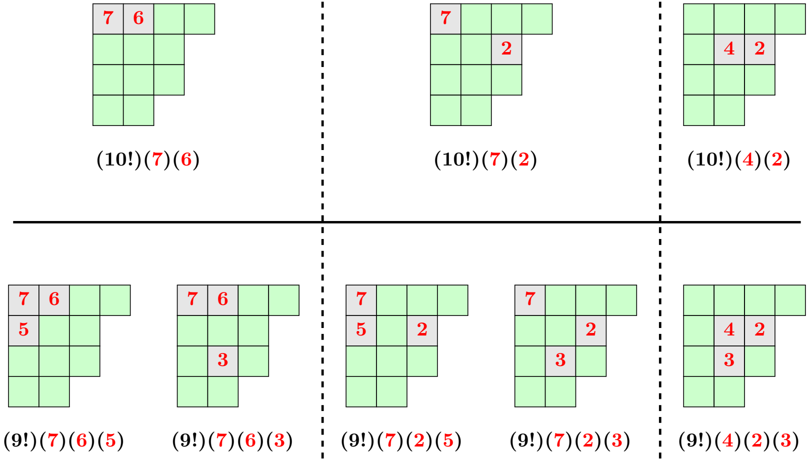


We can now use **NHLF**.

# $p_1$ is greater than $p_2$



**Thus we complete the proof of this theorem.**

### Theorem

*There always exists $x, y$:*

$$\frac{1}{3} \quad \leq \quad \mathbf{P}\left[x \preccurlyeq y\right] \quad \leq \quad \frac{2}{3},$$

*for poset $P_\lambda$ of Young diagram of shape $\lambda$.*

# What we will do next

Previously, we want to find $x, y$:

$$\frac{1}{3} \quad \leq \quad \mathbf{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{2}{3}\,,$$

Now, we want to find $x, y$:

$$\frac{1}{2} \quad \approx \quad \mathbf{P}\big[\, x \preccurlyeq y \,\big] \quad \approx \quad \frac{1}{2}\,,$$

# Kahn–Saks Conjecture

**Conjecture (Kahn-Saks '84)**

*For every finite poset $P$, there exists $x, y$:*

$$\frac{1}{2} - \delta(P) \quad \leq \quad \mathbf{P}\left[x \preccurlyeq y\right] \quad \leq \quad \frac{1}{2} + \delta(P),$$

*with $\delta(P) \to 0$ as width$(P) \to \infty$.*

Here width$(P)$ is the largest size of anti-chains in $P$.

Komlós '90 proved such a result for posets with large number of minimal elements.

# Our results

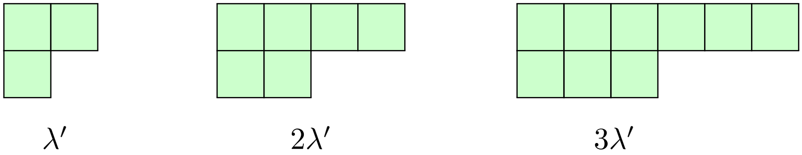
# First result

## Theorem (C.-Pak-Panova '20+)

*There exists $C > 0$ and $x, y$:*

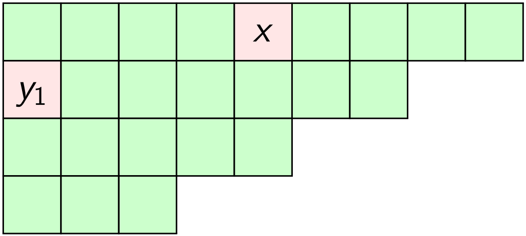
$$\frac{1}{2} - \frac{C}{\sqrt{n}} \quad \le \quad \mathbf{P}\big[x \preccurlyeq y\big] \quad \le \quad \frac{1}{2} + \frac{C}{\sqrt{n}},$$

*for poset $P_\lambda$ of Young diagram of shape $\lambda = \lambda' n$.*



# Where is the improvement?

$x$ and $y$ are different from Olson–Sagan.




$\mathbf{P}\left[\, x \preccurlyeq y_1 \,\right] =$

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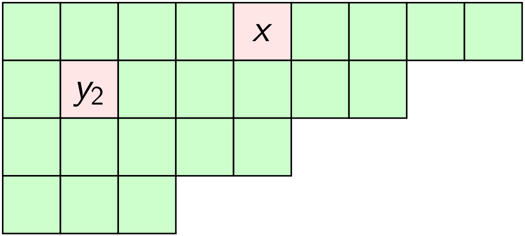


$$\mathbf{P}\left[\, x \preccurlyeq y_2 \,\right] \;=$$

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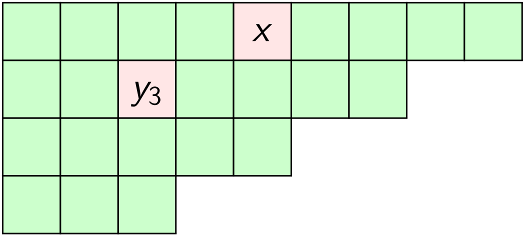
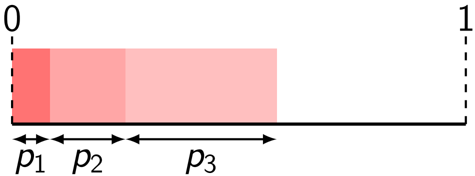



$$\mathbf{P}\left[\, x \preccurlyeq y_3 \,\right] \;=$$



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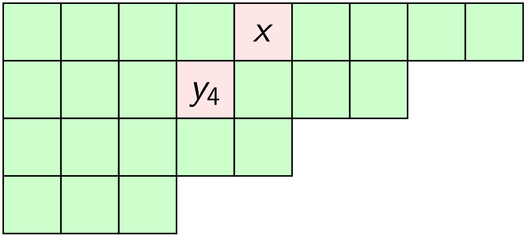
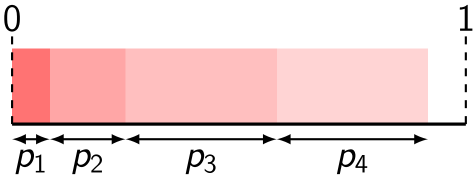
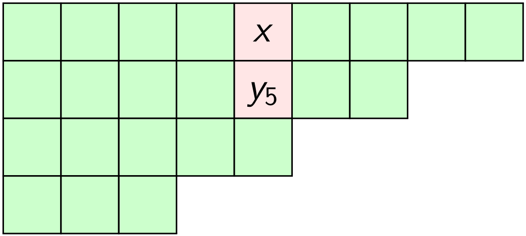



$$\mathbf{P}\left[\, x \preccurlyeq y_4 \,\right] \;=$$

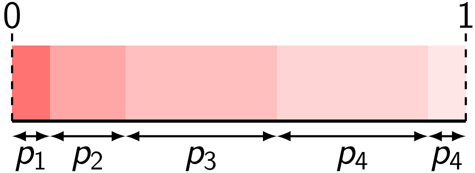


# Where is the improvement?

$x$ and $y$ are different from Olson–Sagan.



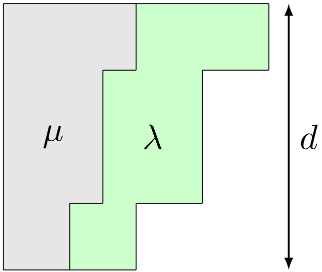

$\mathbf{P}\left[\, x \preccurlyeq y_5 \,\right] =$



# Key steps in the proof

After several asymptotic reductions,

$$\left| \mathbf{P}\left[\, x \preccurlyeq y \,\right] - \tfrac{1}{2} \right| \;\leq\; \text{sums of}$$




$$(\mu_1, \ldots, \mu_d) \;\approx\; \left( \frac{\lambda_1}{2} \pm \sqrt{\lambda_1}, \ldots, \frac{\lambda_d}{2} \pm \sqrt{\lambda_d} \right).$$

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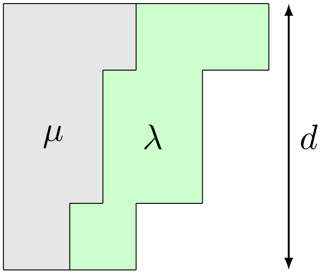


$$\left(\mu_1, \ldots, \mu_d\right) \;\approx\; \left(\frac{\lambda_1}{2} \pm \sqrt{\lambda_1}, \ldots, \frac{\lambda_d}{2} \pm \sqrt{\lambda_d}\right).$$

By NHLF, reduces to asymptotic of Schur polynomials.

# Schur polynomial

Schur polynomial is

$$s_\lambda(z_1, \ldots, z_d) \;=\; \sum_{T \in \mathrm{SSYT}(\lambda)} z_1^{t_1(T)} \ldots z_n^{t_d(T)},$$

summed over semistandard Young tableau.

| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 2 |
| 2 | 2 | 2 | |
| 3 | 3 | 4 | |
| 4 | | | |

$$z_1^3\, z_2^4\, z_3^2\, z_4^2$$

# Key lemma

**Lemma**

*For $d > 0$ and $z_1 > \ldots > z_d > \varepsilon z_1 > 0$,*

$$\frac{s_\lambda(z_1, \ldots, z_d)}{z_1^{\lambda_1} \ldots z_d^{\lambda_d}} \lesssim \prod_{1 \le i < j \le d} \left\{ \lambda_i - \lambda_j + 1, \frac{z_i}{z_i - z_j} \right\},$$

*under some technical conditions.*

# Back to first result

**Theorem (C.-Pak-Panova '20+)**

*There exists $C > 0$ and $x, y$:*

$$\frac{1}{2} - \frac{C}{\sqrt{n}} \quad \leq \quad \mathbf{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{1}{2} + \frac{C}{\sqrt{n}},$$

*for poset $P_\lambda$ of Young diagram of shape $\lambda = \lambda' n$.*

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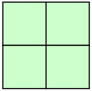
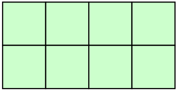
But we can do better for Catalan posets!

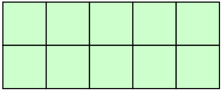
# Catalan posets



$n = 2$

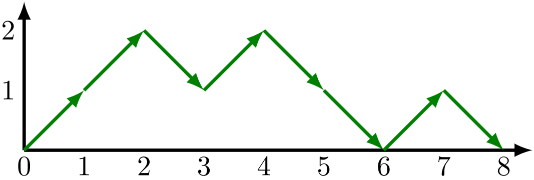


$n = 4$



$n = 5$

| | | | |
|---|---|---|---|
| 1 | 2 | 4 | 7 |
| 3 | 5 | 6 | 8 |



# Second result

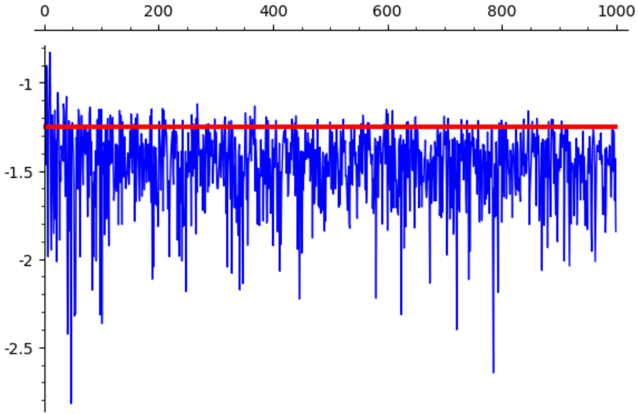
**Theorem (C.-Pak-Panova '20+)**

*There exists $C > 0$ and $x, y$:*

$$\frac{1}{2} - \frac{C}{n^{1.25}} \quad \leq \quad \mathbf{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{1}{2} + \frac{C}{n^{1.25}}\,,$$

*For Catalan posets with $2n$ cells.*

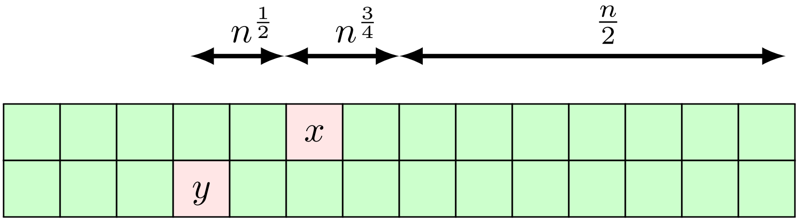
# How good is this bound?




Exponent of the error for $n \leq 1000$.

# How to improve to $n^{1.25}$?

Calculations are done with **HLF** instead of **NHLF**.



# What is next?

## Theorem (C.-Pak-Panova '20+)

*There exists $x, y$:*

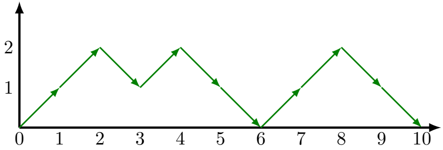
$$\frac{1}{2} - \delta(n) \quad \leq \quad \mathbf{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{1}{2} + \delta(n)\,,$$

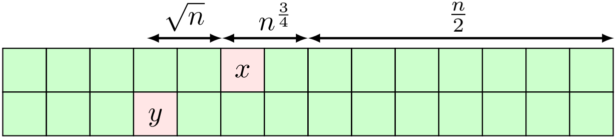
*where $\delta(n) \to 0$ for the poset of Young diagram of shape $\lambda n$.*

## Open Problem

*Prove same result for other families of posets, e.g., k-dimensional Young diagrams and periodic posets.*

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 4 | 7 | 8 |
| 3 | 5 | 6 | 9 | 10 |



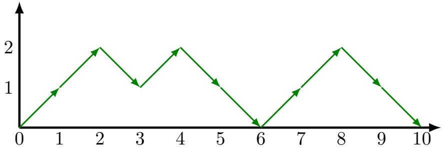



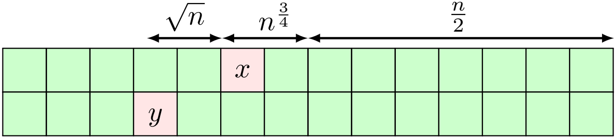

ArXiv preprints: 2005.08390 and 2005.13686.

Webpage: http://math.ucla.edu/∼sweehong/

# THANK YOU!

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 4 | 7 | 8 |
| 3 | 5 | 6 | 9 | 10 |

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