Combinatorial Atlas for Log-concave Inequalities

Swee Hong Chan (UCLA)

joint with Igor Pak

What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \geq a_{k+1}\, a_{k-1} \qquad (1 \leq k < n).$$

Equivalently,

$$\log a_k \geq \frac{\log a_{k+1} + \log a_{k-1}}{2} \qquad (1 \leq k < n).$$

1 4 9 15 20 22 20 15 9 4 1

Example: binomial coefficients

$$a_k \;=\; \binom{n}{k} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\,a_{k-1}} \;=\; \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} \;=\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example: permutations with k inversions

$$a_k \;=\; \text{number of } \pi \in S_n \text{ with } k \text{ inversions},$$

where inversion of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \le k \le \binom{n}{2}} a_k\, x^k \;=\; (1+x)\ldots(1+x+\ldots+x^{n-1})$$

is a product of log-concave polynomials.

Log-concavity appears in many objects:

algebras, matroids, mixed volumes,

measures, posets, random walks.

Log-concavity appears in many objects:

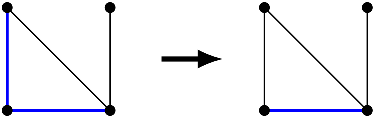
algebras, **matroids**, mixed volumes,
measures, **posets**, random walks

Today we focus on **matroids** and **posets**.

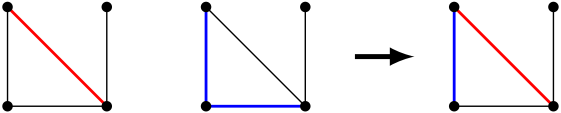
Matroids

Matroid $\mathcal{M}$ is ground set X with collection of independent sets $\mathcal{I} \subseteq 2^X$,

- $S \subseteq T$ and $T \in \mathcal{I}$ implies $S \in \mathcal{I}$.



- If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.



Examples: Matroids

Graphical matroids

- X = edges of a graph G,
- $\mathcal{I}$ = forests in G.

Realizable matroids

- X = finite set of vectors over field $\mathbb{F}$,
- $\mathcal{I}$ = sets of linearly independent vectors.

Mason's Conjecture (1972)

For every matroid and $k \geq 1$,

(1) $I_k{}^2 \;\geq\; I_{k+1}\, I_{k-1}$;

(2) $I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right) I_{k+1}\, I_{k-1}$;

(3) $I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1}$.

I_k is number of ind. sets of size k, and $n = |X|$.

Why $\left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right)$?

Mason (3) is equivalent to ultra/binomial log-concavity,

$$\frac{{I_k}^2}{\binom{n}{k}^2} \;\geq\; \frac{I_{k+1}}{\binom{n}{k+1}}\,\frac{I_{k-1}}{\binom{n}{k-1}}.$$

Equality occurs **if** every $(k+1)$-subset is independent.

Solution to Mason (1)

Theorem (Adiprasito-Huh-Katz '18)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; I_{k+1}\, I_{k-1}.$$

Proof used combinatorial Hodge theory for matroids.

Solution to Mason (2)

Theorem (Huh-Schröter-Wang '18)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right) I_{k+1}\, I_{k-1}.$$

Proof used combinatorial Hodge theory for correlation bound on matroids.

Solution to Mason (3)

Theorem

(Anari-Liu-Oveis Gharan-Vinzant, Brändén-Huh '20)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1}.$$

Proof used theory of strong log-concave polynomials / Lorentzian polynomials.

Solution to Mason (3)

Theorem

(Anari-Liu-Oveis Gharan-Vinzant, Brändén-Huh '20)
For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1}.$$

Theorem (Murai-Nagaoka-Yazawa '21)

Equality occurs if and only if every $(k+1)$*-subset is independent.*

Our contribution

Method: Combinatorial atlas

Results: Log-concave inequalities, and
if and only if conditions for equality

- Matroids (refined);
- Morphism of matroids (refined);
- Discrete polymatroids;
- Stanley's poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex).

Method: Combinatorial atlas

Results: Log-concave inequalities, and

if and only if conditions for equality

- **Matroids (refined)**;
- Morphism of matroids (refined);
- Discrete polymatroids;
- **Stanley's poset inequality (refined);**
- **Poset antimatroids**;
- Branching greedoid (log-convex).

Matroids

Warmup: graphical matroids refinement

Corollary (C.-Pak)

For graphical matroid of simple connected graph $G = (V, E)$ *that is not tree, and* $k = |V| - 2$,

$$(I_k)^2 \;\geq\; \frac{3}{2}\left(1 + \frac{1}{k}\right) I_{k+1}\, I_{k-1},$$

with equality if and only if G *is cycle graph.*

Numerically better than Mason (3), because

$$\frac{3}{2} \;\geq\; 1 + \frac{1}{n-k} \;=\; 1 + \frac{1}{|E| - |V| + 2}.$$

Comparison with Mason (3)

Our bound gives

$$\frac{(I_k)^2}{I_{k+1}\,I_{k-1}} \quad \geq \quad \frac{3}{2} \qquad \text{when} \;\; |E| - |V| \to \infty,$$

Meanwhile, Mason (3) bound only gives

$$\frac{(I_k)^2}{I_{k+1}\,I_{k-1}} \quad \geq \quad 1 \qquad \text{when} \;\; |E| - |V| \to \infty.$$

Our bound is better numerically and asymptotically.

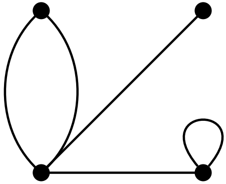
Parallel classes of matroid $\mathcal{M}$

Loop is $x \in X$ such that $\{x\} \notin \mathcal{I}$.

Non-loops x, y are parallel if $\{x, y\} \notin \mathcal{I}$.

Parallelship equiv. relation: $x \sim y$ if $\{x, y\} \notin \mathcal{I}$.

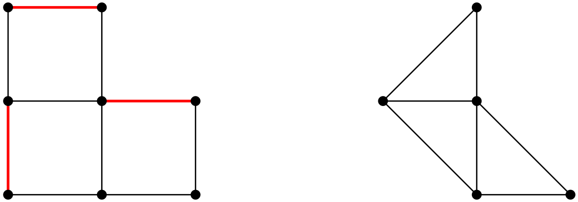
Parallel class $=$ equivalence class of $\sim$.



Matroid contraction

Contraction of $S \in \mathcal{I}$ is matroid $\mathcal{M}_S$ with

$$X_S = X \setminus S, \qquad \mathcal{I}_S = \{T \setminus S : S \subseteq T\}.$$

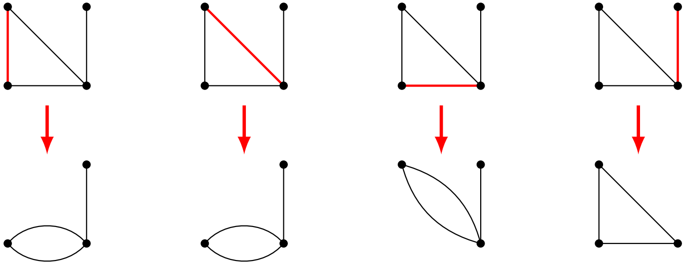


$$\mathrm{prl}(S) := \text{ number of parallel classes of } \mathcal{M}_S$$

Parallel number

The *k*-parallel number is

$$\mathrm{prl}(k) \ := \ \max\{\mathrm{prl}(S) \mid S \in \mathcal{I} \text{ with } |S| = k\}.$$



$\mathrm{prl}(S) = 2$ $\mathrm{prl}(S) = 2$ $\mathrm{prl}(S) = 2$ $\mathrm{prl}(S) = 3$

$\mathrm{prl}(1) = 3$

Refinement for Mason (3)

Theorem 1 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k{}^2 \ \geq \ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{prl}(k-1)+1}\right) I_{k+1}\, I_{k-1}.$$

This refines Mason (3),

$$I_k{}^2 \ \geq \ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1},$$

since

$$\mathrm{prl}(k-1) \ \leq \ n - k + 1.$$

When is equality achieved?

- When every $(k+1)$-subset is independent,
  $$\mathrm{prl}(k-1) \;=\; n-k+1.$$
- Graphical matroid when G is a cycle,
  $$\mathrm{prl}(k-1) \;=\; 3.$$
- Full realizable matroids over finite field $\mathbb{F}_q$,
  $$\mathrm{prl}(k-1) \;=\; \frac{n}{q^{k-1}} - 1.$$
- (k, m, n)-Steiner system matroid,
  $$\mathrm{prl}(k-1) \;=\; \frac{n-k+1}{m-k+1}.$$

Equality conditions

Theorem 2 (C.-Pak)

For every matroid and $k \geq 1$,

$$\mathcal{I}_k^{\,2} = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\operatorname{prl}(k-1) + 1}\right) \mathcal{I}_{k+1}\, \mathcal{I}_{k-1}$$

if and only if

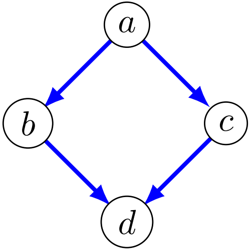
for every $S \in \mathcal{I}$ *with* $|S| = k - 1$,

- S *has* $\operatorname{prl}(k-1)$ *parallel classes; and*
- *Every parallel class of* S *has same size.*

Stanley's poset inequality

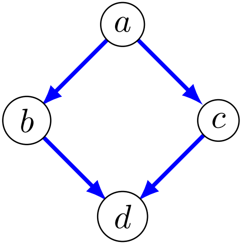
Partially ordered sets

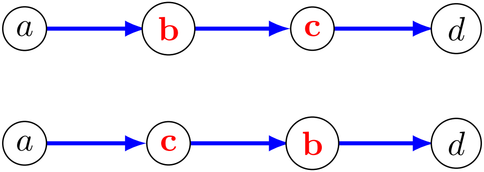
A poset P is a set X with a partial order $\prec$ on X.



Linear extension

A linear extension L is a complete order of $\prec$.





We write $L(x) = k$ if x is k-th smallest in L.

Stanley's inequality

Fix $z \in P$.

N_k *is* number of linear extensions with $L(z) = k$.

Theorem (Stanley '81)

For every poset and $k \geq 1$,

$${N_k}^2 \;\geq\; N_{k+1}\, N_{k-1}.$$

Proof used Aleksandrov-Fenchel inequality for mixed volumes.

When is equality achieved?

Theorem (Shenfeld-van Handel)

Suppose $N_k > 0$. *Then*

$$N_k{}^2 = N_{k+1}\, N_{k-1}$$

if and only if

$$N_k \;=\; N_{k+1} \;=\; N_{k-1}.$$

Proof used classifications of extremals of Aleksandrov-Fenchel inequality for convex polytopes.

Our contribution

We give new combinatorial proof for Stanley's ineq.
and extend to weighted version.

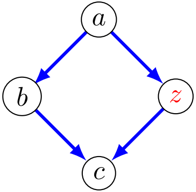
Order-reversing weight

A weight $w : X \to \mathbb{R}_{>0}$ is order-reversing if

$$w(x) \;\geq\; w(y) \qquad \text{whenever} \qquad x \prec y.$$

Weight of linear extension L is

$$w(L) \;:=\; \prod_{L(x)<L(z)} w(x).$$



$$w(L) = w(a)$$

$$w(L) = w(a)w(b)$$

Weighted Stanley's inequality

Fix $z \in P$.

$N_{w,k}$ *is* w-weight of linear extensions with $L(z) = k$.

Theorem 3 (C. Pak)

For every poset and $k \geq 1$,

$$N_{w,k}{}^2 \;\geq\; N_{w,k+1}\, N_{w,k-1}.$$

When is equality achieved?

Theorem 4 (C.-Pak)

Suppose $N_{w,k} > 0$. *Then*

$$ {N_{w,k}}^2 = N_{w,k+1}\, N_{w,k-1} $$

if and only if

for every linear extension L *with* $L(z) = k$,

$$ w(L^{-1}(k+1)) \; = \; w(L^{-1}(k-1)) \; =: \; s, $$

and

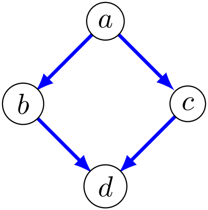
$$ \frac{N_{w,k}}{s^k} \; = \; \frac{N_{w,k+1}}{s^{k+1}} \; = \; \frac{N_{w,k-1}}{s^{k-1}}. $$

Poset antimatroids

Feasible words of a poset

A word $\alpha \in X^*$ is feasible if no repeating elements, and

y occurs in α and $x \prec y \quad \Rightarrow \quad x$ occurs in α before y.



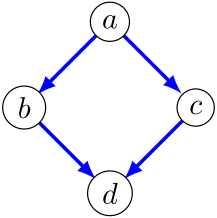
Feasible: $\varnothing$, a, ab, ac, abc, acb, $abcd$.

Not feasible: aa, bc, ba.

Chain weight

For $x \in P$, chain weight is

$\omega(x) =$ number of maximal chains that starts with x.

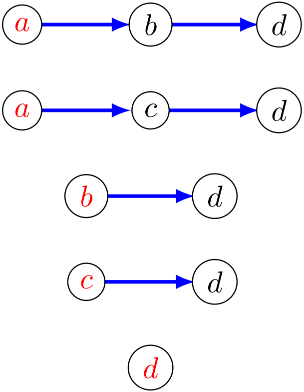


$\omega(a) = 2$

$\omega(b) = 1$

$\omega(c) = 1$

$\omega(d) = 1$



Weight of word α is $\omega(\alpha) := \omega(\alpha_1) \dots \omega(\alpha_\ell)$.

Log-concave inequality for poset antimatroids

$F_{\omega,k}$ is sum of ω-weight of feasible words of length k.

Theorem 5 (C.-Pak)

For every poset and $k \geq 1$,

$$F_{\omega,k}{}^2 \;\geq\; F_{\omega,k+1}\, F_{\omega,k-1}.$$

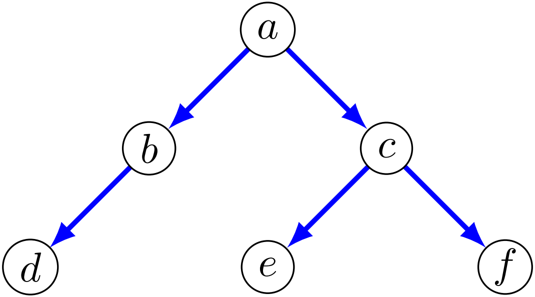
When is equality achieved?

Theorem 6 (C.-Pak)

Equality occurs for $k = 1, \ldots, \mathrm{height}(P) - 1$

if and only if

Hasse diagram of P *is a forest where every leaf is of the same level.*



Method: Combinatorial atlas

Results: Log-concave inequalities, and
if and only if conditions for equality

- Matroids (refined);
- Morphism of matroids (refined);
- Discrete polymatroids;
- Stanley's poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex).

Method: **Combinatorial atlas**

Results: Log-concave inequalities, and
if and only if conditions for equality

- Matroids (refined);
- Morphism of matroids (refined);
- Discrete polymatroids;
- Stanley's poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex).

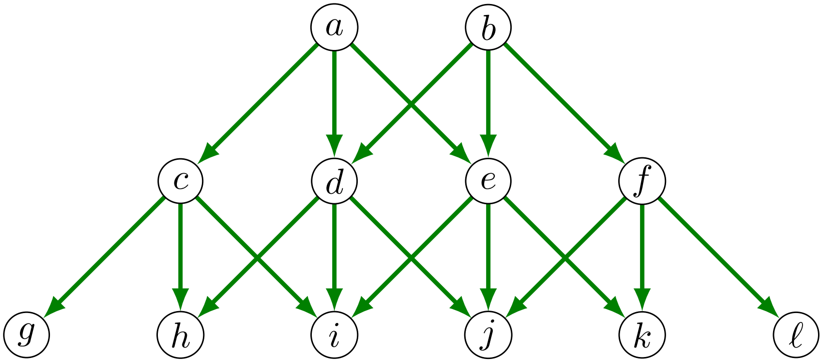
Combinatorial atlas

The strategy

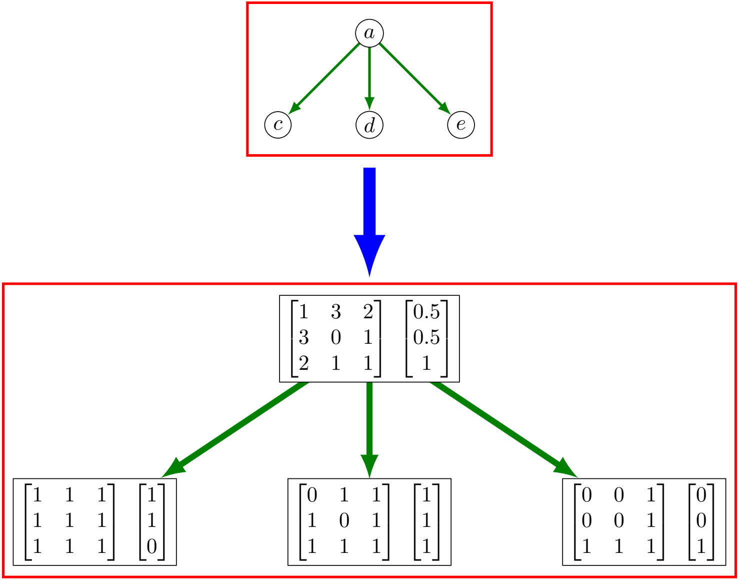
Input: Acyclic digraph $\mathcal{A}$, where each vertex v is associated with

- $r \times r$ nonnegative symmetric matrix $\boldsymbol{M}$;
- nonnegative r-vector $\boldsymbol{h}$.

Combinatorial atlas: example

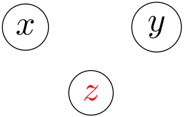


Combinatorial atlas: example (zoomed in)

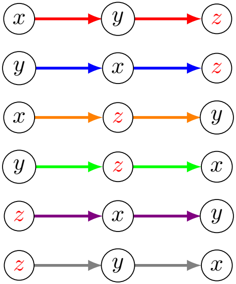


Combinatorial atlas: Example Stanley's Inequality

Poset $=$



	x_{min}	y_{min}	x_{max}	y_{max}
x_{min}	0	1	0	1
y_{min}	1	0	1	0
x_{max}	0	1	0	1
y_{max}	1	0	1	0



The strategy

Input: Acyclic digraph $\mathcal{A}$, where each vertex v is associated with

- $r \times r$ nonnegative symmetric matrix $\boldsymbol{M}$;
- A nonnegative r-vector $\boldsymbol{h}$.

Goal: Show every $\boldsymbol{M}$ has hyperbolic inequality.

Hyperbolic inequality

$\boldsymbol{M}$ has hyperbolic inequality property if

$$\langle \boldsymbol{x}, \boldsymbol{M}\boldsymbol{y} \rangle^2 \geq \langle \boldsymbol{x}, \boldsymbol{M}\boldsymbol{x} \rangle \langle \boldsymbol{y}, \boldsymbol{M}\boldsymbol{y} \rangle,$$

for every $\boldsymbol{x} \in \mathbb{R}^r$, $\boldsymbol{y} \in \mathbb{R}^r_{\geq 0}$.

Note: This property already known to be important in Lorentzian polynomials and Bochner's method proof of Aleksandrov-Fenchel inequality.

How to get log-concave inequalities?

Assume a_{k-1}, a_k, a_{k+1} can be computed by

$$a_k = \langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{h} \rangle, \; a_{k+1} = \langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{g} \rangle, \; a_{k-1} = \langle \boldsymbol{h}, \boldsymbol{M}\boldsymbol{h} \rangle,$$

for specific $\boldsymbol{M}, \boldsymbol{g}, \boldsymbol{h}$ in the atlas.

$$\langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{h} \rangle^2 \geq \langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{g} \rangle \langle \boldsymbol{h}, \boldsymbol{M}\boldsymbol{h} \rangle \quad \text{(hyperbolic ineq.)}$$

then implies

$$a_k^2 \geq a_{k+1} a_{k-1} \quad \text{(log-concave ineq.)}$$

The strategy

Input: Acyclic digraph $\mathcal{A}$, where each vertex v is associated with

- $r \times r$ nonnegative symmetric matrix $\boldsymbol{M}$,
- A nonnegative r-vector $\boldsymbol{h}$.

Goal: Show every $\boldsymbol{M}$ has hyperbolic inequality.

Method: Verify three conditions:

- **Irreducibility condition**;
- **Inheritance condition**;
- **Subdivergence condition**.

Irreducibility condition

- Matrix $\boldsymbol{M}$ associated to v is irreducible when restricted to its support;
- Vector $\boldsymbol{h}$ is associated to v is a positive vector.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Inheritance condition

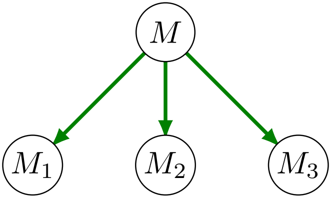
The i-th edge $e = (v, v_i)$ of v is associated with

$$\text{linear map} \quad T_i : \mathbb{R}^r \to \mathbb{R}^r$$

such that, for every $\boldsymbol{x} \in \mathbb{R}^r$,

$$i\text{-th coordinate of } \boldsymbol{Mx} \;=\; \langle T_i\boldsymbol{x}, \boldsymbol{M}_i T_i \boldsymbol{h}\rangle,$$

where $\boldsymbol{M}$ and $\boldsymbol{h}$ are associated to v, while $\boldsymbol{M}_i$ is associated to v_i.



Subdivergence condition

For every $\boldsymbol{x} \in \mathbb{R}^r$,

$$\sum_{i=1}^{r} h_i \langle T_i \boldsymbol{x}, \boldsymbol{M}_i T_i \boldsymbol{x} \rangle \ \geq \ \langle \boldsymbol{x}, \boldsymbol{M} \boldsymbol{x} \rangle,$$

where h_i = i-th coordinate of $\boldsymbol{h}$.

Note: Often hardest condition to check, usually done through injective arguments.

Note: Equality occurs for matroids.

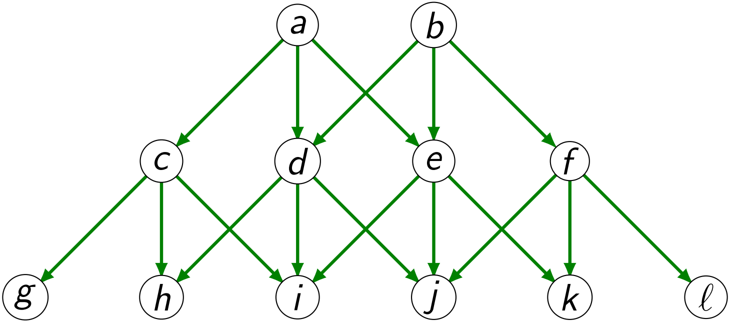
Bottom-to-top principle for inequalities

Proposition

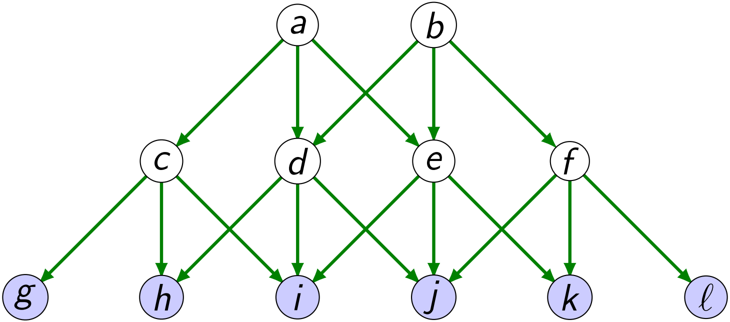
Assume irreducibility, inheritance, subdivergence. If $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_r$ has hyperbolic inequality property, then so does $\boldsymbol{M}$.

Bottom-to-top principle reduces **Goal** to checking hyperbolic inequality only for sink vertices, which are usually easy to check.

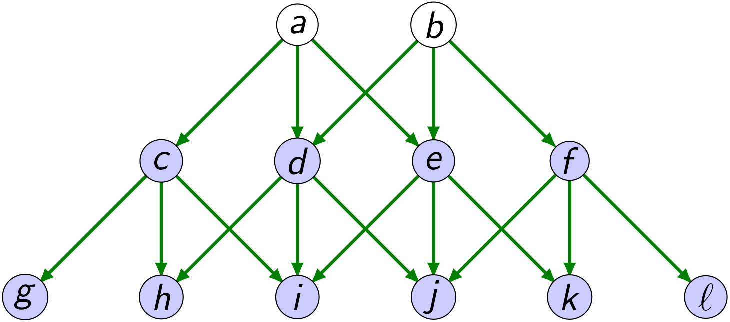
Bottom-to-top principle



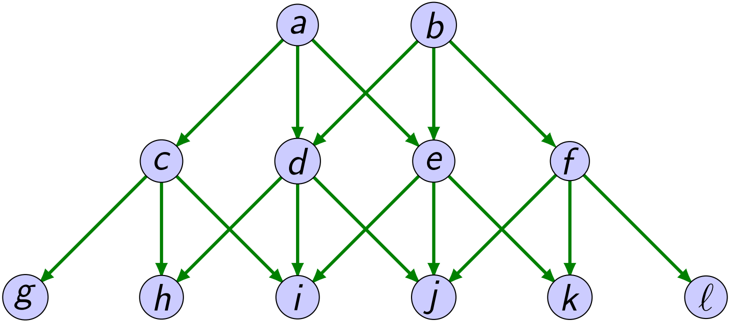
Bottom-to-top principle



Bottom-to-top principle



Bottom-to-top principle



How about equalities?

The strategy

Input:

- An acyclic digraph $\mathcal{A} := (\mathcal{V}, \mathcal{E})$ satisfying previous conditions;
- Vectors $\boldsymbol{g}, \boldsymbol{h} \in \mathbb{R}_{\geq 0}$;

Goal: Show "every" $\boldsymbol{M}$ has hyperbolic equality,

$$\langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{h} \rangle^2 = \langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{g} \rangle \langle \boldsymbol{h}, \boldsymbol{M}\boldsymbol{h} \rangle.$$

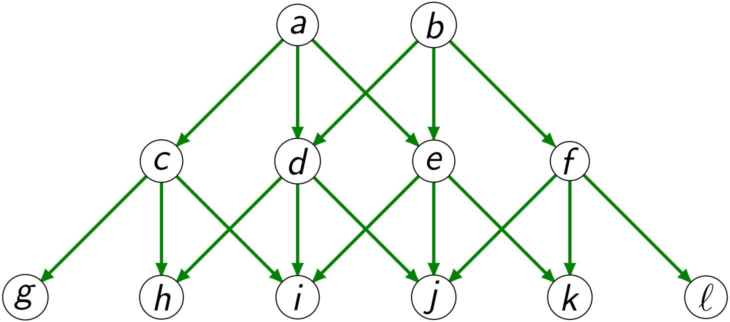
Top-to-bottom principle for equalities

Proposition

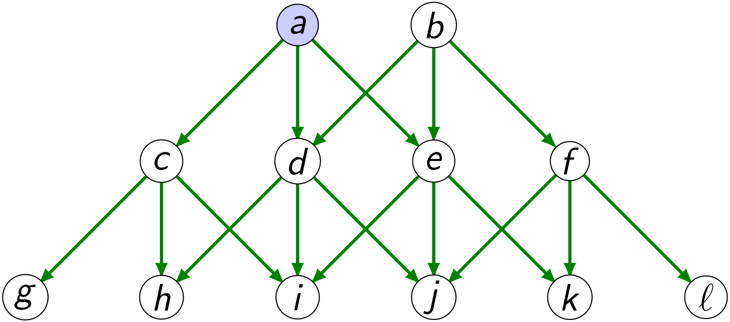
Assume regularity condition. If $\boldsymbol{M}$ has hyperbolic equality property, then so do $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_r$.

Top-to-bottom principle expands hyperbolic equality to sink vertices, which usually gives combinatorial characterizations.

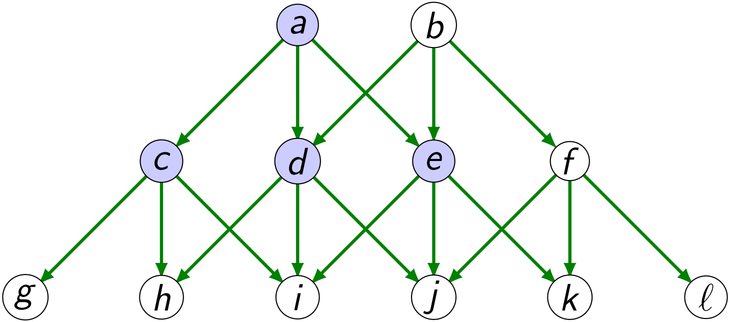
Top-to-bottom principle



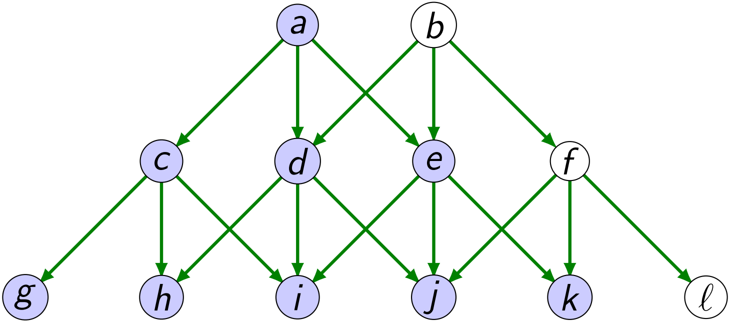
Top-to-bottom principle



Top-to-bottom principle



Top-to-bottom principle



Conclusion

Problem: Log-concave inequalities and equalities.

Strategy:

- Build a combinatorial atlas;
- Verify the required conditions;
- Use hyperbolic inequality property to derive log-concave inequalities;
- Use hyperbolic equality to derive log-concave equalities.

THANK YOU!

Preprint to appear soon in your nearest arXiv server.

Webpage: http://math.ucla.edu/~sweehong/

Email: sweehong@math.ucla.edu