Gröbner geometry of Schubert polynomials through ice

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The complete flag variety

The **complete flag variety** $\mathcal{F}\ell(\mathbb{C}^n)$ is the set of complete flags of nested vector subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n,$$

where $\dim V_i = i$.

Example

Standard flag $\mathcal{SF}$: $\quad \subset \quad \subset \quad \subset \quad \subset \quad = \mathbb{C}^4$

Since $\mathcal{F}\ell(\mathbb{C}^n)$ has transitive action of GL_n, we can identify it with $\mathrm{GL}_n(\mathbb{C})/\mathrm{Stab}(\mathcal{SF}) = \mathrm{GL}_n(\mathbb{C})/U$, where $U =$ upper triangular matrices.

Schubert varieties

Bruhat decomposition: $\mathrm{GL}_n = \coprod_{w \in S_n} LwU$

Schubert cells: $X_w^\circ = LwU/U \subset \mathcal{F}\ell(\mathbb{C}^n)$

Schubert varieties: $X_w = \overline{X_w^\circ}$ give a complex cell decomposition of $F\ell(\mathbb{C}^n)$.

Schubert classes $[X_w]$ form a basis for the cohomology ring

$$H^\star(\mathcal{F}\ell(\mathbb{C}^n)) \cong \mathbb{Z}[x_1, \ldots, x_n]/\mathbb{Z}[x_1, \ldots, x_n]_+^{S_n}.$$

Lascoux and Schützenberger (1982) introduced a "nice" choice for polynomial representatives of these classes:

$$[X_w] \mapsto \mathfrak{S}_w(x_1, \ldots, x_n), \text{ a } \textbf{Schubert polynomial}.$$

Schubert polynomials are nice

No cleanup: $\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w \Leftrightarrow [X_u] \cdot [X_v] = \sum_w c_{u,v}^w [X_w]$

Stability: For $w \in S_n$, $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$ where $\iota : S_n \to S_{n+1}$ is the natural inclusion.

Monomial positivity:
$\mathfrak{S}_{14523} = x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_2^2x_3^2$

The definition of $\mathfrak{S}_w(\mathbf{x})$

- Start with the **longest** permutation in S_n

$$w_0 = n\,n-1\ldots 1 \qquad\qquad \mathfrak{S}_{w_0}(\mathbf{x}) := x_1^{n-1}x_2^{n-2}\ldots x_{n-1}$$

- The rest are defined recursively by **divided difference operators:**

$$\partial_i f := \frac{f - s_i \cdot f}{x_i - x_{i+1}} \quad \text{and} \quad \mathfrak{S}_{ws_i}(\mathbf{x}) := \partial_i \mathfrak{S}_w(\mathbf{x}) \text{ if } w(i) > w(i+1)$$

Double Schubert polynomials

The diagonal matrices $D \curvearrowright \mathrm{GL}_n(\mathbb{C})/U$ preserving the Schubert varieties.

D-equivariant Schubert classes $[X_w]_D \in H^\star_D(\mathcal{F}\ell(\mathbb{C}^n))$.

There are also **double Schubert polynomials**, which are defined by the same operators, with the initial condition

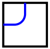
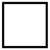
$$\mathfrak{S}_{w_0}(\mathbf{x};\mathbf{y}) := \prod_{i+j\leq n} (x_i - y_j).$$

$\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ represents $[X_w]_D$.

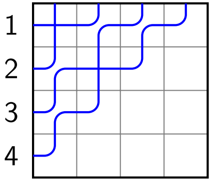
Pipe dreams

A **pipe dream** is a tiling of the $n \times n$ grid with the four tiles so that there are n pipes which

- start at the left edge of the grid,
- end at the top edge,
- pairwise cross at most once.

We also require that only  appear on the main anti-diagonal and only  appear below.

Write PD(w) for the set of pipe dreams which trace out the permutation w.



If a $\boxplus$ tile sits in row i and column j, assign it the weight $(x_i - y_j)$. The weight of a pipe dream is the product of the weights of its crossing tiles.

$$\mathtt{wt}(\mathcal{P}) =$$

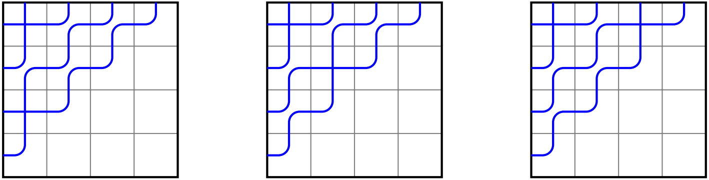
Pipe dreams for double Schuberts

Theorem (Fomin-Kirillov 1996, Knutson-Miller 2005)

The double Schubert polynomial $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ *is the weighted sum*

$$\mathfrak{S}_w(\mathbf{x};\mathbf{y}) = \sum_{\mathcal{P} \in \mathrm{PD}(w)} \mathtt{wt}(\mathcal{P}).$$

$$\mathfrak{S}_{2143} = (x_1 - y_1)(x_3 - y_1) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_3).$$



Equivariant cohomology

The group $D \times D$ acts on $\mathrm{Mat}(n) = \left\{ \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} \right\}$ by

$$(d, \delta) \cdot M := dM\delta^{-1}.$$

Since $\mathrm{Mat}(n)$ is contractible,

$$H_{D \times D}(\mathrm{Mat}(n)) = H_{D \times D}(\mathrm{pt}) = \mathcal{O}(\mathfrak{d} \times \mathfrak{d}) = \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

Whenever $X \in \mathrm{Mat}(n)$ is stable under the action of $D \times D$, it has a class $[X]_{D \times D} \in H_{D \times D}(\mathrm{Mat}(n)) = \mathbb{Z}[\mathbf{x}, \mathbf{y}]$.

Matrix Schubert varieties

The **matrix Schubert variety** (Fulton 1992) $\tilde{X}_w = \overline{LwU} \subseteq \mathrm{Mat}(n)$ is defined by rank conditions on maximal northwest submatrices.

Theorem (Knutson-Miller 2005)

The $D \times D$-equivariant cohomology class of a matrix Schubert variety is the corresponding double Schubert polynomial:

$$[\tilde{X}_w]_{D\times D} = \mathfrak{S}_w(\mathbf{x}; \mathbf{y}).$$

Another goal of Knutson-Miller was to exhibit a geometrically natural explanation for the pipe dream formula for Schubert polynomials.

Computing equivariant classes

Axioms for equivariant cohomology:
For $B \subset n \times n$, let C_B be the coordinate subspace $\langle z_{ij} : (i,j) \notin B \rangle$.

- $H_{D\times D}(C_B) = \prod_{(i,j)\in B}(x_i - y_j)$.
- if $X = \bigcup_{i=1}^m X_i$, is a reduced scheme, then
$$H_{D\times D}(X) = \sum_j H_{D\times D}(X_j),$$
where the sum is over X_j with $\dim X_j = \dim X$.
- If $\mathtt{in}_\prec X$ is a **Gröbner degeneration** of X, then $H_{D\times D}(X) = H_{D\times D}(\mathtt{in}_\prec X)$.

Gröbner degeneration

Choose a term order $\prec$ so that we have a **initial term** $\mathtt{in}_\prec f$ for every $f \in \mathbb{C}[\mathbf{z}]$.

Write $\mathtt{in}_\prec S = \langle \mathtt{in}_\prec f : f \in S \rangle$.

We write $\mathtt{in}_\prec X$ for the vanishing locus of $\mathtt{in}_\prec I(X)$, where $I(X)$ is the ideal of all polynomials vanishing on X. $\mathtt{in}_\prec X$ is a **Gröbner degeneration** of X.

Since $\mathtt{in}_\prec I(X)$ is generated by monomials, $\mathtt{in}_\prec X$ is a union of coordinate subspaces. But which ones?

Gröbner bases

If $I = \langle S \rangle$, then
$$\mathtt{in}_\prec I \supseteq \mathtt{in}_\prec S.$$

Definition (Buchberger 1965)

S is a **Gröbner basis** for the ideal I if $I = \langle S \rangle$ and $\mathtt{in}_\prec I = \mathtt{in}_\prec S$.

A **(anti)diagonal** term order on $\mathbb{C}[\mathbf{z}]$ is one where the initial term of any minor is the product of its main (anti)diagonal.

Example

Take the Schubert determinantal ideal

$$I(\tilde{X}_{2143}) = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

and degenerate with respect to an **antidiagonal** term order. It turns out these two polynomials are a Gröbner basis, so:

$$\begin{aligned} \mathtt{in}_{\prec_a} I(\tilde{X}_{2143}) &= \langle z_{11}, z_{13}z_{22}z_{31} \rangle \\ &= \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle. \end{aligned}$$

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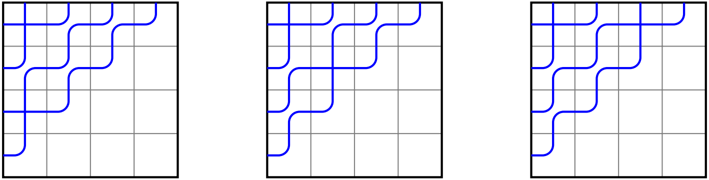


Pipe dreams are natural

Theorem (Knutson-Miller 2005)

With respect to an antidiagonal term order, the 'obvious' generators for $I(\tilde{X}_w)$ are a Gröbner basis.

Theorem (Knutson-Miller 2005)

The initial scheme of $\tilde{X}_w$ with respect to an antidiagonal term order is a union of coordinate subspaces indexed by $\mathrm{PD}(w)$.

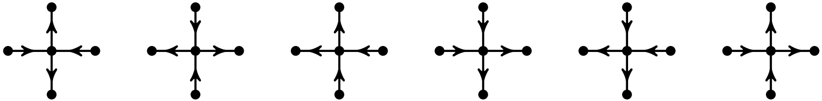
$$\mathtt{in}_{\prec_a}(\tilde{X}_w) = \bigcup_{\mathcal{P} \in \mathrm{PD}(w)} C_{\boxplus(\mathcal{P})}.$$

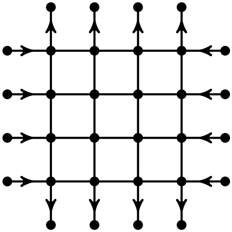
Hence,

$$\mathfrak{S}_w(\mathbf{x}; \mathbf{y}) = \sum_{\mathcal{P} \in \mathrm{PD}(w)} H_{D \times D}(C_{\boxplus(\mathcal{P})}) = \sum_{\mathcal{P} \in \mathrm{PD}(w)} \mathtt{wt}(\mathcal{P}).$$

Lascoux's ice formula

In an unpublished manuscript "Chern and Yang through ice," Lascoux (2002) gave a different combinatorial formula for Schubert polynomials in terms of **square ice**.





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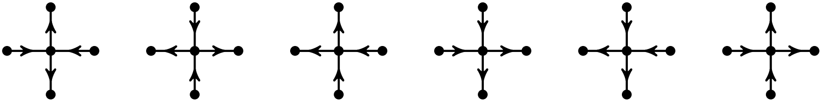

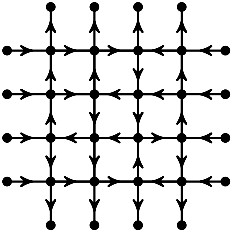

Why ice?

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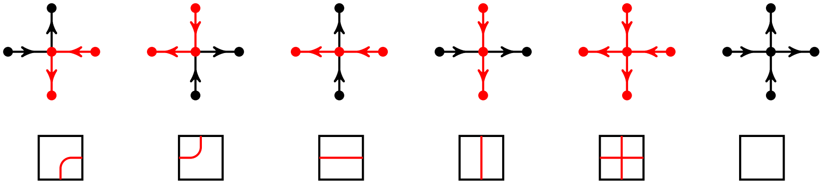
Do not consume!

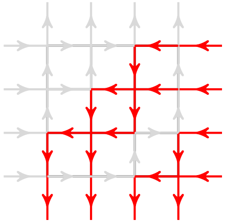
What is Lascoux's formula?

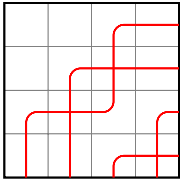
$$\mathfrak{asm} = \begin{array}{cccccccccc} \blacksquare & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & 1 \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & 0 & 0 & 0 \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & 1 & 0 & 0 & 0 \\ \heartsuit & \heartsuit & 0 & 0 & 0 & 0 & 0 & & & \\ \heartsuit & 1 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & \cdot & \cdot & & & & & & \\ 0 & 0 & \cdot & \cdot & & & & & & \\ 0 & 0 & \cdot & 1 & & & & & & \\ 0 & 0 & & & & & & & & \\ 1 & 0 & & & & & & & & \end{array} \quad , \quad K(\sigma) = \begin{array}{cccccccccc} \blacksquare & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \boxed{1} \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & & & \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \boxed{1} & & & & \\ \heartsuit & \heartsuit & \cdot & \cdot & & & & & & \\ \heartsuit & \heartsuit & \cdot & \cdot & & & & & & \\ \heartsuit & \heartsuit & \cdot & 1 & & & & & & \\ \heartsuit & \heartsuit & & & & & & & & \\ \heartsuit & \boxed{1} & & & & & & & & \end{array}$$

Let's use different notation...

From ice to bumpless pipe dreams



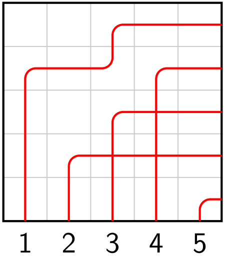




Bumpless pipe dreams

A **bumpless pipe dream** (Lam-Lee-Shimozono 2018) is a tiling of the $n \times n$ grid with the six tiles pictured above so that there are n pipes which

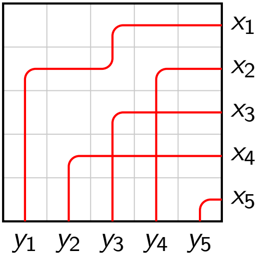
1. start at the right edge of the grid,
2. end at the bottom edge, and
3. pairwise cross at most one time.



Write BPD(w) for the set of bumpless pipe dreams of the permutation w.

The weight of a bumpless pipe dream

If a $\square$ tile sits in row i and column j, assign it the weight $(x_i - y_j)$. The weight of a bumpless pipe dream is the product of the weights of its $\square$ tiles.



$$\mathtt{wt}(\mathcal{P}) =$$

Another Schubert formula

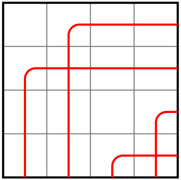
Theorem (Lam-Lee-Shimozono 2018, Lascoux 2002, Weigandt 2020)

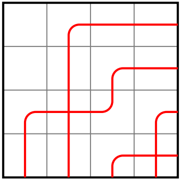
The double Schubert polynomial $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ *is the partition function*

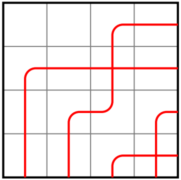
$$\mathfrak{S}_w(\mathbf{x};\mathbf{y}) = \sum_{\mathcal{P}\in\mathsf{BPD}(w)} \mathtt{wt}(\mathcal{P}).$$

Example

$$\mathfrak{S}_{2143} = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2).$$







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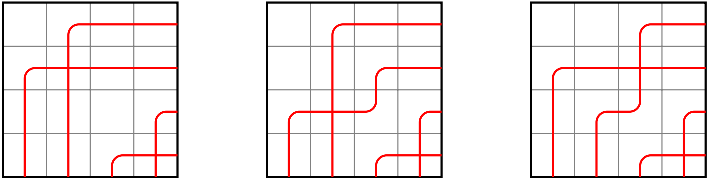

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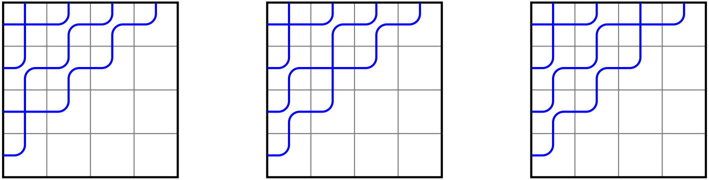

A different degeneration

Take the Schubert determinantal ideal

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and degenerate with respect to a **diagonal** term order.

$$\mathtt{in}_{\prec_d}(I_{2143}) = \langle z_{11}, z_{12}z_{21}z_{33} \rangle = \langle z_{11}, z_{33} \rangle \cap \langle z_{11}, z_{21} \rangle \cap \langle z_{11}, z_{12} \rangle.$$

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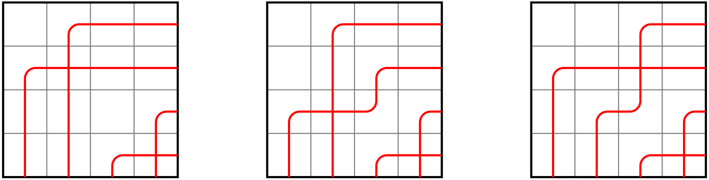


First conjecture

Conjecture (Hamaker-P-Weigandt 2020)

If $\mathtt{in}_{\prec_d}(X_w)$ *is reduced, then*

$$\mathtt{in}_{\prec_d}(\tilde{X}_w) = \bigcup_{\mathcal{P} \in \mathsf{BPD}(w)} C_{\square(\mathcal{P})}.$$

True for vexillary permutations

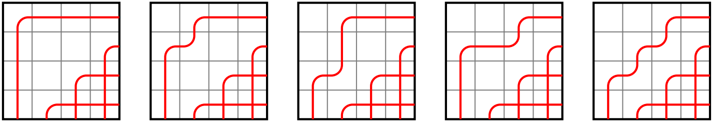
A permutation is **vexillary** if it avoids the pattern 2143.

Theorem (Knutson-Miller-Yong 2009)

With respect to a diagonal term order, the 'obvious' defining equations for $I(\tilde{X}_w)$ are a Gröbner basis $\Leftrightarrow$ w is vexillary.

Theorem (Knutson-Miller-Yong 2009, Weigandt 2020)

The conjecture holds for w vexillary.



2	2
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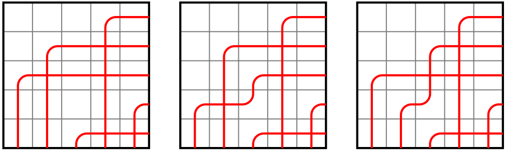
Different generators

- Take the 'obvious' defining equations for $I(\tilde{X}_w)$. Some of them may be single variables z_{ij}. Throw away all other terms that contain them. These are the **CDG generators** (Conca-De Negri-Gorla 2015).
- A permutation is **predominant** if its Lehmer code is of the form $\lambda 0^m \ell$ for some partition λ and $m, \ell \in \mathbb{N}$.

Theorem (Hamaker-P-Weigandt 2020)

If w is predominant, then $\mathtt{in}_{\prec_d}(X_w)$ is reduced and CDG and the main conjecture holds for w.

$w = 42153$:



CDG permutations

Theorem (Conj: Hamaker-P-Weigandt 2020, Proved: Klein 2020)

With respect to any diagonal term order, the CDG generators for $I(\tilde{X}_w)$ are Gröbner $\Leftrightarrow$ w avoids the eight patterns 13254, 21543, 214635, 215364, 241635, 315264, 215634, 4261735 *in* S_7.

Corollary (Hamaker-P-Weigandt 2020, Klein 2020)

If $\mathfrak{S}_w$ is a multiplicity-free sum of monomials, then the CDG generators for $I(\tilde{X}_w)$ are diagonal Gröbner.

Proof.

By comparing with the pattern conditions for multiplicity-freeness of Fink-Mészáros-St. Dizier (2020). □

Non-reduced degenerations

Given a scheme X and a reduced irreducible variety Y, write $\mathrm{mult}_Y(X)$ for the **multiplicity** of X along Y.

Equivariant cohomology satisfies:

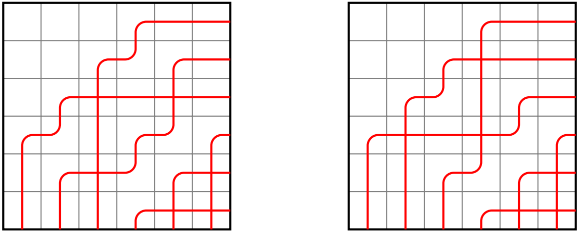
$$H_{D \times D}(X) = \sum_j \mathrm{mult}_{X_j}(X) \, H_{D \times D}(X_j),$$

where the sum is over the top-dimensional components of X.

Example: $X = \mathrm{Spec}(\mathbb{C}[x,y]/\langle x^2 y \rangle)$. Then, $\mathrm{mult}_{\text{x-axis}}(X) = 1$ and $\mathrm{mult}_{\text{y-axis}}(X) = 2$.

Duplicitous permutations

Write $\mathrm{Dup}(n)$ for the set of permutations in S_n which have multiple BPDs with the same $\square$ tiles.



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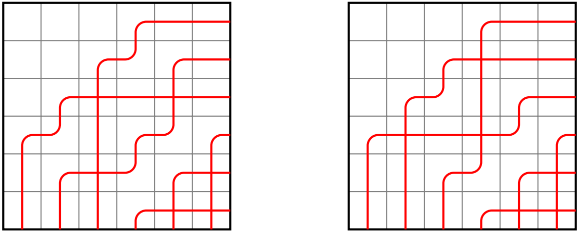


$$\mathrm{Dup}(6) = \{214365, 321654\}.$$

Computations of Heck-Weigandt suggest this phenomenon is also governed by pattern avoidance, but we don't know all the patterns (includes the Billey-Pawlowski (2014) multiplicity-free patterns).

The full conjecture

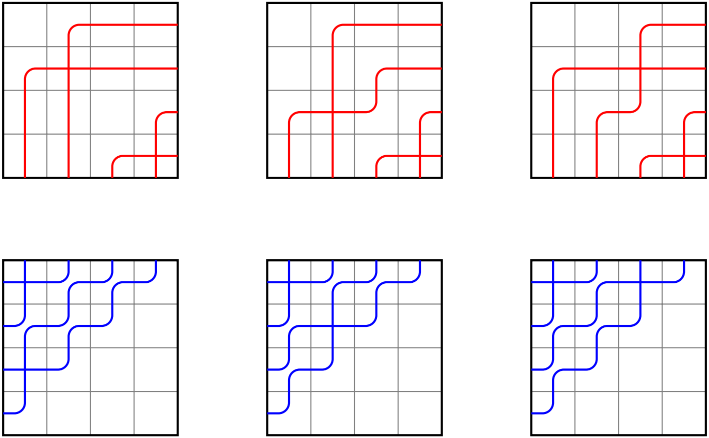
Conjecture (Hamaker-P-Weigandt 2020)

For all (set-theoretic) components C_B of $\mathtt{in}_{\prec_d}(X_w)$,

$$\mathrm{mult}_{C_B}(\mathtt{in}_{\prec_d}(X_w)) = \#\{\mathcal{P} \in \mathsf{BPD}(w) : \square(\mathcal{P}) = B\}.$$

So bumpless pipe dreams for w **with multiplicity** *correspond to components of the diagonal Gröbner degeneration $\mathtt{in}_{\prec_d}(X_w)$* **with multiplicity***.*

Thanks!



Thank you!!