

Combinatorics and Geometry of Markov Numbers

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(jointly with Matty van Son)

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I. Introduction

II. Markov tree: all-in-one

III. Categorification (extention) of Markov tree

History.

- ▶ L. Euler (1744) & J.-L. Lagrange (1770) – Periodic c.f. and quadratic irrationals
- ▶ J. P. G. Dirichlet (1840) - Dirichlet's approximation theorem
- ▶ A. Markov (1872) – Markov spectrum via c.f.
- ▶ A. Hurwitz (1891) - Hurwitz's theorem
- ▶ F.G. Frobenius (1913) – Uniqueness conjecture
- ▶ H. Davenport (1938), first steps in higher dimensional Markov Spectrum (this is related to Littlewood conjecture)

Tools and development

- ▶ Cohn Matrices (1955)
- ▶ Geometry of c.f. (F. Klein, H. Minkowski, G. Voronoi, V. Arnold, E. Korkina, O. German, O.K., etc.)
- ▶ Reduced integer matrices (Y. I. Manin, M. Marcolli, O.K.)
- ▶ M. Aigner (2013) "Markov's Theorem and 100 Years of the Uniqueness Conjecture"

M. van Son, O.K.(2020) Geometrisation of Markov Numbers (and its extension)

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- ▶ 4. “Anomal” operation on numbers

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- ▶ 1. Markov triples, Markov trees
- ▶ 2. Discrete Markov spectrum
- ▶ 3. Semigroup of reduced matrices
- ▶ 4. “Anomal” operation on numbers
(= concatenation of continued fractions)

Markov triples

Markov equation (1879):

$$x^2 + y^2 + z^2 = 3xyz.$$

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Solutions of Markov equation are **Markov triples**.

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Example

Direct computation shows that

$$(1, 1, 1), \quad (1, 2, 1), \quad (1, 5, 2), \quad (1, 13, 5), \dots$$

are Markov triples.

Generating Markov triples

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Proposition

Starting with **any** Markov triple, Markov automaton generates **all** Markov triples.

$(1, 1, 1)$

$(1, 1, 1)$



$3 \cdot 1 \cdot 1 - 1$

$(1, 2, 1)$

Markov tree

$(1, 1, 1)$



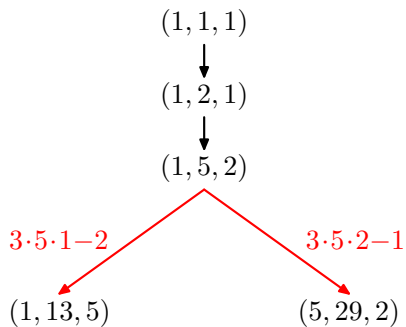
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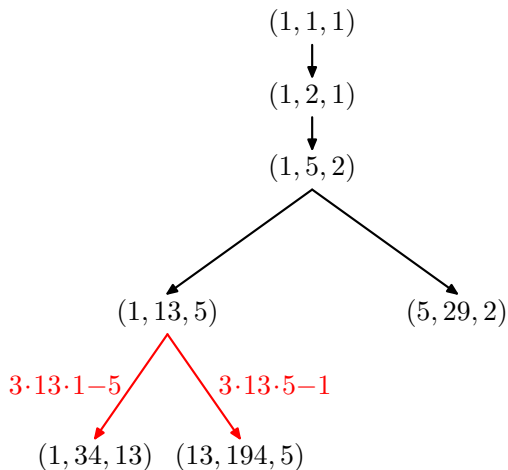
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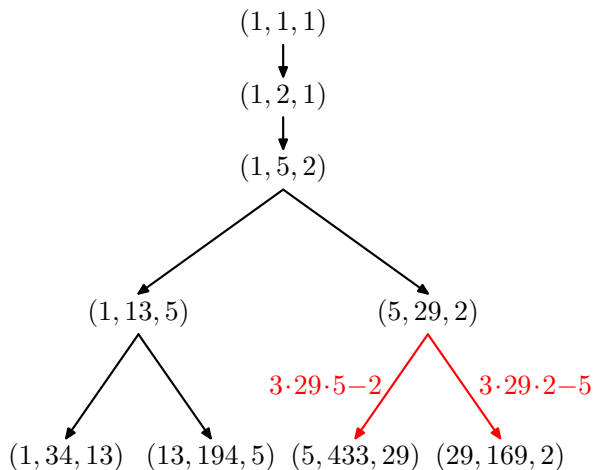
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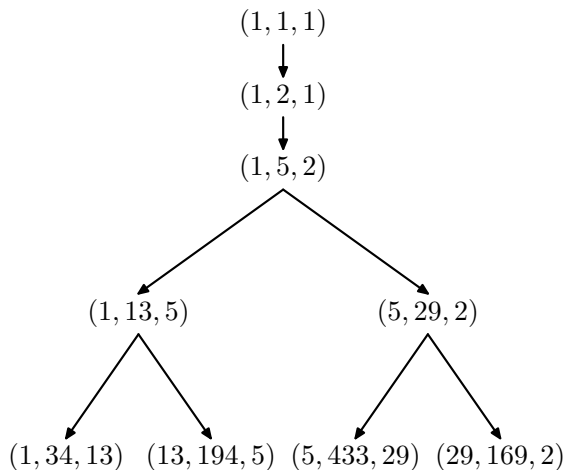
Markov tree



Markov tree



Markov tree



This is a famous **binary rooted Markov tree**.

Discrete Markov spectrum

Markov spectrum

For an indefinite form $f(x, y) = ax^2 + bxy + cy^2$ denote:

- ▶ The **Markov minimum** of f

$$m(f) = \inf_{\mathbb{Z}^2 \setminus \{(0,0)\}} |f(x, y)|.$$

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- ▶ The **Markov spectrum** is

$$\mathcal{M} = \left\{ 1/M(f) \mid f \text{ an indefinite form} \right\}.$$

Example

▶ $f(x, y) = 5x^2 + 11xy - 5y^2$.

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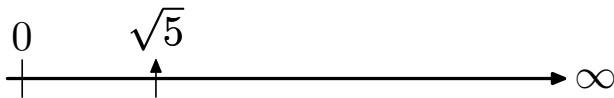
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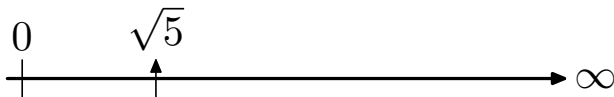
▶ $g(x, y) = x^2 + xy - y^2$.

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Markov spectrum below 3



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Theorem (Hurwitz)

The smallest value on the Markov spectrum is $\sqrt{5}$.

Markov spectrum below 3



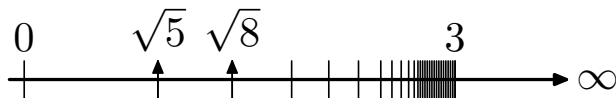
Theorem (Hurwitz)

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Theorem (Markov)

The second smallest value on the Markov spectrum is $\sqrt{8}$.

Markov spectrum below 3



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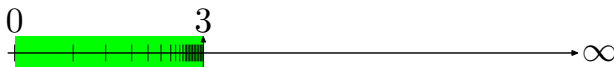
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The first limiting point of the Markov spectrum is 3.

Markov spectrum



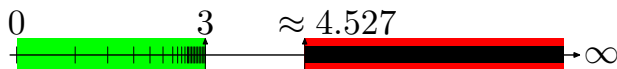
Theorem

Every real number greater than Freiman's constant

$$F = \frac{2221564096 + 283748\sqrt{462}}{491993569} \approx 4.527$$

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We are interested in **discrete spectrum** only.

Semigroup of reduced matrices

Reduced Matrices (Manin, Marcolli, 2002)

Operators A and B are $SL(2, \mathbb{Z})$ -**conjugate** if there exists X such that

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- ▶ complex case: $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.
- ▶ totally real case: Gauss Reduction Theory
- ▶ degenerate case of double roots: $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for $n \geq 0$.

The totally real case of $SL(2, \mathbb{Z})$

Definition

An $SL(2, \mathbb{Z})$ -operator $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is *reduced* if $d > b \geq a > 0$.

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Proposition

If A, B are reduced then AB is reduced.

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} = \begin{pmatrix} 18 & 43 \\ 41 & 98 \end{pmatrix}.$$

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So the space of reduced matrices form a semigroup w.r.t matrix multiplication.

“Anomal” operation on numbers (= concatenation of continued fractions)

Continued fractions for $7/5$

$$\frac{7}{5} =$$

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Continued fractions

The expression (finite or infinite)

$$a_0 + 1/(a_1 + 1/(a_2 + \dots))$$

is an *ordinary continued fraction* if $a_0 \in \mathbb{Z}$, $a_k \in \mathbb{Z}_+$ for $k > 0$.
Denote it $[a_0 : a_1; \dots]$ (or $[a_0 : a_1; \dots; a_n]$).

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Ordinary continued fraction is *odd* (*even*) if it has odd (even) number of elements.

$$\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{1}{2 + \frac{1}{1+1/1}}$$

$$\frac{7}{5} = [1 : 2; 2] = [1 : 2; 1; 1]$$

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Proposition

Any rational number has a unique odd and even ordinary continued fractions.

Any irrational number has a unique infinite ordinary continued fraction

- ▶ **Quadratic numbers (roots of quadratic integer polynomials)**: periodic continued fractions (J.-L. Lagrange, 1770).

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- ▶ **Anomal \star -summation:**

$$[(a_1, \dots, a_n)] \star [(b_1, \dots, b_m)] = [(a_1, \dots, a_n, b_1, \dots, b_m)].$$

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Example: $[(1, 1)] \star [(2, 2)] = [(1, 1, 2, 2)].$

Hence

$$\frac{1 + \sqrt{5}}{2} \star (1 + \sqrt{2}) = \frac{9 + \sqrt{221}}{14}.$$

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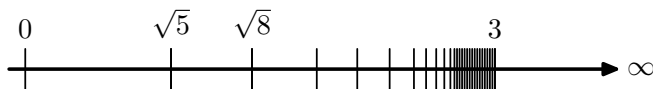
Note: \star -summation is not uniquely defined for numbers, as one can take periods of different lengths.

II. Markov tree: all-in-one

1&2. Markov Triples and forms

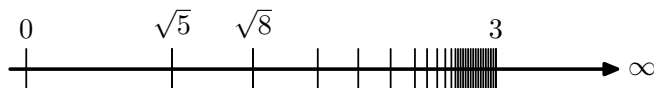
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$$\frac{\sqrt{5}}{1}, \frac{\sqrt{8}}{2}, \frac{\sqrt{221}}{5}, \frac{\sqrt{1517}}{13}, \frac{\sqrt{7565}}{29}, \frac{\sqrt{5200}}{34}, \dots$$

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Theorem (A. Markov, 1879)

The Markov spectrum below 3 is discrete and consists of the numbers

$$\frac{\sqrt{9m^2 - 4}}{m}$$

where m is a positive integer satisfying

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

for positive integers $m_1 \leq m_2 \leq m$.

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Then

$$\frac{\sqrt{\Delta(f)}}{m(f)} = \frac{\sqrt{9m^2 - 4}}{m}$$

2&3. Discrete Markov spectrum via c.f.

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Proposition (Perron)

The Markov spectrum is

$$\mathcal{M} = \left\{ P(\alpha) \mid \alpha = (a_i)_{i \in \mathbb{Z}} \text{ with } a_i \in \mathbb{Z}_+ \right\}.$$

Example

- ▶ Let α be the infinite sequence with period $(2, 2, 1, 1)$. Then

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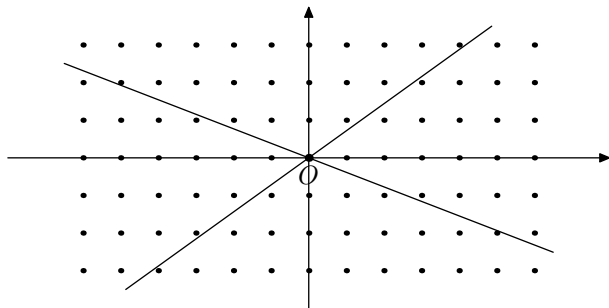
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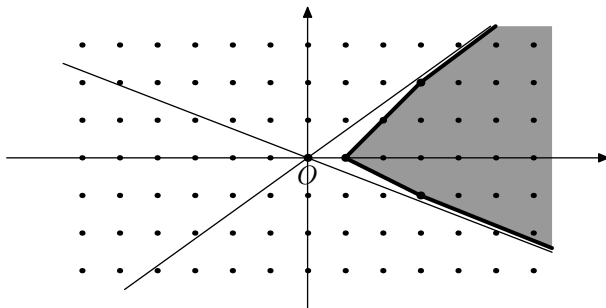
3&4. Reduced matrices and continued fractions

Reduced matrices and continued fractions



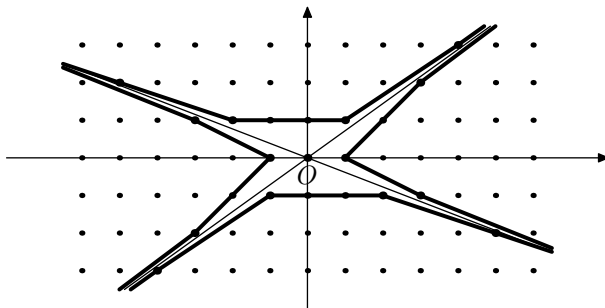
Eigenlines of an operator $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$.

Reduced matrices and continued fractions



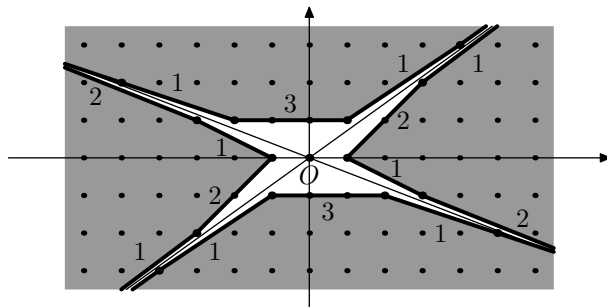
The *sail* for one of the octants, i.e. the boundary of the convex hull of all integer inner points.

Reduced matrices and continued fractions



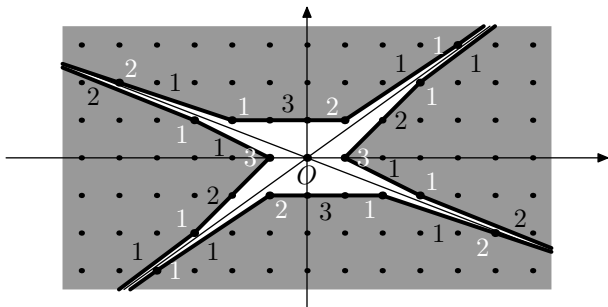
The set of all sails is called *geometric continued fraction* (in the sense of Klein).

Reduced matrices and continued fractions



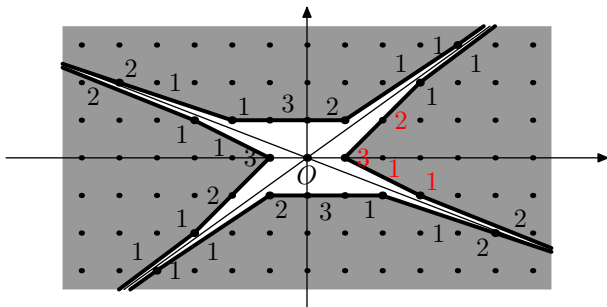
Integer length of a segment is the number of integer inner points in a segment plus one.

Reduced matrices and continued fractions



Geometric continued fraction for the operator $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$.

Reduced matrices and continued fractions



Theorem

A period (up to a shift) is a complete invariant of a conjugacy class of an operator in $SL(2, \mathbb{Z})$.

Matrices and continued fractions

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- ▶ *Tangents of eigenlines has a period*

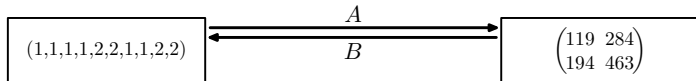
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Markov theory in one diagram

Markov theory in one diagram

$(1,1,1,1,2,2,1,1,2,2)$

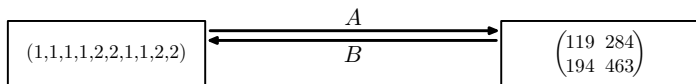
Markov theory in one diagram



$$A: (a_1, \dots, a_n) \mapsto M_{a_1, \dots, a_n} = \begin{pmatrix} K_2^{n-1} & K_2^n \\ K_1^{n-1} & K_1^n \end{pmatrix}$$

where $K_j^i = K(a_i, \dots, a_j)$ -continuant.

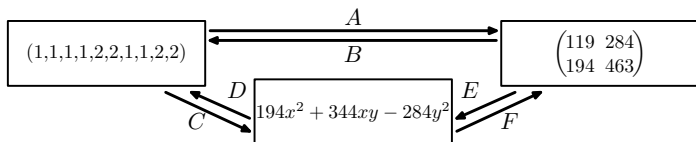
Markov theory in one diagram



$$B: \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto (a_1, \dots, a_{2n}).$$

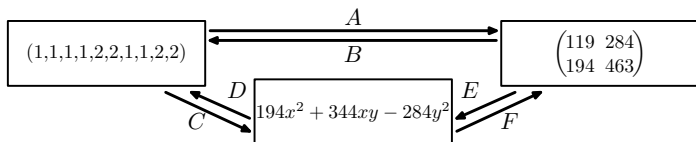
where $d/c = [a_1; \dots; a_{2n-1} : a_{2n}]$.

Markov theory in one diagram



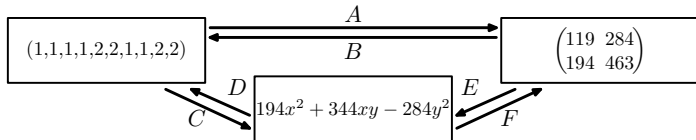
$$C: (a_1, \dots, a_n) \mapsto K_1^{n-1}x^2 + (K_1^n - K_2^{n-1})xy - K_2^n y^2.$$

Markov theory in one diagram



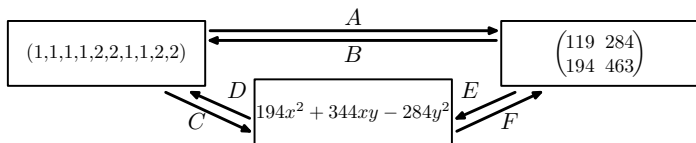
D : $B \circ F$ (we do not have a direct map here).

Markov theory in one diagram



$$E: \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto bx^2 + (d - a)xy - cy^2.$$

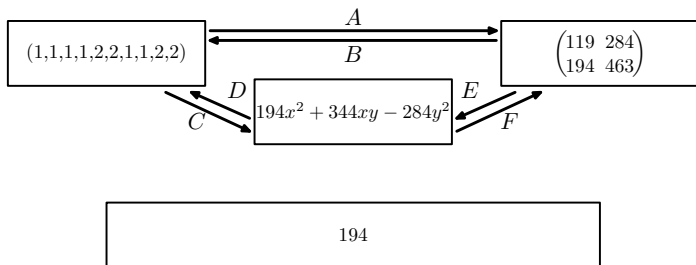
Markov theory in one diagram



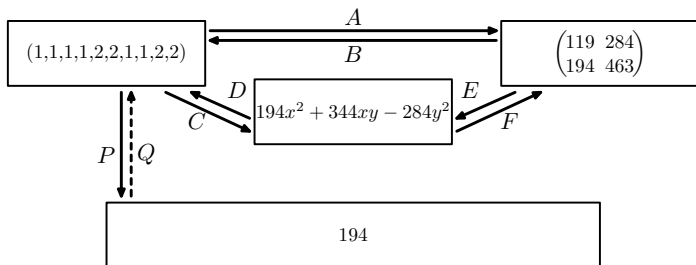
$$F: Ax^2 + Bxy + Cy^2 \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \text{ where}$$

$$a = \frac{-B + \sqrt{B^2 - 4AC + 4}}{2}, \quad b = A, \quad c = -C, \quad d = a + B.$$

Markov theory in one diagram



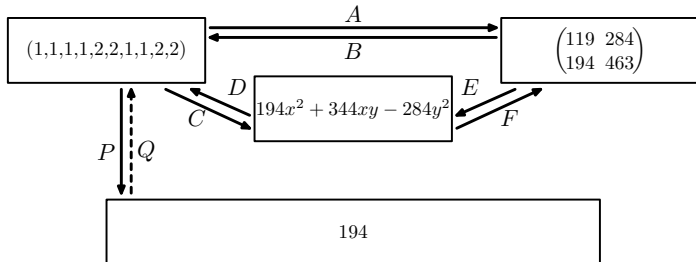
Markov theory in one diagram



$$P: (a_1, \dots, a_n) \mapsto K_1^{n-1}(a_1, \dots, a_n).$$

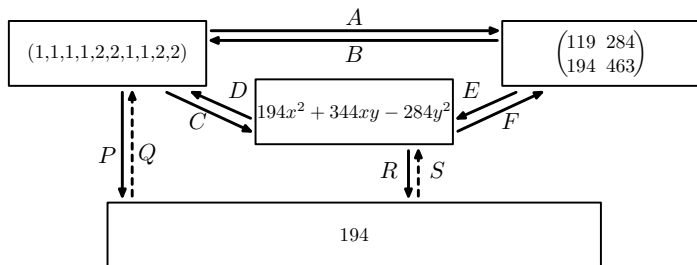
$$\text{E.g., } K_1^{10-1}(1, 1, 1, 1, 2, 2, 1, 1, 2, 2) = 194.$$

Markov theory in one diagram



Q : A reformulation of the **Uniqueness conjecture**.

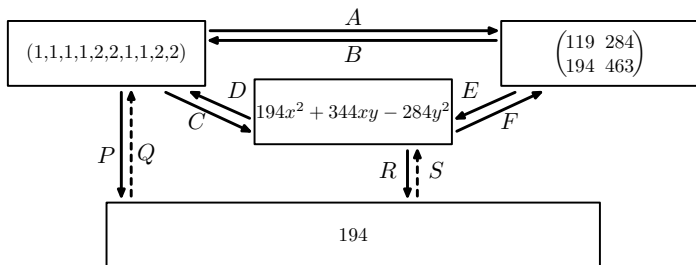
Markov theory in one diagram



$R: f \mapsto f(1, 0).$

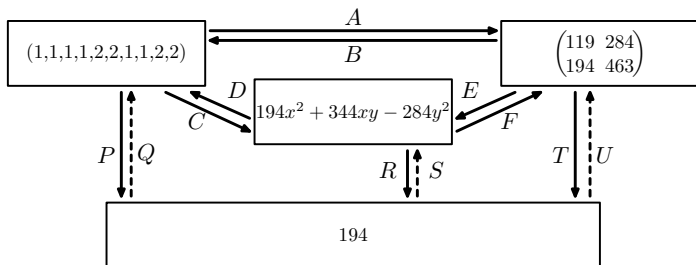
E.g., $m(194x^2 + 432xy - 196y^2) = 194.$

Markov theory in one diagram



S : We have it if we have the **Uniqueness conjecture**.

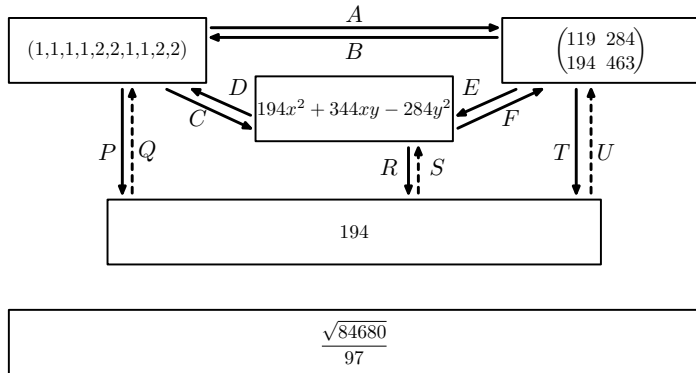
Markov theory in one diagram



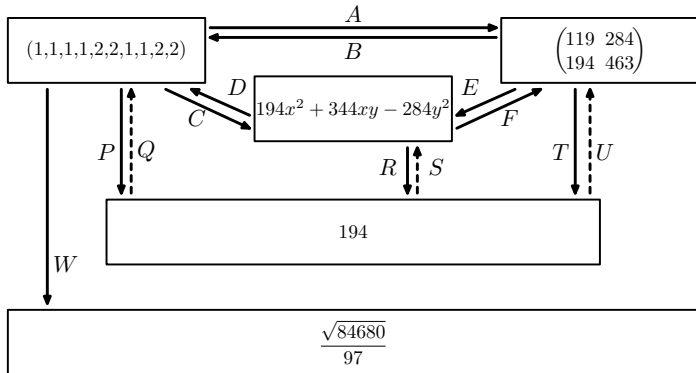
$$T: M \mapsto \frac{\text{tr}(M)}{3} (= 194).$$

U : We have it if we have the **Uniqueness conjecture**.

Markov theory in one diagram



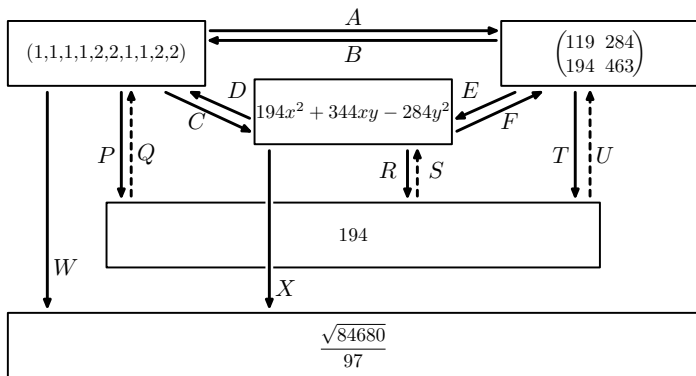
Markov theory in one diagram



$$W: (a_1, \dots, a_n) \mapsto \frac{\sqrt{(K_1^n + K_2^{n-1})^2 - 4}}{K_1^{n-1}}.$$

$$W(1, 1, 1, 1, 2, 2, 1, 1, 2, 2) = \frac{\sqrt{84680}}{97}.$$

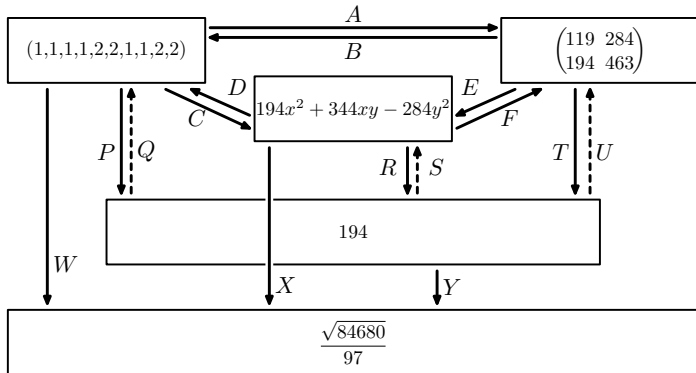
Markov theory in one diagram



$$X: f \mapsto \frac{\sqrt{\Delta(f)}}{f(1,0)}.$$

$$\left(\frac{m(194x^2 + 432xy - 196y^2)}{\sqrt{\Delta(194x^2 + 432xy - 196y^2)}} \right)^{-1} = \frac{\sqrt{84680}}{97}.$$

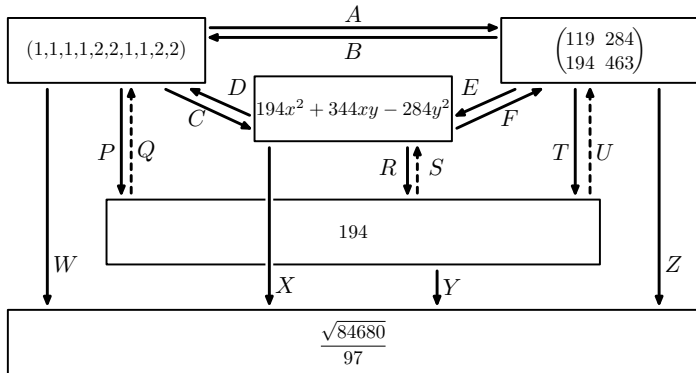
Markov theory in one diagram



$$Y: M \mapsto \frac{\sqrt{9 \cdot M^2 - 4}}{M}$$

$$\frac{\sqrt{9 \cdot 194^2 - 4}}{194} = \frac{\sqrt{84680}}{97}$$

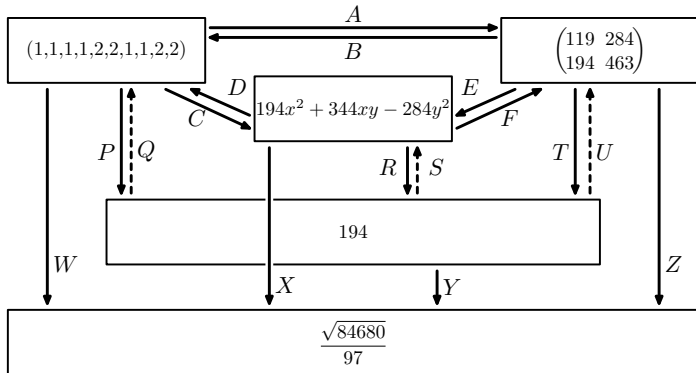
Markov theory in one diagram



$$Z: \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto \frac{\sqrt{(a+d)^2 - 4}}{b}.$$

$$\frac{\sqrt{(119 + 463)^2 - 4}}{194} = \frac{\sqrt{84680}}{97}.$$

Markov theory in one diagram



To the best of our knowledge this diagram is new.

In General

Let

- ▶ (a, b, c) be a Markov triple (let $c > a$, $c > b$);
- ▶ α, β, σ – corresponding periods;
- ▶ A, B, C corresponding matrices.

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Then

- ▶ $\alpha \star \beta = \sigma$;
- ▶ $AB = C$.

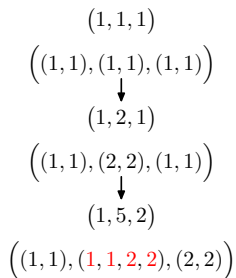
Markov tree via periods

$$(1, 1, 1)$$
$$\left((1, 1), (1, 1), (1, 1) \right)$$

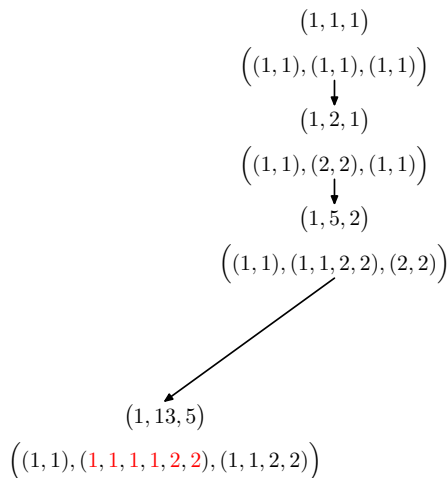
Markov tree via periods

$$\begin{array}{c} (1, 1, 1) \\ \left((1, 1), (1, 1), (1, 1) \right) \\ \downarrow \\ (1, 2, 1) \\ \left((1, 1), (2, 2), (1, 1) \right) \end{array}$$

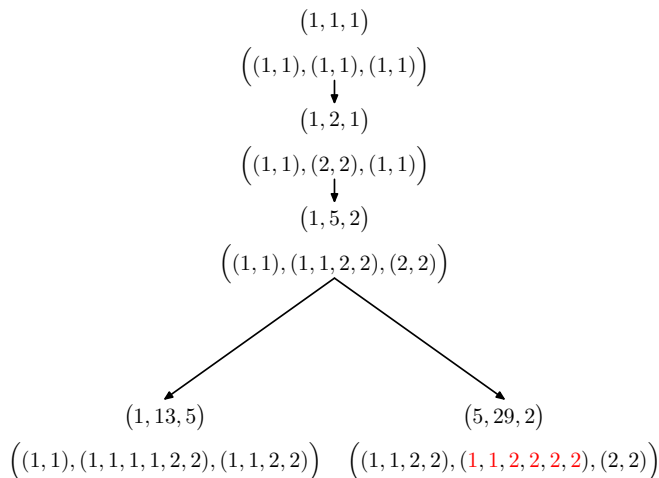
Markov tree via periods



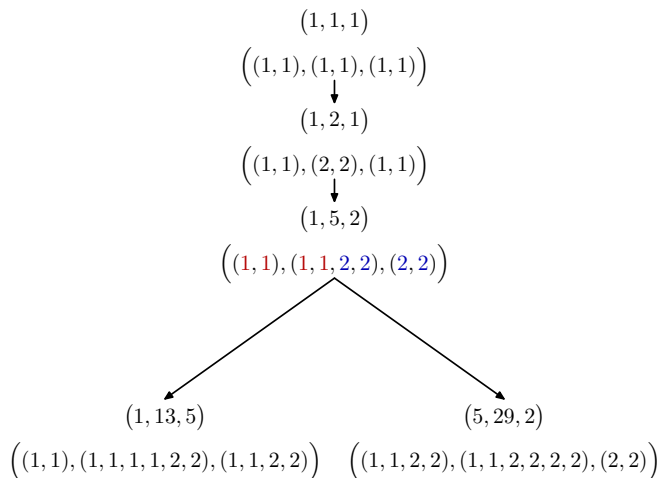
Markov tree via periods



Markov tree via periods



Markov tree via periods



We denote such a graph of sequences by $\mathcal{G}((1, 1), (2, 2))$

$\mathcal{G}(w_1, w_2)$

Similar to $\mathcal{G}((1, 1), (2, 2))$ one can consider $\mathcal{G}(w_1, w_2)$ starting with arbitrary periods w_1 and w_2 .

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In special cases of (w_1, w_2) (discussed further) we have an analogous Markov minimum theory.

We ask whether $\mathcal{G}((2, 2), (3, 3))$ includes the discrete subset of Markov spectrum intersection with numbers whose c.f. does not contain "1".

III. Categorification (extention) of Markov tree

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Example

$$(1, 1, 1, 1, 1, \dots) \prec (1, 1, 2, 2, 1, 1, 2, 2, \dots).$$

Graphs of Markov sequences

For $\alpha = (a_1, \dots, a_{2n})$ set

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$$\langle \alpha \rangle = \alpha \alpha \alpha \dots$$

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Recall: $\alpha \mapsto f_\alpha$ as

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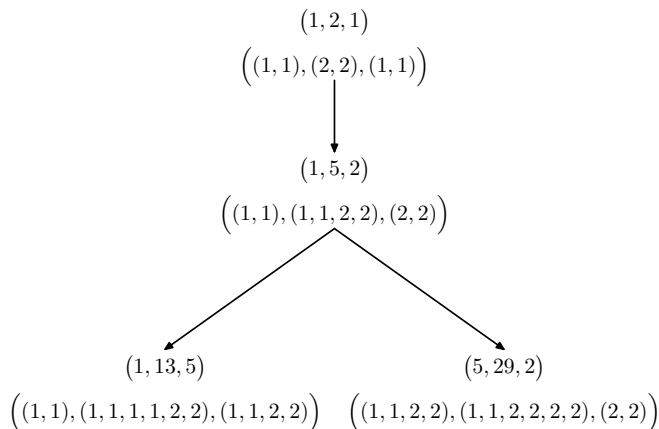
We call $\mathcal{G}(\alpha, \beta)$ a *graph of Markov sequences*.

Example

$\mathcal{G}((1, 1), (2, 2))$ is a graph of Markov sequences.

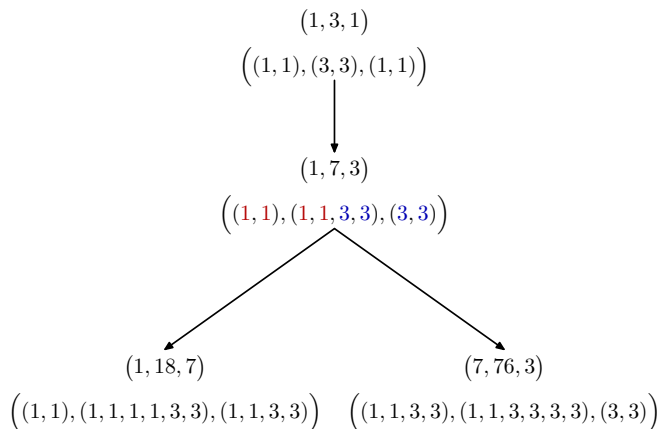
$\mathcal{G}((1, 1), (3, 2, 1, 1, 2, 3))$ is a graph of Markov sequences.

Two graphs



$\mathcal{G}((1, 1), (2, 2))$ and **Markov numbers**.

Two graphs



$\mathcal{G}((1, 1), (3, 3))$ and the corresponding **generalized Markov numbers**.

Automata to generate generalized Markov trees

- Consider triples of pairs $\left((\alpha, a), (\beta, b), (\gamma, c) \right)$ where
- α, β, γ : sequences of integers and
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$$a = \check{K}(\alpha), \quad b = \check{K}(\beta), \quad c = \check{K}(\gamma).$$

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Recall that

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Theorem: for Markov tree $\frac{\check{K}(\alpha \oplus \alpha)}{\check{K}(\alpha)} = 3a.$

Minima in graphs of Markov sequences (M. van Son, O.K. '20)

Theorem

Let $\gamma = (c_1, \dots, c_{2n})$ be a sequence in a graph of Markov sequences. Let $[c_1; c_2 : \dots : c_{2n-1}] = \frac{p}{q}$.

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$$f_\mu(x, y) = 1367x^2 + 5315xy - 2435y^2,$$

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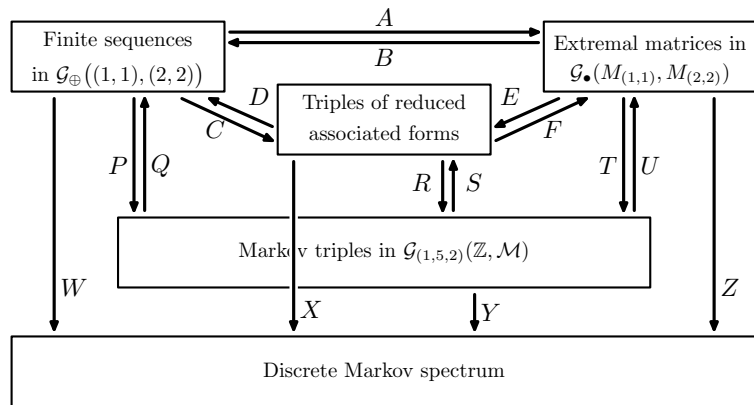
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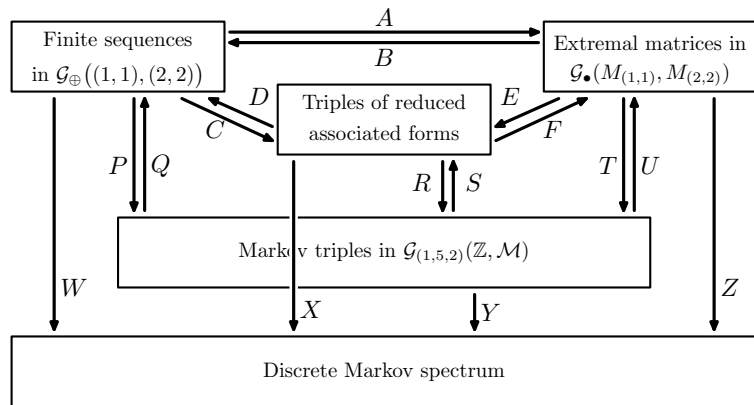
$$\begin{aligned}f_\mu(x, y) &= 1367x^2 + 5315xy - 2435y^2, \\ [2, 2, 2, 2, 4, 3, 3] &= \frac{1367}{566}, \\ m(f_\mu) &= f_\mu(1, 0) = 1367.\end{aligned}$$

Generalized Markov theory in one diagram



Extra maps for triples:

Generalized Markov theory in one diagram

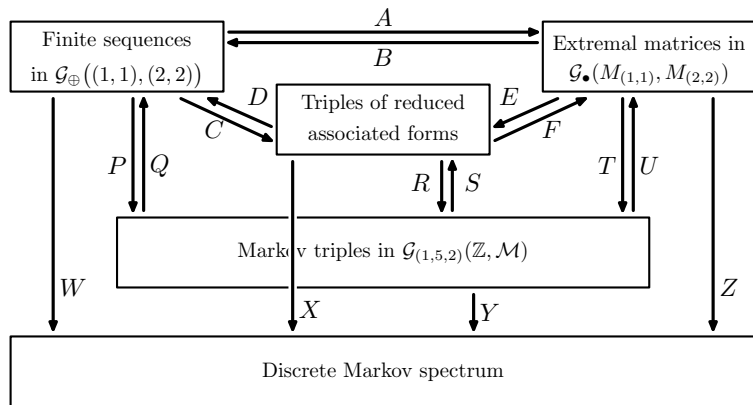


Extra maps for triples:

Q : (A. Markov) $(a, M, b) \mapsto (a_{2n-1}, \dots, a_1, 2)$, where

$$\pm ua \equiv b \pmod{M} \quad \text{and} \quad \frac{M}{u} = [a_1, \dots, a_{2n-1}].$$

Generalized Markov theory in one diagram

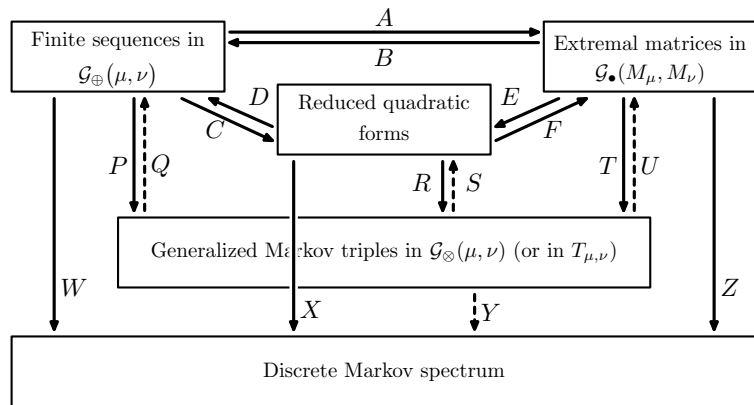


Extra maps for triples:

S : (A. Markov) $(a, M, b) \mapsto Mx^2 + (M+2u)xy + (u+v-2M)y^2$,
 where

$$\pm ua \equiv b \pmod{M} \quad \text{and} \quad v = \frac{u^2 + 1}{M}.$$

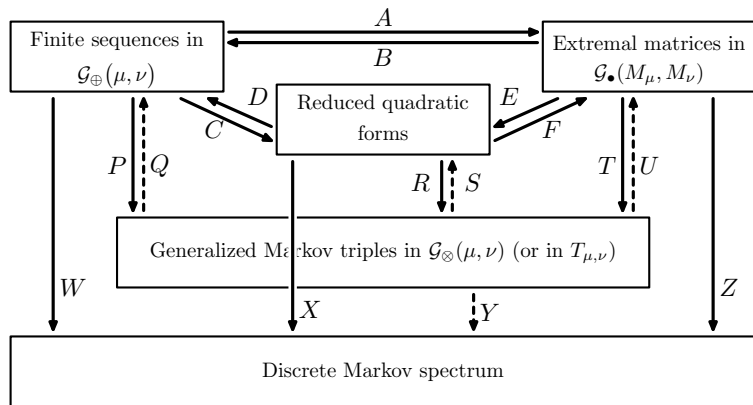
Generalized Markov theory in one diagram



Let $\mathcal{G}(\mu, \nu)$ be graph of Markov sequences satisfying

- ▶ $\langle \mu \rangle \prec \langle \nu \rangle$ and $\langle \bar{\mu} \rangle \prec \langle \bar{\nu} \rangle$;
- ▶ $\bar{\mu} = \mu$ and $\bar{\nu} = \nu$;
- ▶ $m(f_{\mu}) = f_{\mu}(1, 0)$ and $m(f_{\nu}) = f_{\nu}(1, 0)$.

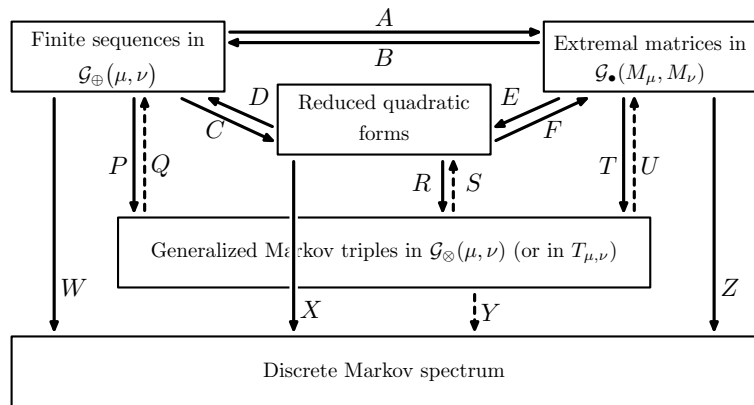
Generalized Markov theory in one diagram



Let $\mathcal{G}(\mu, \nu)$ be graph of Markov sequences satisfying **the conditions**.

Then all bold arrows are as in classical theory (M. van Son, O.K., 2020).

Generalized Markov theory in one diagram

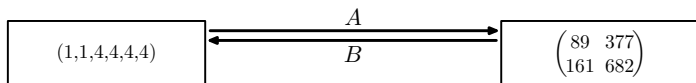


It is not known whether there exist maps for Q , S , U , or Y .

Extended Markov theory in one diagram

$(1,1,4,4,4,4)$

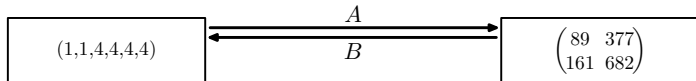
Extended Markov theory in one diagram



$$A: (a_1, \dots, a_n) \mapsto M_{a_1, \dots, a_n} = \begin{pmatrix} K_2^{n-1} & K_2^n \\ K_1^{n-1} & K_1^n \end{pmatrix}$$

where $K_j^i = K(a_i, \dots, a_j)$ -contingent.

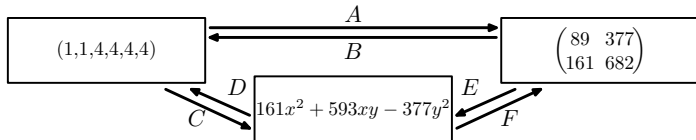
Extended Markov theory in one diagram



$$B: \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto (a_1, \dots, a_{2n-1}, a_{2n} \mid).$$

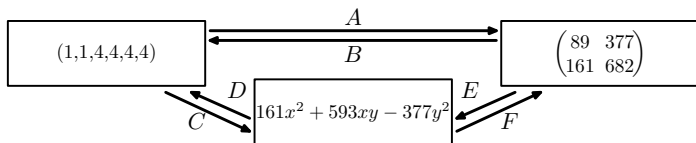
where $d/c = [a_1; \dots; a_{2n-1} : a_{2n}]$.

Extended Markov theory in one diagram



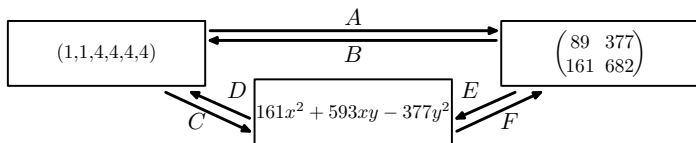
$$C: (a_1, \dots, a_n) \mapsto K_1^{n-1}x^2 + (K_1^n - K_2^{n-1})xy - K_2^n y^2.$$

Extended Markov theory in one diagram



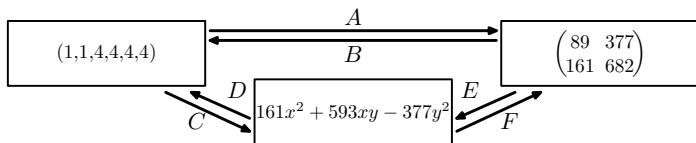
$D: B \circ F$ (we do not have a direct map here).

Extended Markov theory in one diagram



$$E: \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto bx^2 + (d - a)xy - cy^2.$$

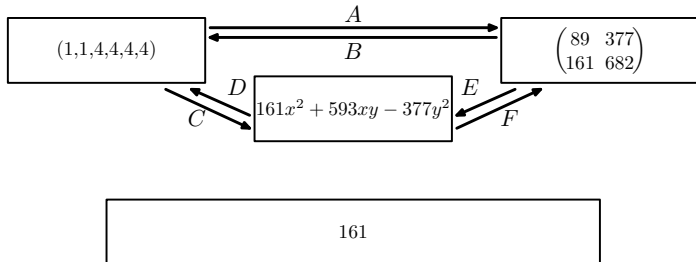
Extended Markov theory in one diagram



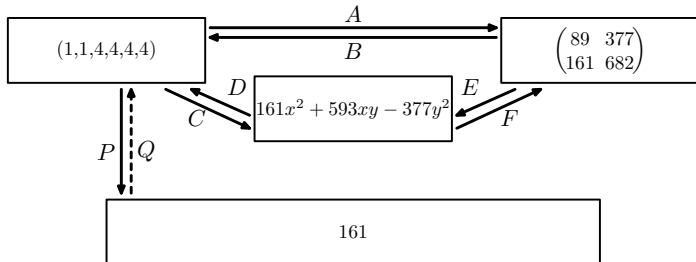
$$F: Ax^2 + Bxy + Cy^2 \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \text{ where}$$

$$a = \frac{-B + \sqrt{B^2 - 4AC + 4}}{2}, \quad b = A, \quad c = -C, \quad d = a + B.$$

Extended Markov theory in one diagram



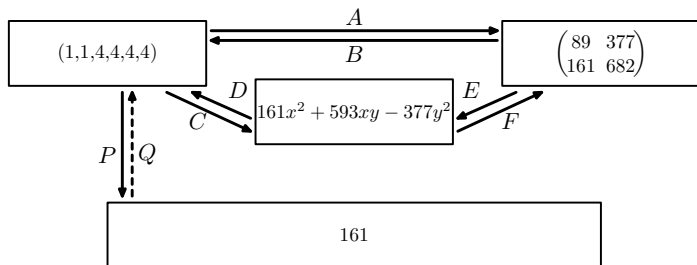
Extended Markov theory in one diagram



$$P: (a_1, \dots, a_n) \mapsto K_1^{n-1}(a_1, \dots, a_n).$$

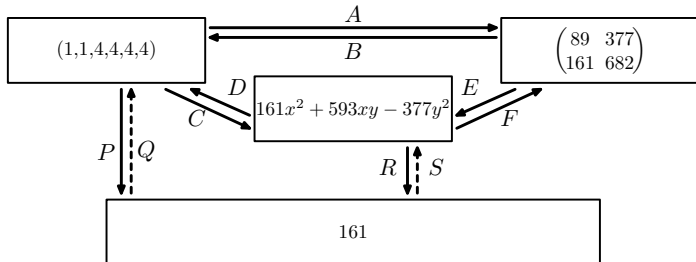
$$\text{E.g., } K_1^{6-1}(1, 1, 4, 4, 4, 4) = 161.$$

Extended Markov theory in one diagram



Q : A reformulation of the **Uniqueness conjecture**.

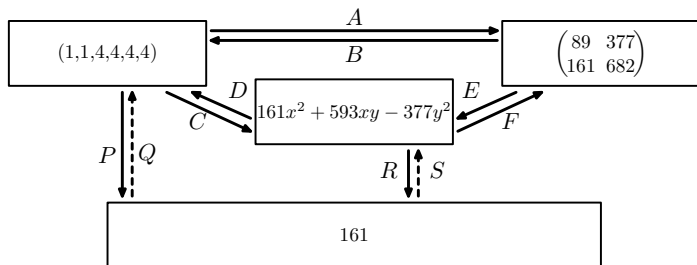
Extended Markov theory in one diagram



$R: f \mapsto f(1, 0).$

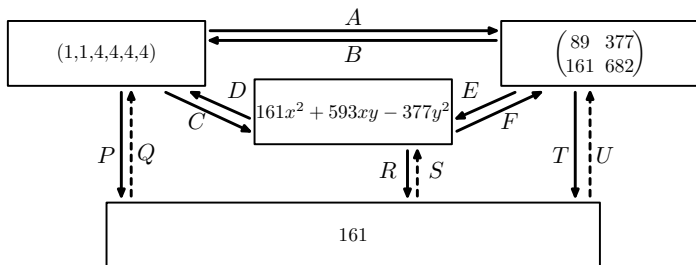
E.g., $m(161x^2 + 593xy - 377y^2) = 161.$

Extended Markov theory in one diagram



S : We have it if we have the **Uniqueness conjecture**.

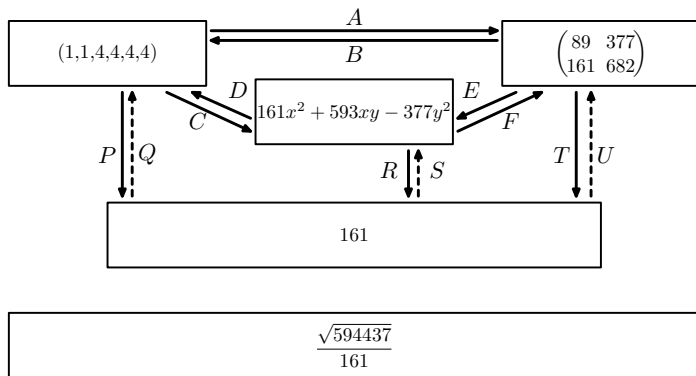
Extended Markov theory in one diagram



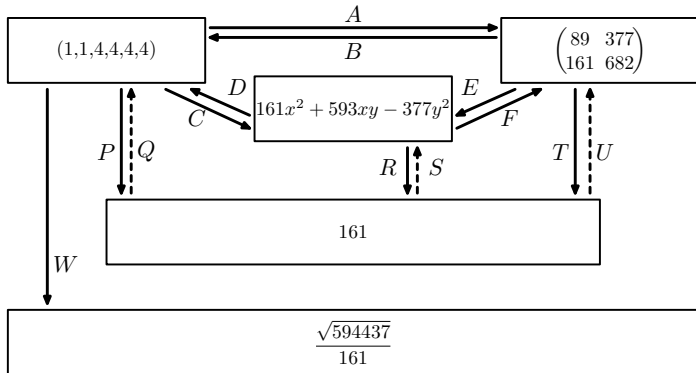
$T: M \mapsto M_{2,1}(= 161)$.

U : We have it if we have the **Uniqueness conjecture**.

Extended Markov theory in one diagram



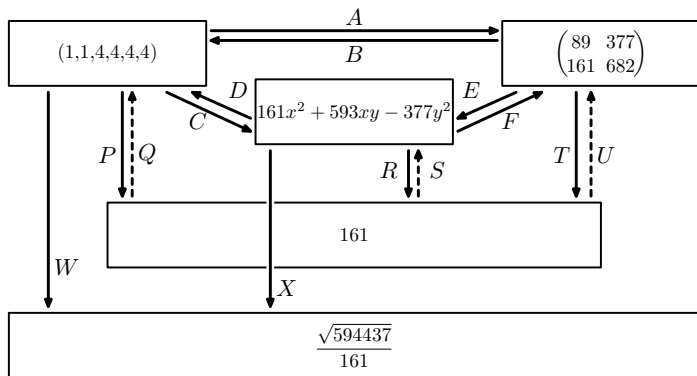
Extended Markov theory in one diagram



$$W: (a_1, \dots, a_n) \mapsto \frac{\sqrt{(K_1^n + K_2^{n-1})^2 - 4}}{K_1^{n-1}}.$$

$$W(1, 1, 4, 4, 4, 4) = \frac{\sqrt{594437}}{161}.$$

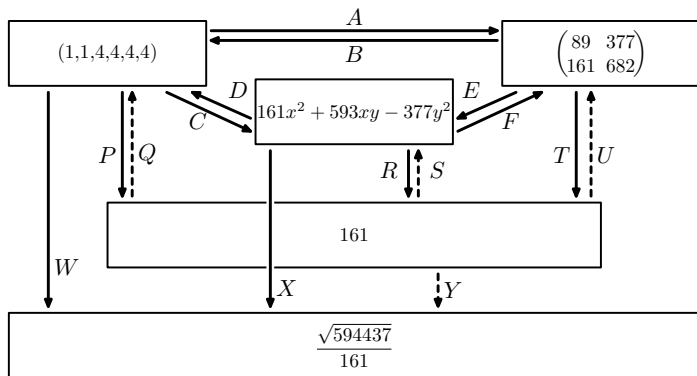
Extended Markov theory in one diagram



$$X: f \mapsto \frac{\sqrt{\Delta(f)}}{f(1,0)}.$$

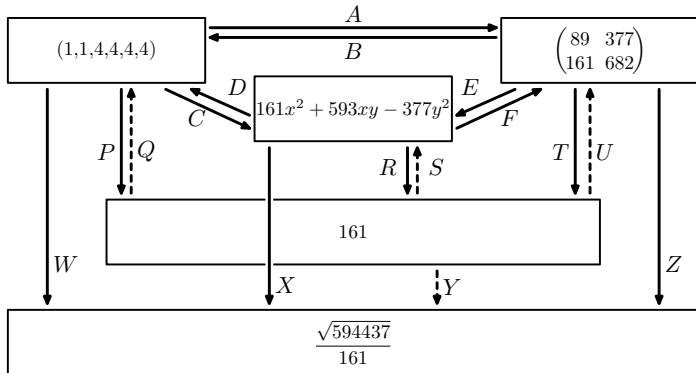
$$\left(\frac{m(161x^2 + 593xy - 377y^2)}{\sqrt{\Delta(161x^2 + 593xy - 377y^2)}} \right)^{-1} = \frac{\sqrt{594437}}{161}.$$

Extended Markov theory in one diagram



Y : Nothing is known here.

Extended Markov theory in one diagram



$$Z: \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto \frac{\sqrt{(a+d)^2 - 4}}{b}.$$

$$\frac{\sqrt{(89 + 682)^2 - 4}}{161} = \frac{\sqrt{594437}}{161}.$$

Generalized Uniqueness Conjecture

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Conjecture (Frobenius 1913)

For $\mathcal{G}((1, 1), (2, 2))$: each Markov number appears exactly once as the maximal element in Markov triples.

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Generalised Uniqueness conjecture (M. van Son, O.K., 2020)

Does the same statement for $\mathcal{G}(\mu, \nu)$ hold?

One counter-example (M. van Son, O.K.)

Example (Counterexample to some cases)

For $n \geq 1$ let

$$a_n = n^2 + 3,$$

$$b_n = n^4 + 5n^2 + 5.$$

One counter-example (M. van Son, O.K.)

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Then the **generalized uniqueness conjecture fails** for the graph of general Markov numbers

$$\mathcal{T}\left(\left(a_n n, a_n n\right), \left(b_n n, b_n n\right)\right).$$

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Let $n = 1$. The vertices

$$(4, 355318099, 19801199),$$

$$(2888956, 355318099, 11)$$

both appear in the graph $\mathcal{T}((4, 4), (11, 11))$.

The end.

Thank you.