From Symmetric Functions, to Knots, & back again

(Nicolle González jt w/ Matt Hogancamp)

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Our main player is the algebra $\mathbb{A}_{q,t}$.

Motivation:

① Shuffle Conjecture:

Space of Diag. Harmonics $\xrightarrow[\text{Character}]{\text{Frobenius}}$ Combinatorial formula over Dyck paths & Parking functions.

$$DH_n = \left\{ f \in \mathbb{C}[X_n, Y_n] \;\middle|\; \sum_{i=1}^{n} \partial^n/\partial x_i^n \, \partial^k/\partial y_i^k (f) = 0 \right\}$$

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Conjecture (Haglund, Haiman, Loehr, Remmel, Ulyanov)

Nabla operator

$$\left(\mathcal{F}r_{DH_n}(X; q,t) = \right) \nabla e_n = \sum_{\pi} \sum_{w \in WP_n} t^{\text{area}(\pi)} q^{\text{dinv}(\pi,w)} x_w$$

monomial

Haiman

nth. elem. Sym. polyn.

Dyck path

Word parking functions

combinatorial stats.

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~~Conjecture~~ Theorem (Haglund, Haiman, Loehr, Remmel, Ulyanov)

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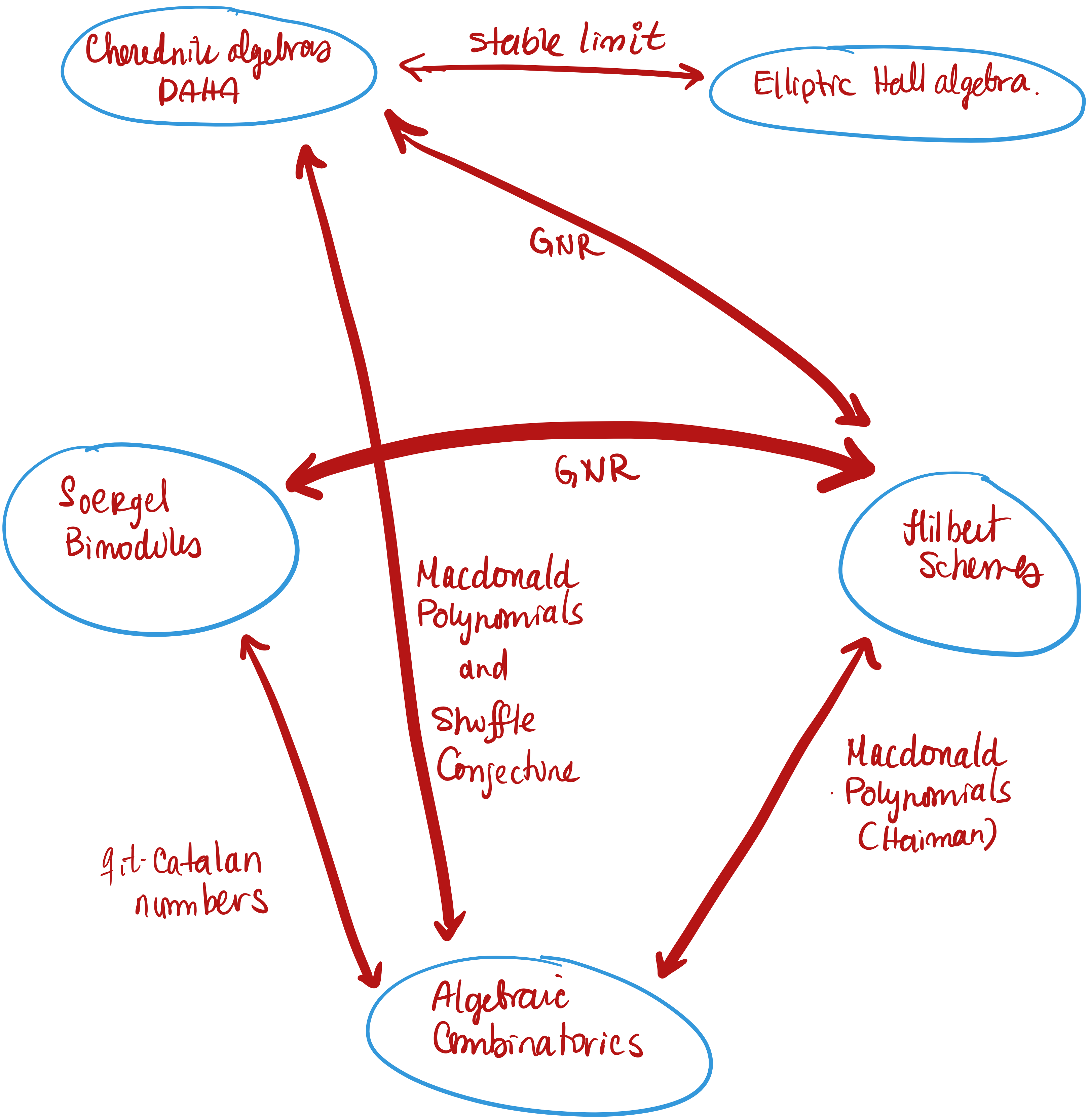
⇒ After 14 years Carlsson-Mellit proved it

Did this by introducing a new algebra $\mathbb{A}_{q,t}$ and defining a rep'n over

$$V_n = \mathbb{Q}[q,t][y_1, \dots y_n][x_1, x_2 \dots]^{\operatorname{Sym}}$$

Road map of $A_{q,t}$.



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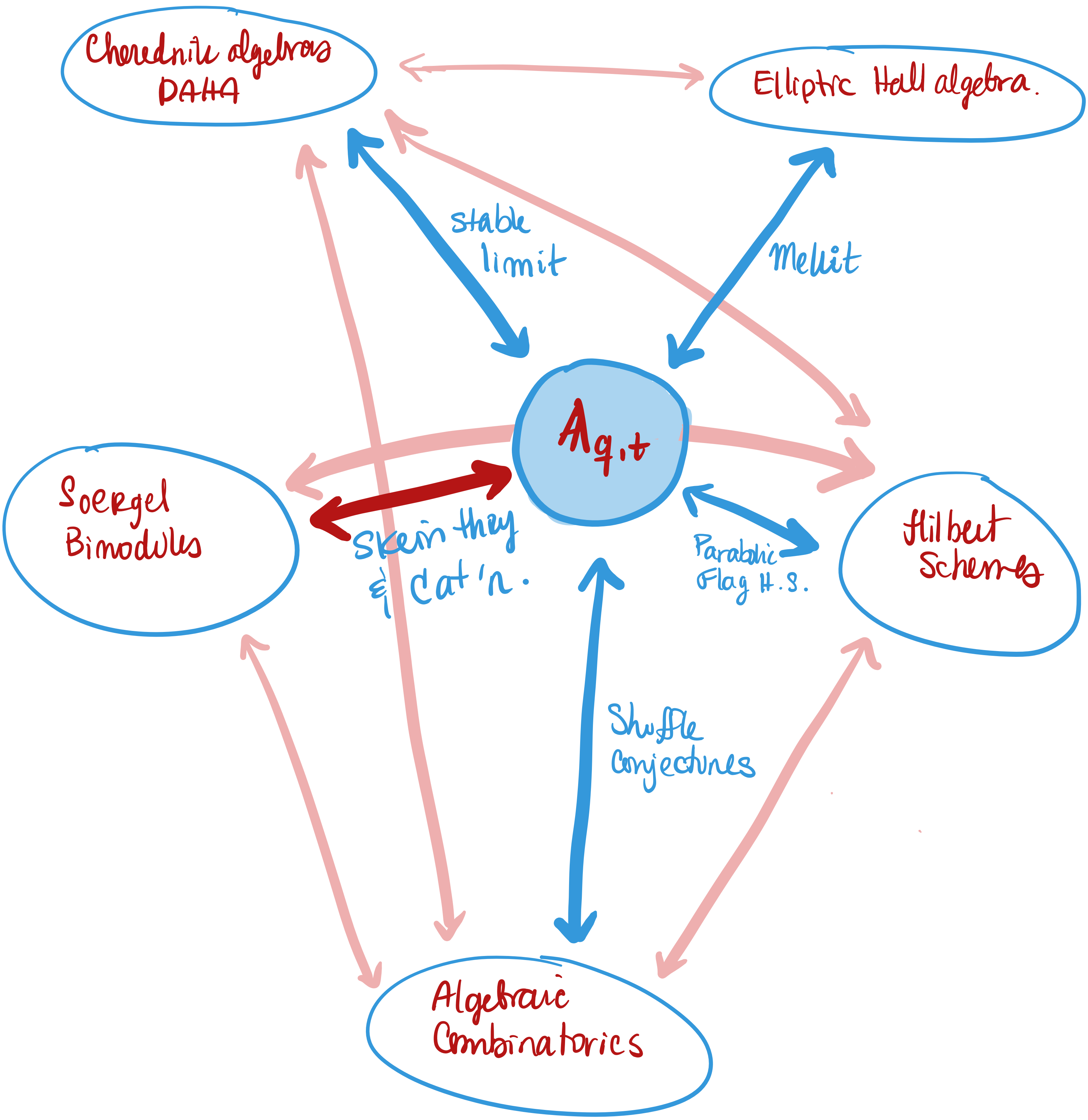
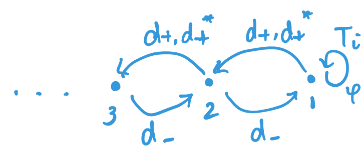


$\mathbb{A}_{q,t}$ algebra (Carlsson-Mellit algebra)

$\mathbb{Q}(q,t)$ algebra w/ idempotents $(\mathbb{1}_n)$ $n \in \mathbb{Z}_n$



gener. by $T_i, T_i^{-1} : n \longrightarrow n$ $(i = 1, \dots, n-1)$

$d_+^*, d_+ : n \longrightarrow n+1$ $\quad d_- : n \longrightarrow n-1$

$\varphi : \dfrac{(tq^{-1})^{1-n}}{tq^{-1} - t^{-1}q} [d_-, d_+]$ $\quad \varphi^* : \dfrac{(tq^{-1})^{n-1}}{t^{-1}q - tq^{-1}} [d_-, d_+^*]$

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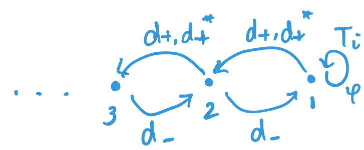

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w/ Rels:

Hecke alg.
$$\begin{cases} (T_i - tq^{-1})(T_i + t^{-1}q) = 0 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{and} \quad T_i T_j = T_j T_i \end{cases}$$

Com. Relns.
$$\left\{ d_- T_i = T_i d_- \ ; \ d_- T_i = T_{i+1} d_+ \right.$$

Braid absorbtion
$$\begin{cases} T_i d_+^2 = (tq^{-1}) d_+ \ ; \quad d_-^2 T_{n-1} = tq^{-1} d_-^2 \\ \varphi d_+ = T_1 d_+ \varphi \ ; \quad d_- \varphi = \varphi d_- T_{n-1} \end{cases}$$

and also same relations w/ $T_i \longrightarrow T_i^{-1}$ $\quad \varphi \longrightarrow \varphi^*$

$d_+ \longrightarrow d_+^*$ $\quad q \longrightarrow q^{-1}$

intertwinned Reln's
$$\begin{cases} d_+ y_i^* = y_{i+1}^* d_+ \qquad d_+^* y_i = y_{i+1} d_+^* \\ -q(tq^{-1}) y_1 d_+^* = -qt\, y_1 d_+^* \end{cases}$$

Properties of $A_{q,t}$:

$A_{q,t}$ contains two affine Hecke subalgebra.

$y_i := T_{i-1} \dots T_1 \varphi T_{n-1}^{-1} \dots T_i^{-1}$ (y_i^* analogously def'n)

$\Rightarrow \{y_i, T_j\}$ & $\{y_i^*, T_j^{-1}\}$ satisfy affine Hecke alg. relations.

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Polynomial Representation

Let $V_n := \mathbb{Q}(q,t)[y_1, \dots, y_n][x_1, \dots]^{Sym}$

Ex: $V_0 = \mathbb{Q}(q,t)[x_1, \dots]^{Sym}$ – Ring of symmetric functions.

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Then: $\mathbb{A}_{q,t}$ acts on $\bigoplus_{n \geq 0} V_n$.

$V_n = \bigoplus_{m \geq 0} V_{n,m}$ $n = \#$ of y's
$m =$ homog. degree of x's

such that:

$$d_+ : V_{n,m} \to V_{n+1,m}$$
$$d_- : V_{n,m} \to V_{n-1,m+1}$$
$$T_i : V_{n,m} \to V_{n,m}$$
$$\varphi : V_{n,m} \to V_{n,m+1}$$
$$y_i : V_{n,m} \to V_{n,m+1}$$

Action on $\bigoplus_{n\geq 0} V_n$ given by:

$$y_i(f) = y_i \cdot f$$

$$T_i(f) = tq^{-1}(f) - \left((tq^{-1}) y_i - (tq^{-1}) y_{i+1} \right) z_i(f)$$

$$d_-(f) = (tq^{-1}) \operatorname{Res}_{y_n} \left(\sum_{k\geq 0} (tq^{-1})^k y_n^{-k} e_k \cdot f[X + (tq^{-1} - qt^{-1}) y_n] \right)$$

$$d_+(f) = (tq^{-1})^n T_1^{-1} \cdots T_n^{-1} f[X - (tq^{-1} - qt^{-1}) y_{n+1}]$$

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Where is ∇?

$\mathcal{N} : \bigoplus_{n\geq 0} V_n \to \bigoplus_{n\geq 0} V_n$ antilinear, degree preserving automorphism uniquely determined by $\mathcal{N}(1) = 1$

$\mathcal{N}$ intertwines the actions of:

$$\mathcal{N} d_+ = d_+^* \mathcal{N} \qquad \mathcal{N} d_- = d_- \mathcal{N} \qquad \mathcal{N} T_i = T_i^{-1} \mathcal{N}$$

$$\mathcal{N} \varphi = \varphi^* \mathcal{N} \qquad \mathcal{N} y_i = y_i^* \mathcal{N}$$

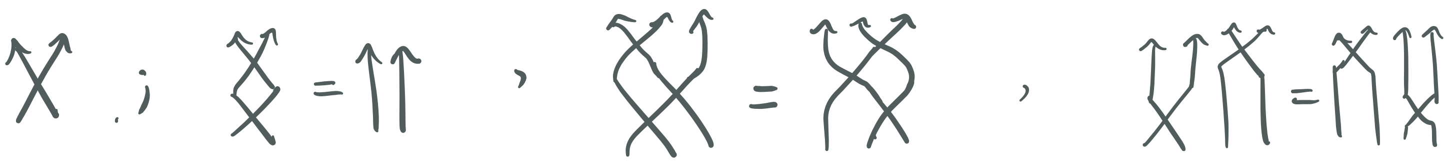
when restricted to $V_0 = \mathrm{Sym}$. $\quad \mathcal{N}\big|_{V_0} = \nabla \circ \overline{\omega}$

standard involution on Sym. functions.

Topological Hecke algebra

S_n - symmetric group.

$$\langle \sigma_i \mid \sigma_i^2=1 \;,\; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \;,\; \sigma_i\sigma_j = \sigma_j\sigma_i \rangle$$



Thus for any permutation

$w \in S_n \rightsquigarrow$ diagram.

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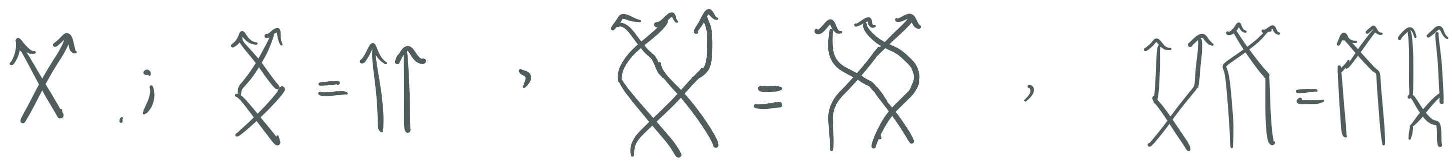
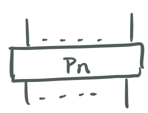
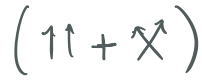


Thus for any permutation

$w \in S_n \rightsquigarrow$ diagram.

$f \in \mathbb{C}[S_n] \rightsquigarrow$ sum of diagram.

Young symmetrizers $e_n = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \rightsquigarrow P_n =$ 

$e_2 = \frac{1}{2!}(id + (12)) \rightsquigarrow P_n = \frac{1}{2!}$ 

$\Rightarrow$ The Hecke algebra is a quantum deformation of $\mathbb{C}[S_n]$

want to lift everything from $\mathbb{C}[S_n]$, including the Young symmetrizers.

Hecke algebra $\mathcal{H}_n$, the $\mathbb{Q}(q,t)$-algebra gen. by:

T_i, T_i^{-1}

w/ relations

- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
- $T_i T_j = T_j T_i$
- $$\underbrace{T_i - T_i^{-1} = (tq^{-1} - qt^{-1})\cdot 1}_{(T_i - tq^{-1})(T_i + t^{-1}q) = 0}$$

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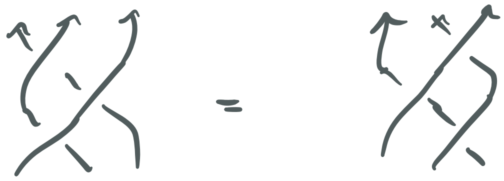
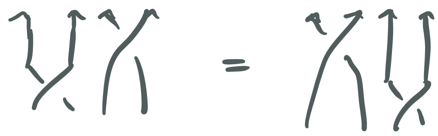
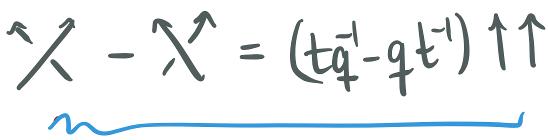
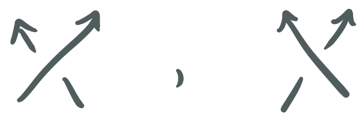
- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ $\longrightarrow$ 
- $T_i T_j = T_j T_i$ $\longrightarrow$ 
- $T_i - T_i^{-1} = (tq^{-1} - qt^{-1}) \cdot 1$ $\longrightarrow$ 

Skein relation

Hecke algebra H_n, the $\mathbb{Q}(q,t)$-algebra gen. by:

T_i, T_i^{-1} $\longrightarrow$ 

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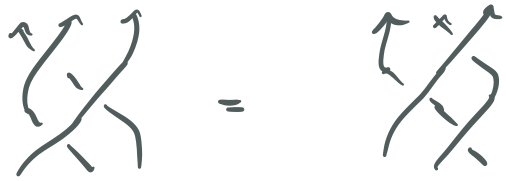
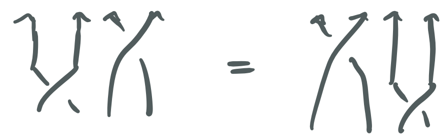
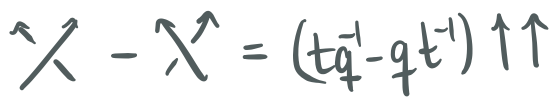
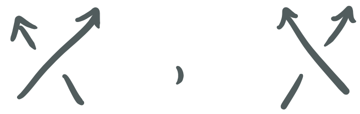
- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ $\longrightarrow$ 
- $T_i T_j = T_j T_i$ $\longrightarrow$ 
- $T_i - T_i^{-1} = (tq^{-1} - qt^{-1}) \cdot 1$ $\longrightarrow$ 

Skein relation

$\Rightarrow$ $w \in H_n \rightsquigarrow$ braid

$H_n = \mathrm{span}_{\mathbb{Q}(q,t)} \left\{ \boxed{D} \;\middle|\; D \text{ braid mod skein} \right\}$

Hecke algebra $\mathcal{H}_n$, the $\mathbb{Q}(q,t)$-algebra gen. by:

$T_i, T_i^{-1} \longrightarrow$ 

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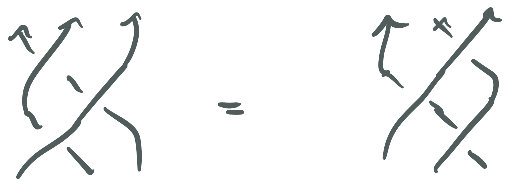
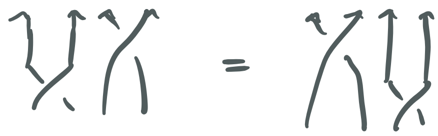
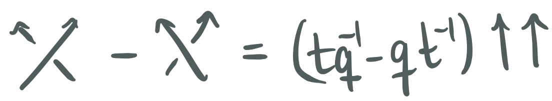
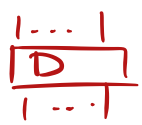
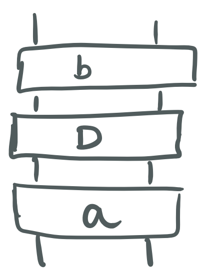
- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \rightarrow$ 
- $T_i T_j = T_j T_i \longrightarrow$ 
- $T_i - T_i^{-1} = (tq^{-1} - qt^{-1}) \cdot 1 \longrightarrow$ 

Skein relation

$\Rightarrow \ w \in \mathcal{H}_n \rightsquigarrow$ braid

$\mathcal{H}_n = \text{span}_{\mathbb{Q}(q,t)} \Big\{$  $\Big|$ D braid mod skein $\Big\}$

$\mathcal{H}_n \circlearrowright \mathcal{H}_n \circlearrowleft \mathcal{H}_n$

$a \cdot w \cdot b \rightsquigarrow$ 

← right action

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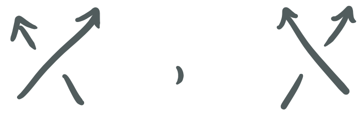
$T_i, T_i^{-1} \longrightarrow$ 

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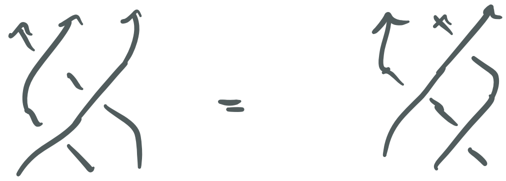
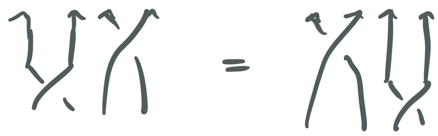
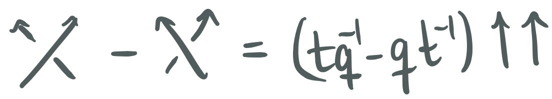
- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \rightarrow$ 
- $T_i T_j = T_j T_i \longrightarrow$ 
- $T_i - T_i^{-1} = (tq^{-1} - qt^{-1}) \cdot 1 \longrightarrow$ 

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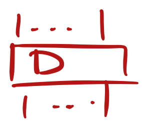
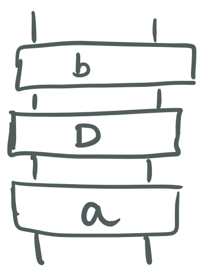
$\mathcal{H}_n = \mathrm{span}_{\mathbb{Q}(q,t)} \Big\{$ 

$\Big|$ D braid mod skein $\Big\}$

$\mathcal{H}_n \curvearrowright_{\mathcal{H}_n} \curvearrowleft \mathcal{H}_n$

$a \cdot w \cdot b \rightsquigarrow$ 

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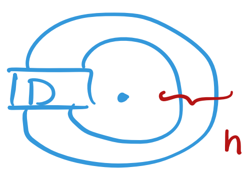
$\mathcal{H}_n / [\mathcal{H}_n, \mathcal{H}_n] = \mathrm{span}_{\mathbb{Q}(q,t)} \Big\{$ 

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Thm (Morton–Aiston) $\bigoplus_{n\geq 0} \mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n] = \mathrm{Sym} \quad (= V_0)$

Recall: $V_n = \mathbb{Q}(q,t)[y_1, \dots y_n][x_1 \dots]^{\mathrm{Sym}}$

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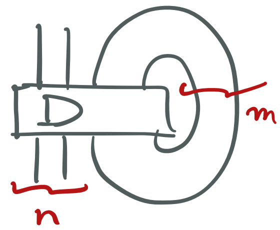
$\boxed{\text{How to incorporate the } y_i\text{'s ?}}$

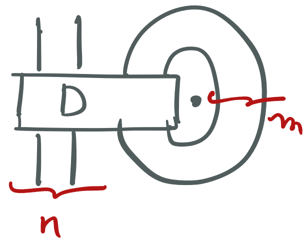
$\mathcal{H}_{n+m}/[\mathcal{H}_m, \mathcal{H}_{n+m}] \simeq \mathrm{Span}_{\mathbb{Q}(q,t)} \Big\{$



$\Big| \ D \text{ braid} \Big\}$ Mod skein

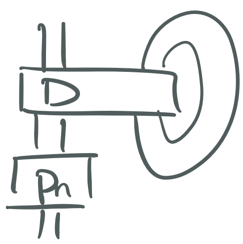
Prop (G-Hogancamp)

$V_n \simeq \bigoplus_{m\geq 0} \mathrm{Span}_{\mathbb{Q}(q,t)} \Big\{$

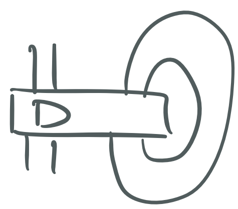


$\Big| \ D \text{ braid} \ni D(p_n \otimes 1) = D \Big\}$ Mod Skein

nth Young symmetrizing q-idempotent



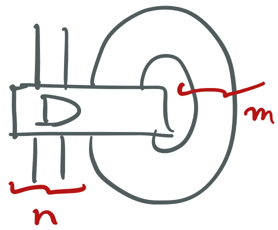
$=$



Thm (Morton–Aiston) $\bigoplus_{n\geq 0} H_n/[H_n, H_n] = \mathrm{Sym} \ (= V_0)$

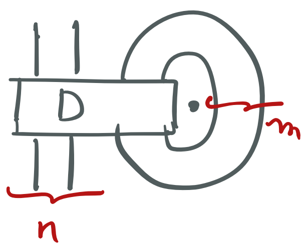
Recall: $V_n = \mathbb{Q}(q,t)[y_1, \dots, y_n][x_1, \dots]^{Sym}$

$\boxed{\text{How to incorporate the } y_i\text{'s?}}$

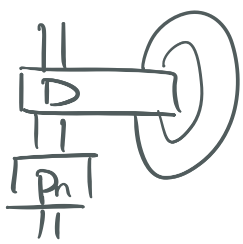
$H_{n+m}/[H_m, H_{n+m}] \simeq \mathrm{Span}_{\mathbb{Q}(q,t)} \Big\{$ 

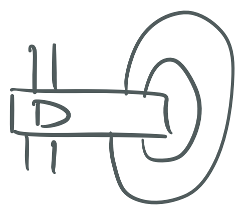
$\Big|$ D braid, Mod skein $\Big\}$

Prop (G–Hogancamp)

$V_n \simeq \bigoplus_{m\geq 0} \mathrm{Span}_{\mathbb{Q}(q,t)} \Big\{$ 

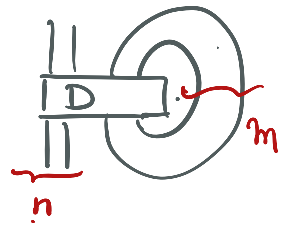
$\Big|$ D braid $\ni$ $D(p_n \otimes 1) = D$, Mod Skein $\Big\}$

nth Young Symmetrizing q-idempotent $\rightarrow$ 

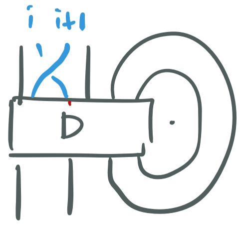
$=$ 

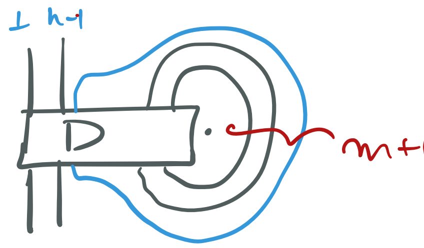
Q1: How does $A_{q,t}$ act on this vector space? Can we recover the polynomial representation?

$A_{q,t}$ action

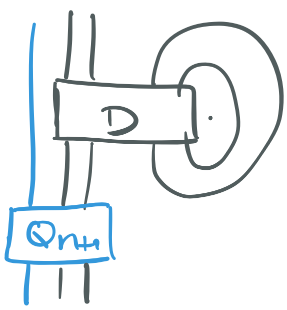
$D =$ 

$\in V_n$

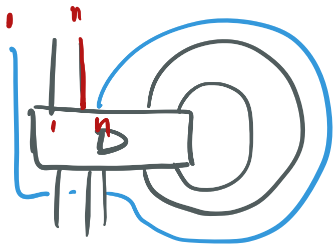
$T_i(D) =$ 

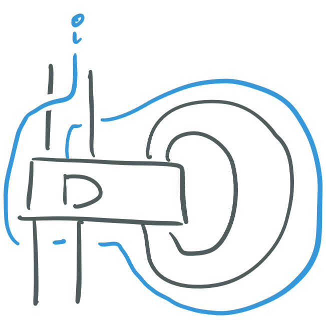
$d_-(D) =$ 

$d_- : V_{\underline{n},m} \longmapsto V_{\underline{n-1},m+1}$

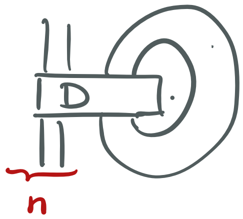
$d_+(D) =$ 

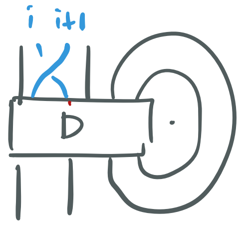
Q_{n+1} is a multiple of p_{n+1} $\rightarrow$

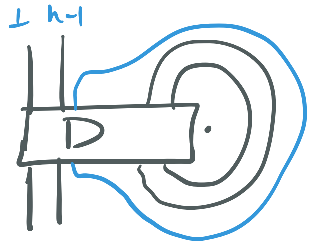
$\varphi(D) =$ 

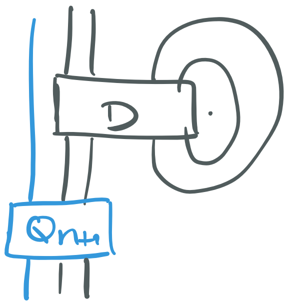
$y_i(D) =$ 

$A_{q,t}$ action

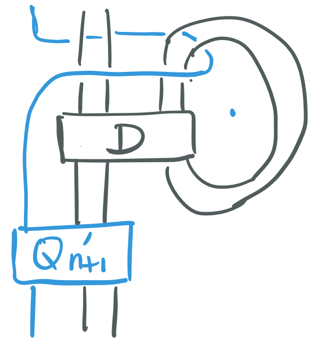
$D =$  $\in V_n$

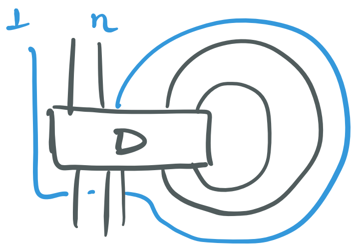
$T_i(D) =$ 

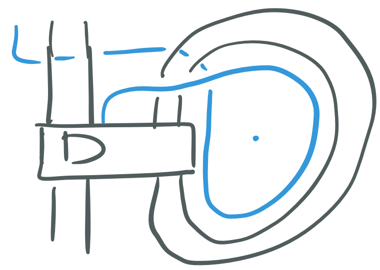
$d_-(D) =$ 

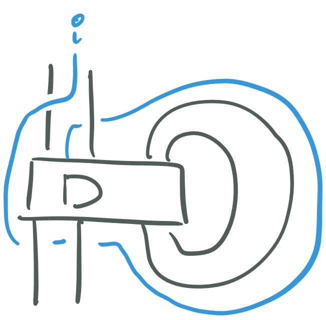
$d_+(D) =$ 

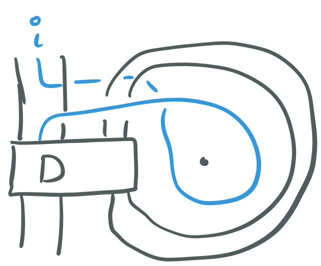
Q_{n+1} is a multiple of p_{n+1} $\rightarrow$

$d_+^*(D) = t(tq^{-1})^{-2}$ 

$\varphi(D) =$ 

$\varphi^*(D) = t(tq^{-1})^{-2}$ 

$y_i(D) =$ 

$y_i^*(D) = t(tq^{-1})^{-2}$ 

Theorem (G-Hogancamp) These diagrams induce a topological action of $A_{q,t}$ on V_n that realizes the polynomial representation.

Proof: diagrammatic.

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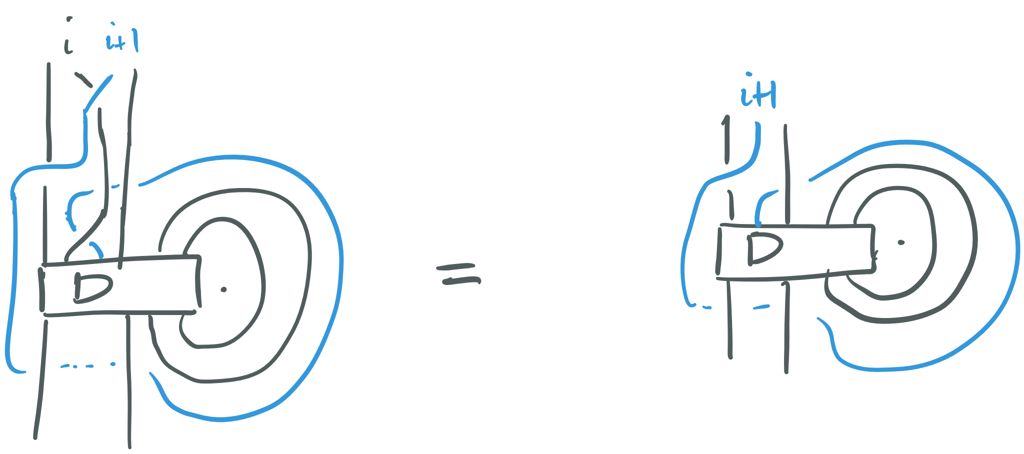
$$d_- \varphi = \varphi d_- T_{n-1}$$

$$T_i y_i T_i = y_{i+1}$$



.....etc - - - --

$\blacksquare$

Recall the algebraic action originally defined...

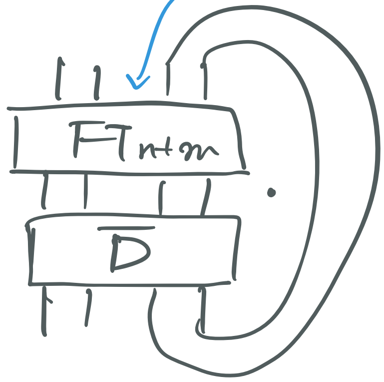
- $y_i(f) = y_i \cdot f$
- $T_i(f) = tq^{-1}(f) - \left((tq^{-1}) y_i - (tq^{-1}) y_{i+1} \right) \partial_i(f)$
- $d_-(f) = (tq^{-1}) \, \mathrm{Res}_{y_n} \left(\sum_{k \geq 0} (tq^{-1})^k y_n^{-k} e_k \cdot f[X + (tq^{-1} - qt^{-1}) y_n] \right)$
- $d_+(f) = (tq^{-1})^n \, T_1^{-1} \cdots T_n^{-1} f[X - (tq^{-1} - qt^{-1}) y_{n+1}]$

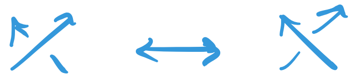
where $\quad \varphi : \dfrac{(tq^{-1})^{1-n}}{tq^{-1} - t^{-1}q} [d_-, d_+]$

Q: What about ∇ or $\mathcal{N}$?

The intertwining operator becomes

FT = full twist

$$\mathcal{N}(D) = t^{n+m}\,(tq^{-1})^{-2(n+m)}$$ 

$\bar{D}$ is D w/ 

$q, t \longleftrightarrow q^{-1}, t^{-1}$

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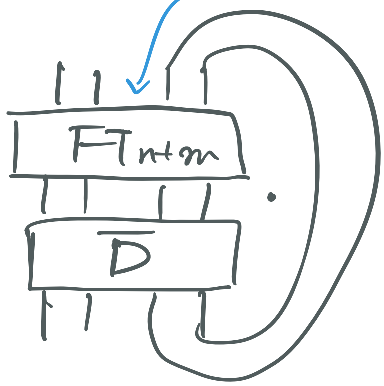
$$\mathcal{N}(D) = t^{n+m}\,(tq^{-1})^{-2(n+m)}$$ 

$\overline{D}$ is D w/ $\longleftrightarrow$

$q, t \leftrightarrow q^{-1}, t^{-1}$

and satisfies $\mathcal{N}^2 = \mathrm{id}$ and

$$\mathcal{N} y_i = y_i^* \mathcal{N} \qquad \mathcal{N}\varphi = \varphi^* \mathcal{N} \qquad \mathcal{N} d_- = d_- \mathcal{N}$$

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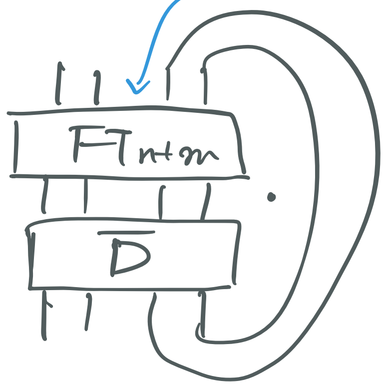
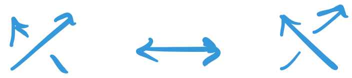
$$\mathcal{N}(D) = t^{n+m} (tq^{-1})^{-2(n+m)}$$ 

$\overline{D}$ is D w/ 

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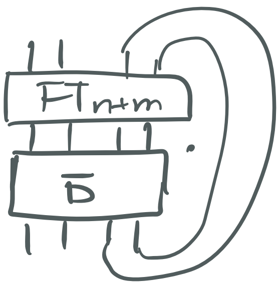
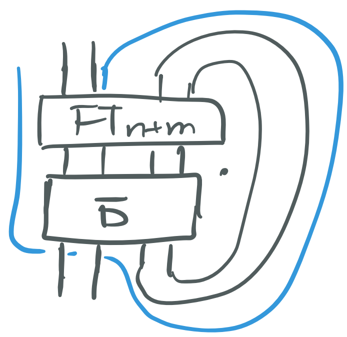
Example: $\mathcal{N} \varphi \mathcal{N} = \varphi^*$

$t^{n+m} (tq^{-1})^{-2(n+m)}$  $\xrightarrow{\varphi}$ $t^{n+m} (tq^{-1})^{-2(n+m)}$ 

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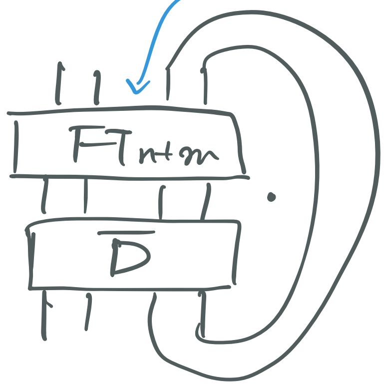
$\mathcal{N}(D) = t^{n+m}(tq^{-1})^{-2(n+m)}$ 

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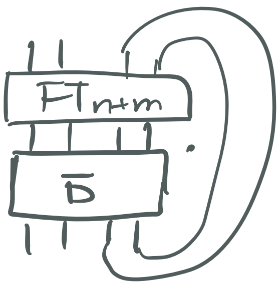
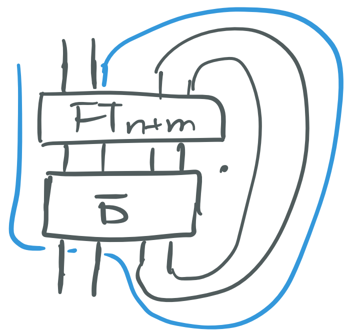
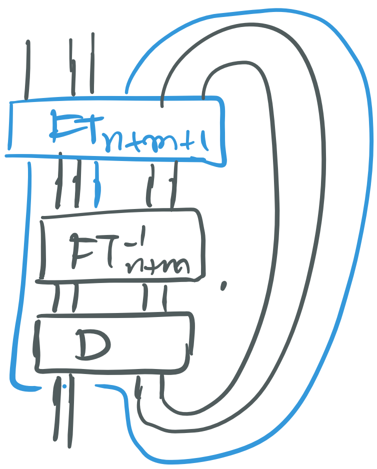
$t^{n+m}(tq^{-1})^{-2(n+m)}$ 

. $\xrightarrow{\varphi}$ $t^{n+m}(tq^{-1})^{-2(n+m)}$ 

.

$\xrightarrow{\mathcal{N}}$ $t^{(n+m+1)}(tq^{-1})^{-2(n+m+1)} \cdot t^{-(n+m)}(tq^{-1})^{2(n+m)}$ 

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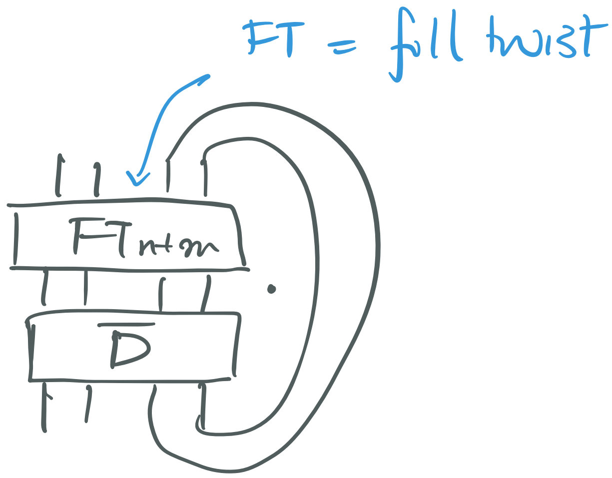
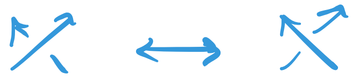
$$\mathcal{N}(D) = t^{n+m}(tq^{-1})^{-2(n+m)}$$ 

$\bar{D}$ is D w/ 

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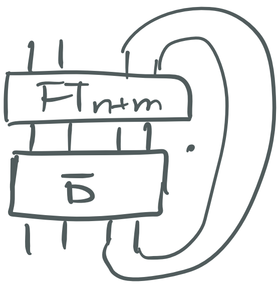
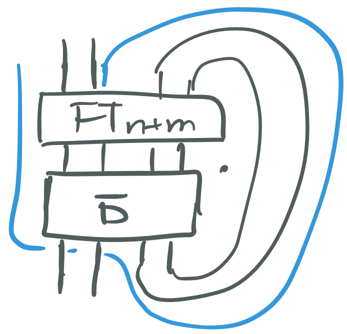
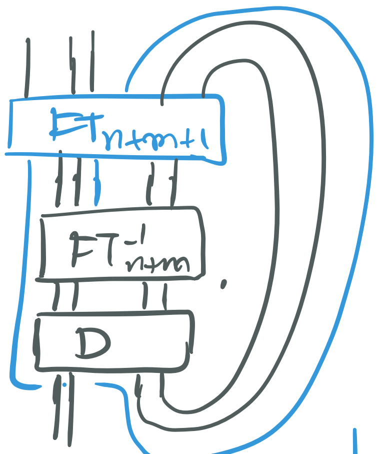
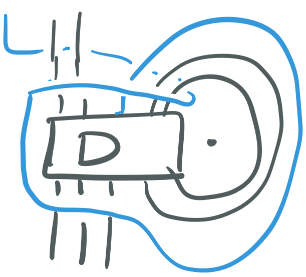
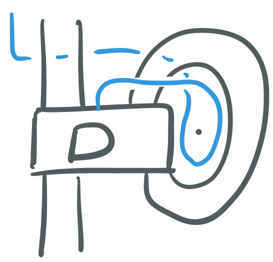
$t^{n+m}(tq^{-1})^{-2(n+m)}$ 

$\xrightarrow{\varphi}$ $t^{n+m}(tq^{-1})^{-2(n+m)}$ 

$\xrightarrow{\mathcal{N}}$ $t^{(n+m+1)}(tq^{-1})^{-2(n+m+1)} \cdot t^{-(n+m)}(tq^{-1})^{2(n+m)}$ 

$= t(tq^{-1})^{-2}$ 

$= t(tq^{-1})^{-2}$ 

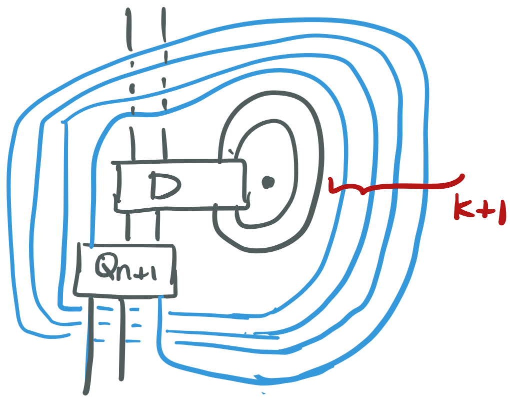
$= \varphi^*$

Symmetric functions

Many of the symmetric functions that appear in the proof of the Shuffle conjecture appear in this context

for $D \in V_n$

- $e_{k+1}(D) = (tq^{-1})^{-(n+k)}$



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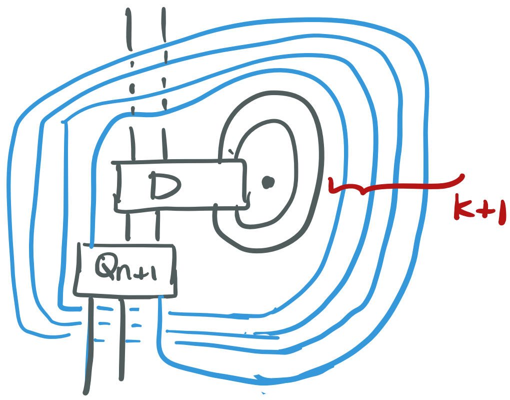
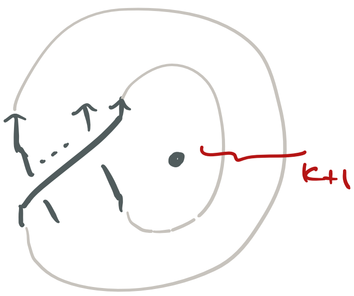
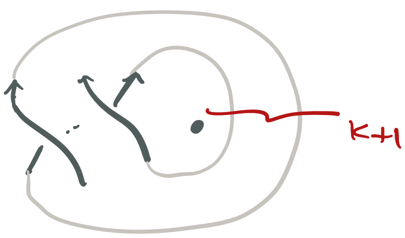


at $n=0$ ($V_0 = Sym$)

→ $e_{k+1} = (tq^{-1})^{k}(tq^{-1} - qt^{-1})$



→ $h_{k+1} = (-tq^{-1})^{k}(tq^{-1} - qt^{-1})$



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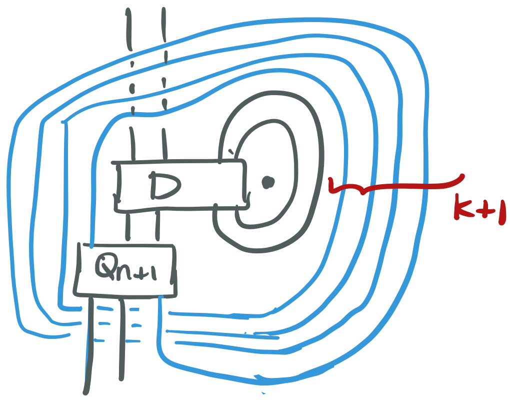

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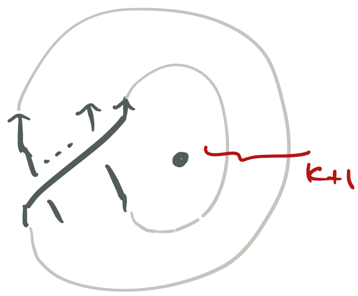

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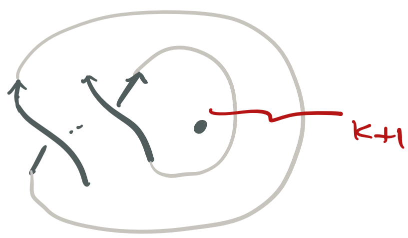
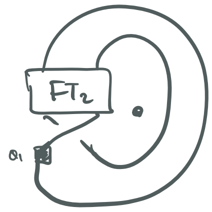
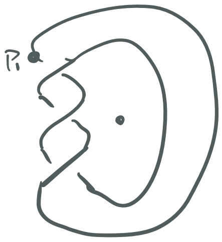


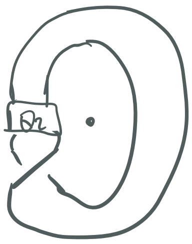
- $\nabla e_n \sim$

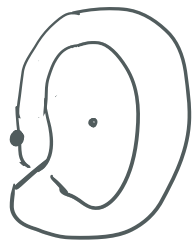
- parking functions, etc all have topological analogues

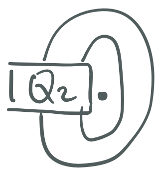
Ex:

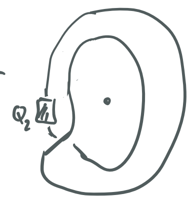
$\nabla e_2 =$ 

$= (\eta^2-1)\eta^{-2}$ 

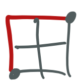
$= (\eta^2-1)\eta^{-2}$ 

$+ (\eta^2-1)\eta^{-2}q^2$ 

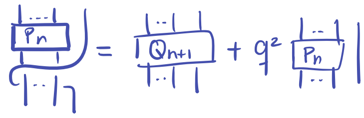
$= \underbrace{(\eta^2-1)\eta^{-1}}$ 

$+ \underbrace{q^2}$ 

$= (\eta^2-1)\eta^{-1}\,\chi(\pi_1, 1) + \dfrac{q^2}{\eta(\eta-\eta^{-1})} \cdot e_2$



Fact: $Q_n = (\eta^{2n}-1)\eta^{-2}P_n$



$\boxed{\eta = tq^{-1}}$

Future Directions

- new operators on Sym
- connections btw q,t combinatorics & knot homology
- categorical understanding

Thanks!