Plabic graphs and cluster structures on positroid varieties

Slides available at `www.math.berkeley.edu/~msb`

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joint work with K. Serhiyenko and L. Williams (arXiv:1902.00807)
C. Fraser (arXiv:2006.10247)

UCLA Combinatorics Seminar

Grassmannian background

Fix $0 < k < n$. $[n] := \{1, \ldots, n\}$ and $\binom{[n]}{k} := \{I \subset [n] : |I| = k\}$.

- $Gr_{k,n} := \{V \subset \mathbb{C}^n : \dim V = k\}$.
- $V \in Gr_{k,n} \rightsquigarrow$ full rank $k \times n$ matrix A whose rows span V
- Plücker coordinates: $I \in \binom{[n]}{k}$, $\Delta_I(V) :=$ max'l minor of A in cols I. Satisfy Plücker relations like

$$\Delta_{Sik}\Delta_{Sjl} = \Delta_{Sij}\Delta_{Skl} + \Delta_{Sil}\Delta_{Sjk}.$$

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- Postnikov, Lusztig: The *totally nonnegative* (TNN) Grassmannian $Gr_{k,n}^{\geq 0} := \{V \in Gr_{k,n} : \Delta_I(V) \geq 0 \text{ for all } I \in \binom{[n]}{k}\}$.

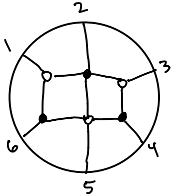
The combinatorics: positroids

$V \in Gr_{k,n}$. The *matroid* of V is $\mathcal{M}_V := \{I \in \binom{[n]}{k} : \Delta_I(V) \neq 0\}$. If V is TNN, $\mathcal{M}_V$ is a *positroid*.

Examples: uniform matroid $\binom{[n]}{k}$, Schubert matroids, lattice path matroids.

Indexed by a menagerie of combinatorial objects.

$\mathcal{M} = \binom{[6]}{3} \setminus \{345, 156\}$



$\pi = 465213$

+	0	+
0	+	+
+	+	+

$\vec{\mathcal{I}} = (123, 234, 346, 456, 256, 126)$

The geometry: positroid varieties

Open positroid variety (Knutson-Lam-Speyer, Lusztig, Rietsch):

$$\Pi^\circ_{\mathcal{M}} := \{V \in Gr_{k,n} : \text{ smallest positroid containing } \mathcal{M}_V \text{ is } \mathcal{M}\}$$

Examples:

$$Gr^\circ_{k,n} = Gr_{k,n} \setminus \{V : \Delta_{12\cdots k}\Delta_{23\cdots k+1}\cdots\Delta_{n1\cdots k-1} = 0\}$$

$$\Pi^\circ_J = \{V : \Delta_J \text{ is lex min nonzero Plücker}\} \setminus \{V : \Delta_J\Delta_{J_2}\cdots\Delta_{J_n} = 0\}$$

(open Schubert variety)

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$\Pi^\circ_{\mathcal{M}}$ is cut out of $Gr_{k,n}$ by the equations

$$\Delta_I = 0 \quad \text{ for } I \notin \mathcal{M}$$
$$\Delta_{I_1}, \dots, \Delta_{I_n} \neq 0$$

so $\mathbb{C}[\Pi^\circ_{\mathcal{M}}]$ is $\mathbb{C}[Gr_{k,n}]/\langle \Delta_I : I \notin \mathcal{M} \rangle$ localized at $\Delta_{I_1}, \dots, \Delta_{I_n}$.

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Positive part: $\Pi^\circ_{\mathcal{M}} \cap Gr_{k,n}^{\geq 0} = \{V \in \Pi^\circ_{\mathcal{M}} : \Delta_I > 0 \text{ for } I \in \mathcal{M}\}$.
It is a cell (Postnikov, Rietsch).

Also, many short "subtraction-free" relations hold in $\mathbb{C}[\Pi^\circ_{\mathcal{M}}]$, which give you information about positivity.

Brief overview of cluster algebras

Introduced by Fomin and Zelevinsky (2000). Commutative rings with distinguished generators, defined recursively.

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Ingredients: Start with coordinate ring $\mathbb{C}[V]$.

- Initial *seed* $\Sigma \subset \mathbb{C}[V]$ of *cluster variables* labeling a directed graph.

$$\begin{tikzcd}
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\Delta_{14} \arrow[r] & \Delta_{24} \arrow[u] \arrow[d] & \Delta_{23} \arrow[l] & \longrightarrow & \Delta_{14} \arrow[ur] \arrow[dr] & \Delta_{13} \arrow[l] \arrow[r] & \Delta_{23} \arrow[ul] \arrow[dl] \\
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- *Mutation*: local move to get new seed. Cluster variable x is exchanged for new cluster variable x' satisfying $x \cdot x' = A + B$.
- Some variables are *frozen*, so can't mutate them.

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Cluster algebra $\mathcal{A}(\Sigma) := \mathbb{C}[\text{frozen variables}^{\pm 1}][\text{cluster variables}]$.
$\mathbb{C}[V]$ is a cluster algebra if $\mathbb{C}[V] = \mathcal{A}(\Sigma)$ for some seed Σ.

What to remember about cluster algebras

- *Cluster monomials* (monomials in elements of a seed, inverses of frozens allowed) are part of a basis for $\mathcal{A}(\Sigma)$ with positive structure constants (Gross-Hacking-Keel-Kontsevich).
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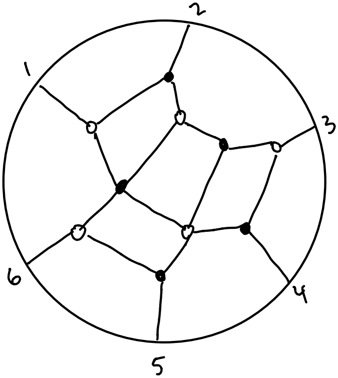
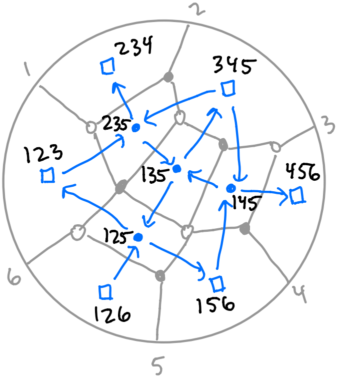
Basic problems: Show $\mathbb{C}[V]$ is a cluster algebra. Then explicitly describe as many cluster monomials (equivalently, seeds) as possible.

Theorem (Scott '06)

$\mathbb{C}[Gr^{\circ}_{k,n}]$ is a cluster algebra and Postnikov's plabic graphs for $Gr^{\circ}_{k,n}$ give seeds (consisting entirely of Plücker coordinates).





Grassmannian case

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Conjecture (Muller-Speyer '16)

Analog of Scott's result should hold for open positroid varieties $\Pi^\circ_{\mathcal{M}}$.

Cluster algebra structure for positroid varieties

Theorem (Serhiyenko-SB-Williams '19)

Let Π_J° be an open Schubert variety. Then plabic graphs for Π_J° give seeds for a cluster algebra structure on $\mathbb{C}[\Pi_J^\circ]$.

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Cluster algebra structure picks out the "right" positive part:
$\{V \in \Pi_{\mathcal{M}}^\circ :$ all cluster variables are positive on $V\} = \Pi_{\mathcal{M}}^\circ \cap Gr_{k,n}^{\geq 0}$.
Plabic graphs give positivity tests.

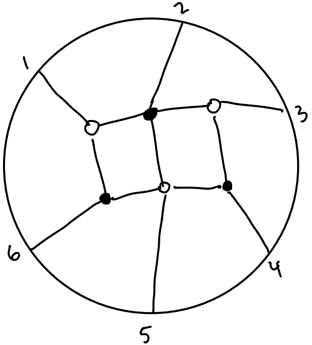
A bit more on seeds from plabic graphs

A *plabic graph* G of type (k, n): planar, embedded in disk, boundary vertices $1, \ldots, n$ going clockwise, internal vertices colored black and white.

To get a seed Σ_G:

- Directed graph is dual graph (boundary faces are frozen).
- For cluster: use *trips* to label faces.
    - For trip $i \rightsquigarrow j$, put the *target* j in faces to left of trip.

  Every face labeled by k-elt subset, which we interpret as Plücker coordinate.



Trip permutation μ tells you which positroid variety G is a plabic graph for. All seeds Σ_G, where G has trip permutation μ, are related by mutation.

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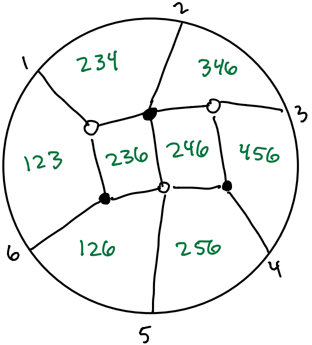

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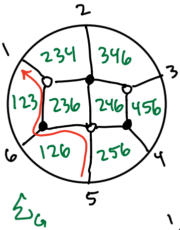

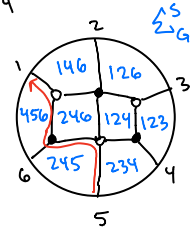

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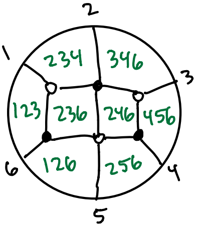
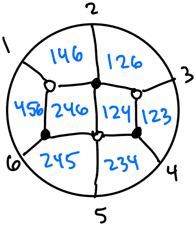
For $\Pi^\circ_\mu \neq Gr^\circ_{k,n}$:

1. Many nonzero P_I do not show up in any plabic seed.
2. $\exists$ seeds whose cluster variables are $P_I \cdot$(*Laurent monomial in frozens*), but don't know combinatorial "source".
3. No sequence of mutations between Σ^S_G and Σ_G! Two convention choices for plabic seeds give different cluster algebras $\mathcal{A}(\Sigma_G)$ and $\mathcal{A}(\Sigma^S_G)$.



Σ_G

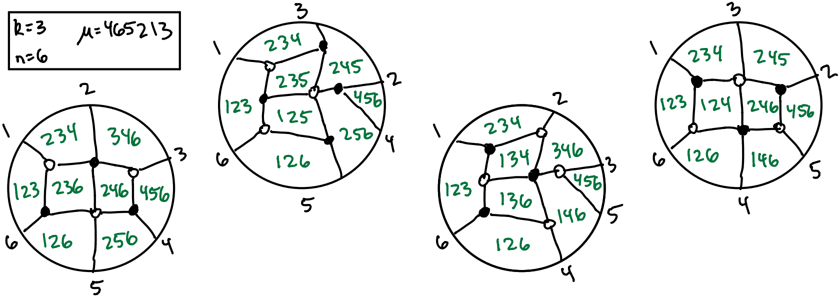


Σ^S_G

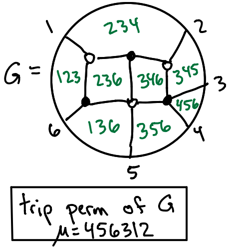
Main results

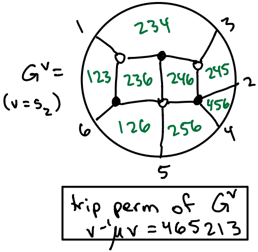
Theorem (Fraser–SB '20)

$\mathbb{C}[\Pi_\mu^\circ]$ *can be identified with many different cluster algebras (with different frozen and cluster variables), with seeds given by certain relabeled plabic graphs with trip permutation* μ.



G a plabic graph of type (k, n), $v \in S_n$. The *relabeled plabic graph* G^v is obtained from G by applying v to its boundary vertex labels.





Trip permutation, face labels, seed of G^v computed according to its boundary labels.

Note: The "source" seed Σ_G^S is the same as $\Sigma_{G(\mu^{-1})}$.

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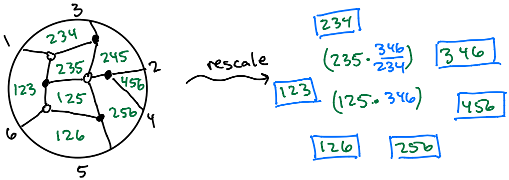
For open Schubert varieties Π°_J, *these cluster algebras are "the same up to frozens" and have same cluster monomials. The relabeled graph seeds can be rescaled by frozens to get seeds in* $\mathcal{A}(\Sigma_G)$.



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Conjecture (Fraser–SB '20)

Above theorem holds for arbitrary Π°_μ.

Partial results for Π°_μ, including that relabeled plabic graphs give positivity tests.

Summary & questions

Call the cluster structure on $\mathbb{C}[\Pi_\mu^\circ]$ given by usual plabic graphs the *standard* cluster structure.

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Takeaway: Relabeled plabic graphs with trip perm μ give many additional explicit seeds in the standard cluster structure, with cluster variables $P_I \cdot$(Laurent mono. in frozens).

Summary & questions

Call the cluster structure on $\mathbb{C}[\Pi_\mu^\circ]$ given by usual plabic graphs the *standard* cluster structure.

Takeaway: Relabeled plabic graphs with trip perm μ give many additional explicit seeds in the standard cluster structure, with cluster variables $P_I \cdot$(Laurent mono. in frozens).

Questions:

- Let Σ be a seed in standard cluster structure on $\mathbb{C}[\Pi_\mu^\circ]$ with cluster variables $\{P_I \cdot \text{(Laurent mono. in frozens)}\}$. Does Σ come from rescaling a relabeled plabic graph seed?
- Are all Plücker coordinates in $\mathbb{C}[\Pi_\mu^\circ]$ cluster monomials? From a relabeled plabic graph seed?
- Relabeled versions of other combinatorial objects indexing positroids?
- Characterization of relabeled plabic graph seeds?

Thanks for listening!

arXiv:1902.00807
arXiv:2006.10247