Which Schubert varieties are Hessenberg varieties?

(jt work in progress w/ Martha Precup &
John Shareshian)

① The flag variety

$\mathcal{F}_n = \{ (V_1, \ldots, V_n) \mid$ each V_i is a vector space and $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n \}$

Example: $(\langle e_1 \rangle, \langle e_1, e_1+e_3 \rangle, \mathbb{C}^3) \in \mathcal{F}_3$

Represent elements of $\mathcal{F}_n$ by invertible $n \times n$ mats:
$M \longleftrightarrow$ (span of column 1, span of columns 1 & 2, ...)

Note: M & N represent the same flag iff there is an invertible upper triangular matrix b such that $Mb = N$.

Let $G = SL_n(\mathbb{C})$ and $B = \{$upper triangular matrices in $G\}$.
Then $\mathcal{F}_n = G/B$.

Bruhat Stratification of $\mathcal{F}_n$

Let $T = \{$ diagonal matrices in $G\}$

$T \circlearrowright \mathcal{F}_n$ by $t \cdot (gB) = tgB$

permutations of $[n]$ $\longleftrightarrow$ T-fixed points of $\mathcal{F}_n$

$w \longleftrightarrow (\langle e_{w_1}\rangle, \langle e_{w_1}, e_{w_2}\rangle, \dots) =: f_w$

$B \circlearrowright \mathcal{F}_n$ by $b \cdot (gB) = bgB.$

Theorem: $\mathcal{F}_n = \coprod_{w \in S_n} Bf_w$

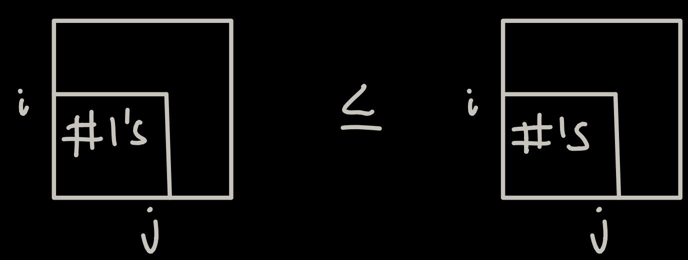
Schubert cells: $C_w := Bf_w$

Schubert varieties: $X_w = \overline{Bf_w}$

In fact: $X_w = \coprod_{v \le w} C_v$, where $\le$ is the Bruhat order

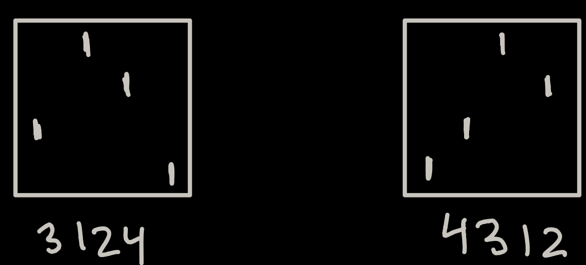
Definition: $v \le w$ if for all $i,j \in [n]$

permutation mtx of v



permutation mtx of w

Example: $3124 \le 4312$



Schubert varieties are combinatorial

$\dim(X_w)$ = number of inversions of w.

A permutation v avoids the permutation 4231 if there do not exist $i_1 < i_2 < i_3 < i_4$ such that $v_{i_4} < v_{i_2} < v_{i_3} < v_{i_1}$.

Examples: 7654321 avoids 4231
7432651 contains the subsequence 7351 so it contains 4231.

Similarly, v avoids 3412 if there do not exist $i_1 < i_2 < i_3 < i_4$ such that $v_{i_3} < v_{i_4} < v_{i_1} < v_{i_2}$.

[Lakshmibai–Sandhya]: X_w is smooth iff w avoids 3412 and 4231.

Corollary of the Bruhat decomposition: $\{[X_w] \mid w \in S_n\}$ is a basis for $H^*(F_n)$.

[Borel]: $H^*(F_n) \cong \mathbb{Z}[x_1, \dots, x_n] / \langle \text{elementary symmetric fcts} \rangle$
$[X_w] \longleftarrow$ Schubert polynomial

Difficult Problem: it is known that $[X_w] \cdot [X_v] = \sum_{u \in S_n} c^u_{w,v} [X_u]$ with $c^u_{w,v} \geq 0$.
Give a combinatorial rule that computes $c^u_{w,v}$.

Hessenberg varieties in $\mathcal{F}_n$ (original definition)

[De Mari, Shayman]

A **Hessenberg function** is a nondecreasing function $h : [n] \longrightarrow [n]$ such that $h(i) \geq i$ for all i.

$\mathfrak{sl}_n(\mathbb{C}) = \{ n \times n \text{ matrices } x \text{ with } \operatorname{trace}(x) = 0 \}$.

Given a Hessenberg function h and $x \in \mathfrak{sl}_n(\mathbb{C})$, the associated **Hessenberg variety** is
$\operatorname{Hess}(x, h) = \{ (V_1, \ldots, V_n) \in \mathcal{F}_n \mid x(V_i) \subseteq V_{h(i)} \text{ for all } i \}$

Examples:

1. Suppose $h(i) = n$ or $x = 0 \Rightarrow \operatorname{Hess}(x, h) = \mathcal{F}_n$
2. Suppose $h(i) = i+1$ for $i < n$, $h(n) = n$, and x is generic $\Rightarrow \operatorname{Hess}(x, h)$ is the toric variety of the braid arrangement.
3. Appropriately chosen x & h yield the Springer fiber and the Peterson variety.

Hessenberg varieties are combinatorial, but less is known

[Precup]: Although not smooth in general, they often have palindromic Poincaré polynomial.

Studying the cohomology rings of Hessenberg varieties leads to interesting combinatorics.

[Many]: when x is generic, $H^*(\mathrm{Hess}(x,h))$ is related to the Stanley-Stembridge conjecture on chromatic

Stanley-Stembridge Conjecture

Let G be a simple graph with n vertices

The chromatic symmetric function of G is

$$X_G(t) = \sum_{\substack{\kappa:[n]\to\mathbb{N} \\ \text{proper}}} t_{\kappa(1)} \cdots t_{\kappa(n)}$$

Note: $X_G(1,\dots,1)$ = classical chromatic polynomial

Given a Hessenberg fct h, let $\Gamma_h = (V, E)$ where $V = [n]$ and $E = \{ ij \mid i<j \text{ and } j \le h(i) \}$.

Ex: $h = (3,3,4,4) \Rightarrow \Gamma_h =$ 

Conjecture: $X_{\Gamma_h}(x)$ is e-positive.

[Tymoczko]: Is every Schubert variety a Hessenberg variety?

Example: $w = 4231$

If $X_w \subseteq \mathrm{Hess}(x, h)$ and $x \neq 0$ we show that $h(i) = 4$ for all i.

It follows that $\mathrm{Hess}(x, h) = \mathcal{F}_4$.

Theorem [E.-Precup-Shareshian]: If there exists $x \in \mathfrak{sl}_n\mathbb{C}$ and a Hessenberg function h such that $X_w = \mathrm{Hess}(x, h)$ then w avoids 4231.

By the Marcus-Tardos Theorem (aka Stanley-Wilf conjecture) the number of $w \in S_n$ such that X_w is a Hessenberg variety is bounded above by an exponential function of n.

The converse of our Theorem does not hold

Example: For $n=4$, X_w is a Hessenberg variety iff w is one of the following permutations:

4231, 4123, 2341, 1423, 2314

More general Hessenberg varieties

[Goresky–Kottwitz–MacPherson]: Let $\Phi : SL_n(\mathbb{C}) \longrightarrow GL(V)$ be a representation, $H \subseteq V$ a B-invariant vector subspace, and $x \in V$. The associated Hessenberg variety is $Hess(x, H) = \{gB \mid \Phi(g^{-1})x \in H\}$.

In the original definition, Φ is the adjoint representation:
$V = \mathfrak{sl}_n \mathbb{C}$ and $\Phi(g^{-1})x = g^{-1}xg$.

Theorem [E–Precup–Shareshian]: Every Schubert variety (of any Lie type) is a Hessenberg variety of this form.

Details:

A single representation and vector works:

λ a strictly dominant weight, e.g. $\lambda = (n-1, n-2, \dots, 2, 1, 0)$

$\Phi : SL_n\mathbb{C} \longrightarrow V$ the irreducible representation with highest weight λ.

v_λ = highest weight vector of this representation.

Given $w \in S_n$ we set $H_w \subseteq V$ to be the Demazure module $H_w = \mathbb{C}\{\Phi(b\dot{w}^{-1})v_\lambda \mid b \in B\}$

THANK
YOU
MUCHAS
GRACIAS