Robinson-Schensted correspondence for natural unit interval orders

Dongkwan Kim

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(joint with Pavlo Pylyavskyy)
1. What does the title mean?
2. Motivation
3. $\mathcal{P}$-Knuth equivalence
4. Ladders and the main theorem
5. $\mathcal{P}$-Robinson-Schensted algorithm
Outline

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Robinson-Schensted correspondence

Robinson-Schensted-Knuth correspondence is a bijection

\[
\{\mathbb{N}\text{-valued matrices}\} \rightarrow \bigsqcup_{\lambda} \text{SSYT}_\lambda \times \text{SSYT}_\lambda
\]

that is usually described in terms of “bumping process”. If we restrict the domain to permutation matrices, then it becomes a bijection

\[
S_n \rightarrow \bigsqcup_{\lambda \vdash n} \text{SYT}_\lambda \times \text{SYT}_\lambda
\]

which is the usual Robinson-Schensted correspondence.
What does the title mean?

Robinson-Schensted correspondence

1

1 1 1 2 2 3 5
2 2 3 3
3 4 6
4

⇒

1 1 1 2 2 3 5
2 2 3 3
3 4 6
4

⇒

1 1 1 1 2 3 5
2 2 2 3
3 4 6
4

⇒

1 1 1 1 2 3 5
2 2 2 3
3 3 6
4

⇒

1 1 1 1 2 3 5
2 2 2 3
3 3 6
4

⇒

1 1 1 1 2 3 5
2 2 2 3
3 3 6
4
Suppose that a partial order $\mathcal{P}$ on $[1, n]$ is given. We assume that

$$a \succ_P b \Rightarrow a > b,$$

i.e. the usual order on $[1, n]$ is a linearization of $\mathcal{P}$. We write:

- $a \rightarrow_P b$ if $a$ is bigger than $b$ with respect to $\mathcal{P}$,
- $a \dashrightarrow_P b$ if $a$ and $b$ are not comparable in $\mathcal{P}$, and
- $a \leftarrow_P b$ if $a > b$ and $a \dashrightarrow_P b$.

**Definition**

We say that $\mathcal{P}$ is a natural unit interval order if for any $a, b, c \in [1, n]$ such that $a \rightarrow_P c$, $a \dashrightarrow_P b$, and $b \dashrightarrow_P c$, we have $a > b > c$. 
Natural unit interval orders

If $\mathcal{P}$ is a natural unit interval order on $[1, n]$, then there exist $y_1, y_2, \ldots, y_n \in \mathbb{R}$ such that $y_1 < y_2 < \cdots < y_n$ and $a \rightarrow_{\mathcal{P}} b \iff y_a > y_b + 1$. 

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {1};
  \node (b) at (1,0) {2};
  \node (c) at (2,0) {3};
  \node (d) at (3,0) {4};
  \node (e) at (4,0) {5};
  \node (f) at (5,0) {6};

  \draw (a) -- (b);
  \draw (b) -- (c);
  \draw (c) -- (d);
  \draw (d) -- (e);
  \draw (e) -- (f);

  \node (a2) at (-1,-2) {3};
  \node (b2) at (0,-2) {4};
  \node (c2) at (1,-2) {5};

  \draw (a2) -- (a);
  \draw (a2) -- (b);
  \draw (b2) -- (b);
  \draw (b2) -- (c);
  \draw (c2) -- (c);
  \draw (c2) -- (d);

  \node (a3) at (-2,-4) {1};
  \node (b3) at (-1,-4) {2};

  \draw (a3) -- (a2);
  \draw (a3) -- (b2);
  \draw (b3) -- (b2);
  \draw (b3) -- (c2);

  \node (a4) at (-3,-6) {6};
  \node (b4) at (-2,-6) {3};
  \node (c4) at (-1,-6) {4};
  \node (d4) at (0,-6) {5};

  \draw (a4) -- (a3);
  \draw (a4) -- (b3);
  \draw (b4) -- (b3);
  \draw (b4) -- (c3);
  \draw (b4) -- (d3);
  \draw (c4) -- (b3);
  \draw (c4) -- (c3);
  \draw (c4) -- (d3);
  \draw (d4) -- (c3);

  \node (a5) at (-4,-8) {6};

  \draw (a5) -- (a4);
  \draw (a5) -- (b4);
  \draw (a5) -- (c4);
  \draw (a5) -- (d4);
\end{tikzpicture}
\end{center}
If $\mathcal{P}$ is a natural unit interval order on $[1, n]$, then there exists a partition $\lambda \subset (n - 1, n - 2, \ldots, 2, 1)$ such that $a \leftarrow_{\mathcal{P}} b$ if and only if $a \leq \lambda_{n+1-b}$. In this case we write $\mathcal{P} = \mathcal{P}_{\lambda,n}$.
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4. Ladders and the main theorem
5. $P$-Robinson-Schensted algorithm
Recall the fundamental quasisymmetric function $F_S$ for $S \in [1, n - 1]$ defined by

$$F_S = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For $w \in S_n$, we define its $\mathcal{P}$-descent to be

$$\text{des}_{\mathcal{P}}(w) = \{ i \in [1, n - 1] \mid w_i \to_{\mathcal{P}} w_{i+1} \}.$$

We consider the following quasisymmetric function

$$\gamma_{\mathcal{P}, S_n} = \sum_{w \in S_n} F_{\text{des}_{\mathcal{P}}(w)}.$$

For example, if $\mathcal{P}$ is the usual order then $\gamma_{\mathcal{P}, S_n} = h_{1^n}$. 
Motivation

Stanley-Stembridge conjecture

Lemma
\[ \gamma_{P,S_n} \text{ is a symmetric function and } p\text{-positive.} \]

Theorem (Haiman, Gasharov)
\[ \gamma_{P,S_n} \text{ is Schur-positive.} \]

Conjecture (Stanley, Stembridge)
\[ \gamma_{P,S_n} \text{ is } h\text{-positive.} \]

This conjecture is still open, but partial progress was made:
Stanley-Stembridge, Gebhard-Sagan, Dahlberg-van Willigenburg, Harada-Precup, Cho-Huh, Cho-Hong, etc.
Graded Stanley-Stembridge conjecture

For $w \in S_n$, set its “fake $\mathcal{P}$-inversion” to be

$$f\text{-inv}_{\mathcal{P}}(w) = \{(w_i, w_j) \in [1, n]^2 \mid i < j, w_i \rightarrow_\mathcal{P} w_j\}.$$

We consider the following weighted version

$$\tilde{\gamma}_{\mathcal{P}, S_n} = \sum_{w \in S_n} t^{|f\text{-inv}_{\mathcal{P}}(w)|} F_{\text{des}_\mathcal{P}}(w).$$

**Theorem (Shareshian-Wachs)**

$\tilde{\gamma}_{\mathcal{P}, S_n}$ is a symmetric function and both $p$-and Schur-positive.

**Conjecture (Shareshian-Wachs)**

$\tilde{\gamma}_{\mathcal{P}, S_n}$ is $h$-positive.
Hessenberg varieties

For $\lambda \subset (n-1, \cdots, 2, 1)$ and regular semisimple $s \in GL_n$, we let

$$\mathcal{H}ess_{\lambda, s} = \{ F_\bullet = [F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n] \mid s \cdot F_i \subset F_{n-\lambda_i'} \}$$

called a Hessenberg variety.

Tymoczko defined a so-called “dot-action” on $\bigoplus_{i \in \mathbb{Z}} H^2_i(\mathcal{H}ess_{\lambda, s}) t^i$ where $T$ is the maximal torus containing $s$, which makes it into a graded $S_n$-module.

**Theorem (Brosnan-Chow, Guay-Paquet)**

*The Frobenius character of $\bigoplus_{i \in \mathbb{Z}} H^2_i(\mathcal{H}ess_{\lambda, s}) t^i$ equals $\tilde{\gamma}_{P_{\lambda, n}, S_n}$.*

This is originally conjectured by Shareshian-Wachs.
Our goal is to understand combinatorics behind this picture.

- Introduce $\mathcal{P}$-Knuth equivalence
- Define $\mathcal{P}$-Robinson-Schensted algorithm
- Use these combinatorial tools to analyze $\tilde{\gamma}_{\mathcal{P},n_S}$ in detail

$\Rightarrow$ refinement of the results of Shareshian-Wachs
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Knuth moves and equivalence

Regard $S_n$ as a set of words with alphabets in $[1, n]$.

**Definition**

We say that $w, w' \in S_n$ are connected by a Knuth move if for $a, b, c \in [1, n]$ such that $a < b < c$ we have either

\[
w = \cdots cab \cdots \leftrightarrow w' = \cdots acb \cdots \text{ or }\]
\[
w = \cdots bca \cdots \leftrightarrow w' = \cdots bac \cdots .
\]

The Knuth equivalence is defined to be the closure of these moves.

For each equivalence class $\Gamma$, consider the following generating function $\gamma_\Gamma = \sum_{w \in \Gamma} F_{\text{des}}(w)$. Then,

- $\gamma_{P,S_n} = \sum_{\Gamma} \gamma_\Gamma$ when $P$ is the usual order, and
- $\gamma_\Gamma = s_\lambda$ for some $\lambda \vdash n$. 

Example: $S_4$
For \( a, b, c \in [1, n] \) such that \( a < b < c \) and \( a \leftrightarrow \mathcal{P} c \), there are four possibilities of \( \mathcal{P}|\{a,b,c\} \):

1. \( \mathcal{P}|_{\{a,b,c\}} \cong \mathcal{P}(1),3 \)
2. \( \mathcal{P}|_{\{a,b,c\}} \cong \mathcal{P}(1,1),3 \)
3. \( \mathcal{P}|_{\{a,b,c\}} \cong \mathcal{P}(2),3 \)
4. \( \mathcal{P}|_{\{a,b,c\}} \cong \mathcal{P}(2,1),3 \)
\(\mathcal{P}\)-Knuth moves and equivalence

In each case, we define the \(\mathcal{P}\)-Knuth move as follows:

1. \(a \leftarrow \mathcal{P} b\) and \(b \leftarrow \mathcal{P} c\):

\[
\cdots bca \cdots \xrightarrow{\mathcal{P}} \cdots cab \cdots .
\]

2. \(a \leftarrow \mathcal{P} b\) and \(b \leftarrow \mathcal{P} c\):

\[
\cdots bca \cdots \xrightarrow{\mathcal{P}} \cdots bac \cdots \quad \text{and} \quad \cdots cba \cdots \xrightarrow{\mathcal{P}} \cdots cab \cdots .
\]

3. \(a \leftarrow \mathcal{P} b\) and \(b \leftarrow \mathcal{P} c\):

\[
\cdots bca \cdots \xrightarrow{\mathcal{P}} \cdots cba \cdots \quad \text{and} \quad \cdots acb \cdots \xrightarrow{\mathcal{P}} \cdots cab \cdots .
\]

4. \(a \leftarrow \mathcal{P} b\) and \(c \leftarrow \mathcal{P} c\) (“usual case”):

\[
\cdots bca \cdots \xrightarrow{\mathcal{P}} \cdots bac \cdots \quad \text{and} \quad \cdots acb \cdots \xrightarrow{\mathcal{P}} \cdots cab \cdots .
\]

The \(\mathcal{P}\)-Knuth equivalence is defined to be the closure of these moves.
Example: $S_3$

(a) $\mathcal{P}_{0,3}$

(b) $\mathcal{P}_{(1),3}$

(c) $\mathcal{P}_{(2),3}$

(d) $\mathcal{P}_{(1,1),3}$

(e) $\mathcal{P}_{(2,1),3}$
For each equivalence class $\Gamma$, consider

$$\tilde{\gamma}_P,\Gamma = \sum_{w \in \Gamma} t^{\text{f-inv}_P(w)} F_{\text{des}_P}(w)$$

so that $\tilde{\gamma}_P, S_n = \sum_{\Gamma} \tilde{\gamma}_P,\Gamma$.

**Question**

Is $\tilde{\gamma}_P,\Gamma$ a symmetric function?

**Question**

Is $\tilde{\gamma}_P,\Gamma$ Schur-positive?

**Question**

Is $\tilde{\gamma}_P,\Gamma$ a single Schur function?
Example: $\mathcal{P} = \mathcal{P}_{(2,1),4}$

$$
\begin{array}{cccc}
\text{s}_4 & \text{ts}_4 & \text{ts}_4 & \text{ts}_{31} \\
1234 & 1243 & 1324 & 1342 \\
1432 & 2134 & 2143 & 2341 \\
2431 & 2413 & 3142 & 3142 \\
3241 & 3421 & 4231 & 4312 \\
4321 & & & \\
\end{array}
$$
Main conjecture

Lemma

If $w$ and $w'$ are $\mathcal{P}$-Knuth equivalent, then $|f\text{-inv}_\mathcal{P}(w)| = |f\text{-inv}_\mathcal{P}(w')|$. 

As a result, $\tilde{\gamma}_\mathcal{P,\Gamma} = t^{f\text{-inv}_\mathcal{P}(w)} \cdot (\tilde{\gamma}_\mathcal{P,\Gamma})_{t=1}$ for any $w \in \Gamma$.

Conjecture

If $\mathcal{P}$ is a natural unit interval order on $[1, n]$, then $(\tilde{\gamma}_\mathcal{P,\Gamma})_{t=1}$ for each $\mathcal{P}$-Knuth equivalence class $\Gamma$ is a Schur positive symmetric function.

This is a refinement of the theorem of Shareshian-Wachs.
Example: $\mathcal{P} = \mathcal{P}_{2,2,1,5}, \tilde{\gamma}_{\mathcal{P}, \Gamma} = t^2(s_{32} + s_{41})$
Example: $\mathcal{P} = \mathcal{P}_{(2,1,1),5}, \tilde{\gamma}_{\mathcal{P},\Gamma} = t^3(s_{32} + 2s_{41})$
Example: $\mathcal{P} = \mathcal{P}_{(3,2,1),5}, \tilde{\gamma}_\mathcal{P}, \Gamma = t^3(2s_{32} + s_{41})$
Example: $\mathcal{P} = \mathcal{P}_{(4,3,2,1),6}, \tilde{\gamma}_\mathcal{P}, \Gamma = t^2(s_{42} + s_{51})$
Example: $\mathcal{P} = \mathcal{P}(5,4,2,1), 6$, $\tilde{\gamma}_{\mathcal{P}, \Gamma} = t^2 (s_{2211} + s_{3111})$
Example: $\mathcal{P} = \mathcal{P}_{(3,3,2,1),6}, \tilde{\gamma}_{\mathcal{P},\Gamma} = t^4(s_{33} + 2s_{42} + s_{51})$
Example: $P = P_{(3,1,1),5}, \tilde{\gamma}_P, \Gamma = t^3(s_{311} + s_{32} + 2s_{41})$
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**Ladder order**

**Definition**
We say that $\mathcal{P}$ is a ladder order if it is isomorphic to $\mathcal{P}_{(n-2, n-3, \ldots, 1), n}$ for some $n$.

For example, any $[a, b] \subseteq [1, n]$ is a ladder in $\mathcal{P}$ if $\mathcal{P}|_X$ is a ladder order.

For example, any $[a, b] \subseteq [1, n]$ is a ladder in $\mathcal{P}_{(n-2, n-3, \ldots, 1), n}$.
Climbing a ladder

Definition

We say that someone is climbing a ladder in $\mathcal{P}$ or $\mathcal{P}$ is ladder-climbing if there exist $x, y_1, \ldots, y_k \in [1, n]$ such that

- $x \not\in \{y_1, y_2, \ldots, y_k\}$,
- $\{y_1, y_2, \ldots, y_k\}$ is a ladder in $\mathcal{P}$, and
- $y_1 \leftarrow_{\mathcal{P}} x \leftarrow_{\mathcal{P}} y_k$.

In such a case, we also say that $x$ is climbing a ladder in $\mathcal{P}$ or $x$ is climbing (the ladder) $\{y_1, \ldots, y_k\}$ in $\mathcal{P}$. Otherwise we say that no one is climbing a ladder in $\mathcal{P}$ or $\mathcal{P}$ is not ladder-climbing.
Ladders and the main theorem

Example: $\mathcal{P}_{(3,1,1),5}$ and $\mathcal{P}_{(4,2,1,1),6}$

Theorem

$\mathcal{P}$ is not ladder-climbing if and only if $\mathcal{P}$ avoids $\mathcal{P}_{(3,1,1),5}$ and $\mathcal{P}_{(4,2,1,1),6}$.
Main theorem

Theorem (Main theorem)

Suppose that \( P \) is a natural unit interval order that is not ladder-climbing. Then for any \( P \)-Knuth equivalence class \( \Gamma \), there exist \( \lambda_1, \ldots, \lambda_k \) such that

\[
\tilde{\gamma}_{P, \Gamma} = t |f-inv_P(\Gamma)| (s_{\lambda_1} + \cdots + s_{\lambda_k}) \text{ where } |f-inv_P(\Gamma)| \text{ is } |f-inv_P(w)| \text{ for any } w \in \Gamma, \text{ and }
\]

\[2\] the lengths of \( \lambda_1, \ldots, \lambda_k \) are all equal.

It is likely that the first part of the main theorem is valid for any natural unit interval order \( P \) albeit the second part is not true.
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A tableau $T$ is said to be a $\mathcal{P}$-tableau if

- the entries of $T$ are exactly $1, 2, \ldots, n$ without repetition,
- the entries are increasing along columns with respect to $\mathcal{P}$ and
- the entries are nondecreasing along rows with respect to $\mathcal{P}$. 

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1 4 3 2
5 8 7 6
9
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Properties of $\mathcal{P}$-Robinson-Schensted algorithm

Theorem

Suppose that $\mathcal{P}$ is not ladder-climbing. Then there exists a bijection $\mathcal{P}$-RS : $\mathcal{S}_n \rightarrow \bigsqcup_{\lambda \vdash n} \mathcal{P}$-Tab$\lambda$ $\times$ SYT$\lambda$ such that:

(A) $\text{des}_\mathcal{P}(w) = \{n - x \mid x \in \text{des}(Q)\} = \text{des}(\text{evac}(Q))$.

(B) $w \sim_\mathcal{P} \text{read}(P)$ (reading word of $P$).

(C) If $w' \in \mathcal{S}_n$ satisfies $\mathcal{P}$-RS($w'$) = ($P'$, $Q'$) and $w \sim_\mathcal{P} w'$, then the first columns of $P$ and $P'$ are of the same length.

Unlike the usual Robinson-Schensted algorithm, in general we do not have the same $\mathcal{P}$-tableaux for $w, w'$ such that $w \sim_\mathcal{P} w'$. 
Proof of the main theorem

Theorem (Main theorem)

Suppose that $\mathcal{P}$ is not ladder-climbing. Then for any $\mathcal{P}$-Knuth equivalence class $\Gamma$, there exist $\lambda_1, \ldots, \lambda_k$ such that

1. $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^{\left| f\text{-inv}_\mathcal{P}(\Gamma) \right|} (s_{\lambda_1} + \cdots + s_{\lambda_k})$ where $\left| f\text{-inv}_\mathcal{P}(\Gamma) \right|$ is $\left| f\text{-inv}_\mathcal{P}(w) \right|$ for any $w \in \Gamma$, and
2. the lengths of $\lambda_1, \ldots, \lambda_k$ are all equal.

Sketch of the proof.

Using (B) one can show that $\mathcal{P}$-RS restricts to a bijection $\Gamma \simeq \bigsqcup_{i=1}^{k} \{ P_i \} \times \text{SYT}_{\text{sh}(P_i)}$ for certain $\mathcal{P}$-tableaux $P_1, \ldots, P_k$. Now by (A), it follows that $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^{\left| f\text{-inv}_\mathcal{P}(\Gamma) \right|} \sum_{i=1}^{n} s_{\text{sh}(P_i)}$, which proves (1). The first columns of $P_1, \ldots, P_k$ are of the same length by (C), and thus (2) follows.
Thank you!