Robinson-Schensted correspondence for natural unit interval orders

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Outline

1. What does the title mean?
2. Motivation
3. $\mathcal{P}$-Knuth equivalence
4. Ladders and the main theorem
5. $\mathcal{P}$-Robinson-Schensted algorithm

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Robinson-Schensted correspondence

Robinson-Schensted-Knuth correspondence is a bijection

$$\{\mathbb{N}\text{-valued matrices}\} \to \bigsqcup_{\lambda} \mathrm{SSYT}_\lambda \times \mathrm{SSYT}_\lambda$$

that is usually described in terms of "bumping process".
If we restrict the domain to permutation matrices, then it becomes a bijection

$$\mathcal{S}_n \to \bigsqcup_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$$

which is the usual Robinson-Schensted correspondence.

Robinson-Schensted correspondence

1

1	1	1	2	2	3	5
2	2	3	3			
3	4	6				
4						

$\Rightarrow$

2

1	1	1	1	2	3	5
2	2	3	3			
3	4	6				
4						

$\Rightarrow$

3

1	1	1	1	2	3	5
2	2	2	3			
3	4	6				
4						

$\Rightarrow$

4

1	1	1	1	2	3	5
2	2	2	3			
3	3	6				
4						

$\Rightarrow$

1	1	1	1	2	3	5
2	2	2	3			
3	3	6				
4	4					

Natural unit interval orders

Suppose that a partial order $\mathcal{P}$ on $[1, n]$ is given. We assume that

$$a \succ_{\mathcal{P}} b \Rightarrow a > b,$$

i.e. the usual order on $[1, n]$ is a linearization of $\mathcal{P}$. We write:

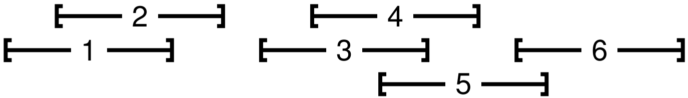
- $a \rightarrow_{\mathcal{P}} b$ if a is bigger than b with respect to $\mathcal{P}$,
- $a \dashrightarrow\!\!\!\!-_{\mathcal{P}} b$ if a and b are not comparable in $\mathcal{P}$, and
- $a \dashrightarrow_{\mathcal{P}} b$ if $a > b$ and $a \dashrightarrow\!\!\!\!-_{\mathcal{P}} b$.

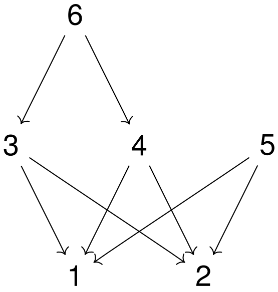
Definition

We say that $\mathcal{P}$ is a natural unit interval order if for any $a, b, c \in [1, n]$ such that $a \rightarrow_{\mathcal{P}} c$, $a \dashrightarrow\!\!\!\!-_{\mathcal{P}} b$, and $b \dashrightarrow\!\!\!\!-_{\mathcal{P}} c$, we have $a > b > c$.

Natural unit interval orders

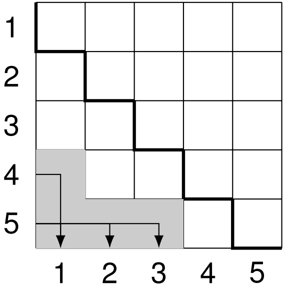
If $\mathcal{P}$ is a natural unit interval order on $[1, n]$, then there exist $y_1, y_2, \ldots, y_n \in \mathbb{R}$ such that $y_1 < y_2 < \cdots < y_n$ and $a \to_{\mathcal{P}} b \Leftrightarrow y_a > y_b + 1$.

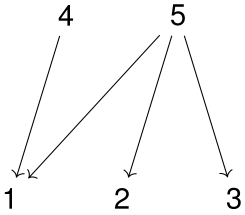




Natural unit interval orders

If $\mathcal{P}$ is a natural unit interval order on $[1, n]$, then there exists a partition $\lambda \subset (n-1, n-2, \ldots, 2, 1)$ such that $a \leftarrow_{\mathcal{P}} b$ if and only if $a \leq \lambda_{n+1-b}$. In this case we write $\mathcal{P} = \mathcal{P}_{\lambda,n}$.





Outline

Stanley-Stembridge conjecture

Recall the fundamental quasisymmetric function F_S for $S \in [1, n-1]$ defined by

$$F_S = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ i_a < i_{a+1} \Leftrightarrow a \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For $w \in \mathcal{S}_n$, we define its $\mathcal{P}$-descent to be

$$\mathrm{des}_{\mathcal{P}}(w) = \{i \in [1, n-1] \mid w_i \to_{\mathcal{P}} w_{i+1}\}.$$

We consider the following quasisymmetric function

$$\gamma_{\mathcal{P},\mathcal{S}_n} = \sum_{w \in \mathcal{S}_n} F_{\mathrm{des}_{\mathcal{P}}(w)}.$$

For example, if $\mathcal{P}$ is the usual order then $\gamma_{\mathcal{P},\mathcal{S}_n} = h_{1^n}$.

Stanley-Stembridge conjecture

Lemma

$\gamma_{\mathcal{P},\mathcal{S}_n}$ *is a symmetric function and p-positive.*

Theorem (Haiman, Gasharov)

$\gamma_{\mathcal{P},\mathcal{S}_n}$ *is Schur-positive.*

Conjecture (Stanley, Stembridge)

$\gamma_{\mathcal{P},\mathcal{S}_n}$ *is h-positive.*

This conjecture is still open, but partial progress was made: Stanley-Stembridge, Gebhard-Sagan, Dahlberg-van Willigenburg, Harada-Precup, Cho-Huh, Cho-Hong, etc.

Graded Stanley-Stembridge conjecture

For $w \in \mathcal{S}_n$, set its "fake $\mathcal{P}$-inversion" to be

$$\text{f-inv}_{\mathcal{P}}(w) = \{(w_i, w_j) \in [1, n]^2 \mid i < j, w_i \dashrightarrow_{\mathcal{P}} w_j\}.$$

We consider the following weighted version

$$\tilde{\gamma}_{\mathcal{P},\mathcal{S}_n} = \sum_{w \in \mathcal{S}_n} t^{|\text{f-inv}_{\mathcal{P}}(w)|} F_{\text{des}_{\mathcal{P}}(w)}.$$

Theorem (Shareshian-Wachs)

$\tilde{\gamma}_{\mathcal{P},\mathcal{S}_n}$ is a symmetric function and both p-and Schur-positive.

Conjecture (Shareshian-Wachs)

$\tilde{\gamma}_{\mathcal{P},\mathcal{S}_n}$ is h-positive.

Hessenberg varieties

For $\lambda \subset (n-1, \cdots, 2, 1)$ and regular semisimple $s \in GL_n$, we let

$$\mathcal{H}\text{ess}_{\lambda,s} = \{F_\bullet = [F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n] \mid s \cdot F_i \subset F_{n-\lambda'_i}\}$$

called a Hessenberg variety.
Tymoczko defined a so-called "dot-action" on $\bigoplus_{i\in\mathbb{Z}} H_T^{2i}(\mathcal{H}\text{ess}_{\lambda,s})t^i$ where T is the maximal torus containing s, which makes it into a graded $\mathcal{S}_n$-module.

Theorem (Brosnan-Chow, Guay-Paquet)

The Frobenius character of $\bigoplus_{i\in\mathbb{Z}} H^{2i}(\mathcal{H}\text{ess}_{\lambda,s})t^i$ *equals* $\tilde{\gamma}_{\mathcal{P}_{\lambda,n},\mathcal{S}_n}$.

This is originally conjectured by Shareshian-Wachs.

Goal

Our goal is to understand combinatorics behind this picture.

- Introduce $\mathcal{P}$-Knuth equivalence
- Define $\mathcal{P}$-Robinson-Schensted algorithm
- Use these combinatorial tools to analyze $\tilde{\gamma}_{\mathcal{P}_{\lambda,n},\mathcal{S}_n}$ in detail

$\rightsquigarrow$ refinement of the results of Shareshian-Wachs

Outline

Knuth moves and equivalence

Regard $\mathcal{S}_n$ as a set of words with alphabets in $[1, n]$.

Definition

We say that $w, w' \in \mathcal{S}_n$ are connected by a Knuth move if for $a, b, c \in [1, n]$ such that $a < b < c$ we have either

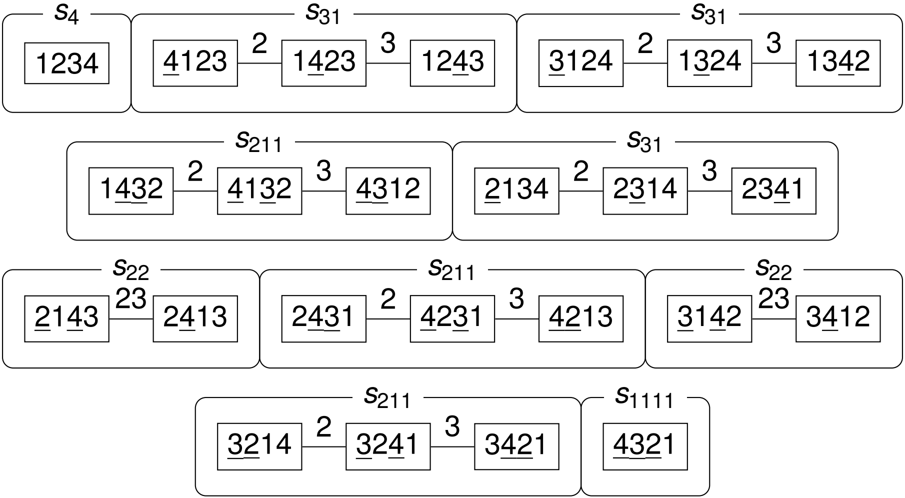
$$w = \cdots cab \cdots \leftrightsquigarrow w' = \cdots acb \cdots \text{ or}$$
$$w = \cdots bca \cdots \leftrightsquigarrow w' = \cdots bac \cdots .$$

The Knuth equivalence is defined to be the closure of these moves.

For each equivalence class Γ, consider the following generating function $\gamma_\Gamma = \sum_{w \in \Gamma} F_{\mathrm{des}(w)}$. Then,

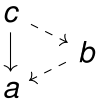
- $\gamma_{\mathcal{P},\mathcal{S}_n} = \sum_\Gamma \gamma_\Gamma$ when $\mathcal{P}$ is the usual order, and
- $\gamma_\Gamma = s_\lambda$ for some $\lambda \vdash n$.

Example: $\mathcal{S}_4$

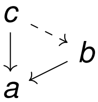


$\mathcal{P}$-Knuth moves and equivalence

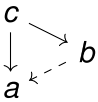
For $a, b, c \in [1, n]$ such that $a < b < c$ and $a \leftarrow_{\mathcal{P}} c$, there are four possibilities of $\mathcal{P}|_{\{a,b,c\}}$:



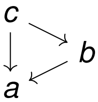
(1) $\mathcal{P}|_{\{a,b,c\}} \simeq \mathcal{P}_{(1),3}$



(2) $\mathcal{P}|_{\{a,b,c\}} \simeq \mathcal{P}_{(1,1),3}$



(3) $\mathcal{P}|_{\{a,b,c\}} \simeq \mathcal{P}_{(2),3}$



(4) $\mathcal{P}|_{\{a,b,c\}} \simeq \mathcal{P}_{(2,1),3}$

$\mathcal{P}$-Knuth moves and equivalence

In each case, we define the $\mathcal{P}$-Knuth move as follows:

1. $a \dashleftarrow_{\mathcal{P}} b$ and $b \dashleftarrow_{\mathcal{P}} c$:

$$\cdots b\underline{c}a \cdots \overset{\mathcal{P}}{\leftrightsquigarrow} \cdots \underline{c}ab \cdots .$$

2. $a \leftarrow_{\mathcal{P}} b$ and $b \dashleftarrow_{\mathcal{P}} c$:

$$\cdots b\underline{c}a \cdots \overset{\mathcal{P}}{\leftrightsquigarrow} \cdots \underline{b}ac \cdots \quad \text{and} \quad \cdots c\underline{b}a \cdots \overset{\mathcal{P}}{\leftrightsquigarrow} \cdots \underline{c}ab \cdots .$$

3. $a \dashleftarrow_{\mathcal{P}} b$ and $b \leftarrow_{\mathcal{P}} c$:

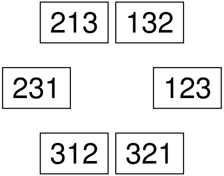
$$\cdots b\underline{c}a \cdots \overset{\mathcal{P}}{\leftrightsquigarrow} \cdots \underline{c}ba \cdots \quad \text{and} \quad \cdots a\underline{c}b \cdots \overset{\mathcal{P}}{\leftrightsquigarrow} \cdots \underline{c}ab \cdots .$$

4. $a \leftarrow_{\mathcal{P}} b \leftarrow_{\mathcal{P}} c$ (“usual case”):

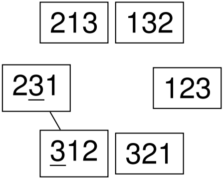
$$\cdots b\underline{c}a \cdots \overset{\mathcal{P}}{\leftrightsquigarrow} \cdots \underline{b}ac \cdots \quad \text{and} \quad \cdots a\underline{c}b \cdots \overset{\mathcal{P}}{\leftrightsquigarrow} \cdots \underline{c}ab \cdots .$$

The $\mathcal{P}$-Knuth equivalence is defined to be the closure of these moves.

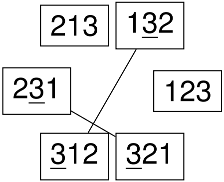
Example: $\mathcal{S}_3$



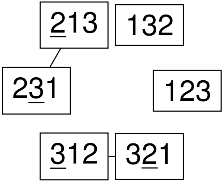
(a) $\mathcal{P}_{\emptyset,3}$



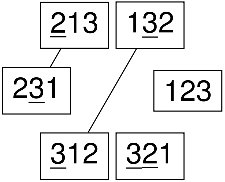
(b) $\mathcal{P}_{(1),3}$



(c) $\mathcal{P}_{(2),3}$



(d) $\mathcal{P}_{(1,1),3}$



(e) $\mathcal{P}_{(2,1),3}$

$\mathcal{P}$-Knuth moves and equivalence

For each equivalence class Γ, consider

$$\tilde{\gamma}_{\mathcal{P},\Gamma} = \sum_{w \in \Gamma} t^{|\text{f-inv}_{\mathcal{P}}(w)|} F_{\text{des}_{\mathcal{P}}(w)}$$

so that $\tilde{\gamma}_{\mathcal{P},\mathcal{S}_n} = \sum_{\Gamma} \tilde{\gamma}_{\mathcal{P},\Gamma}$.

Question

Is $\tilde{\gamma}_{\mathcal{P},\Gamma}$ a symmetric function?

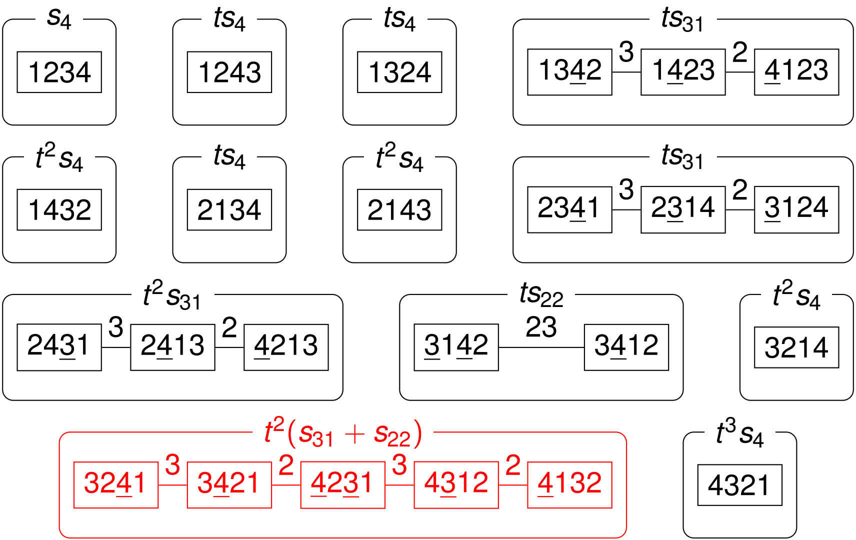
Question

Is $\tilde{\gamma}_{\mathcal{P},\Gamma}$ Schur-positive?

Question

Is $\tilde{\gamma}_{\mathcal{P},\Gamma}$ a single Schur function?

Example: $\mathcal{P} = \mathcal{P}_{(2,1),4}$



Main conjecture

Lemma

If w and w' are $\mathcal{P}$-Knuth equivalent, then $|\,\text{f-inv}_{\mathcal{P}}(w)| = |\,\text{f-inv}_{\mathcal{P}}(w')|$.

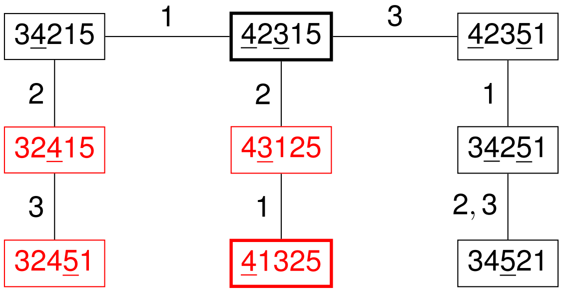
As a result, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^{|\,\text{f-inv}_{\mathcal{P}}(w)|} \cdot (\tilde{\gamma}_{\mathcal{P},\Gamma})_{t=1}$ for any $w \in \Gamma$.

Conjecture

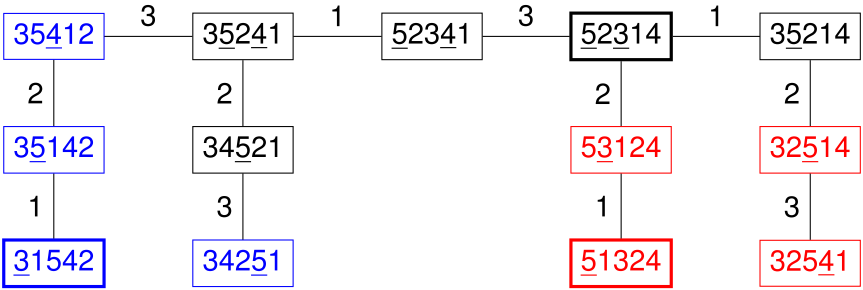
If $\mathcal{P}$ is a natural unit interval order on $[1, n]$, then $(\tilde{\gamma}_{\mathcal{P},\Gamma})_{t=1}$ for each $\mathcal{P}$-Knuth equivalence class Γ is a Schur positive symmetric function.

This is a refinement of the theorem of Shareshian-Wachs.

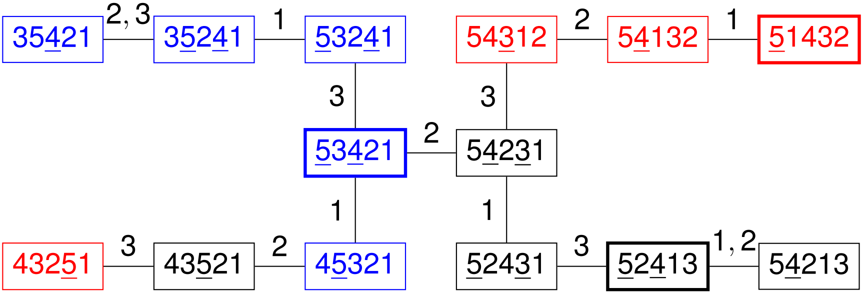
Example: $\mathcal{P} = \mathcal{P}_{(2,2,1),5}, \tilde{\gamma}_{\mathcal{P},\Gamma} = t^2(s_{32} + s_{41})$



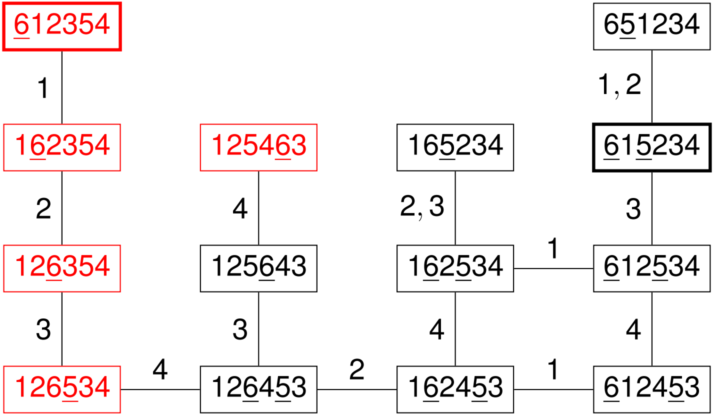
Example: $\mathcal{P} = \mathcal{P}_{(2,1,1),5}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^3(s_{32} + 2s_{41})$



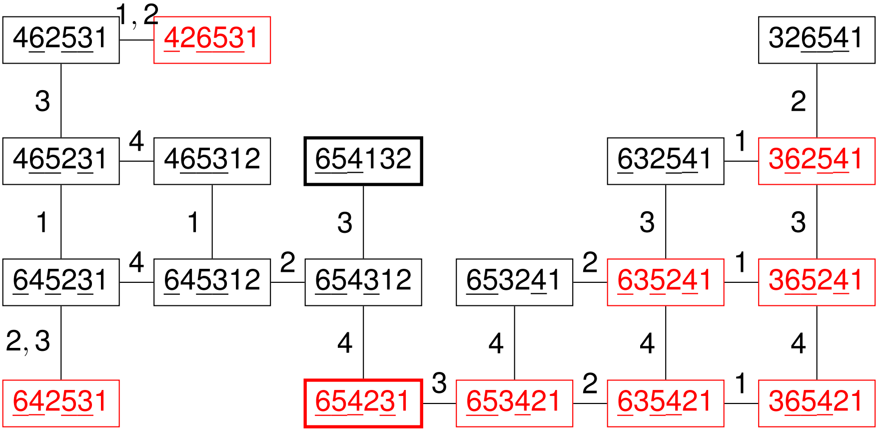
Example: $\mathcal{P} = \mathcal{P}_{(3,2,1),5}, \tilde{\gamma}_{\mathcal{P},\Gamma} = t^3(2s_{32} + s_{41})$



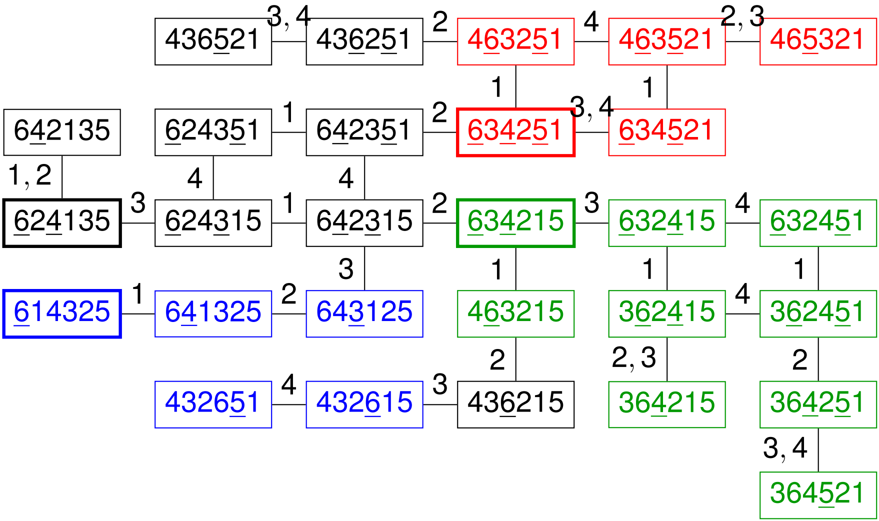
Example: $\mathcal{P} = \mathcal{P}_{(4,3,2,1),6}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^2(s_{42} + s_{51})$



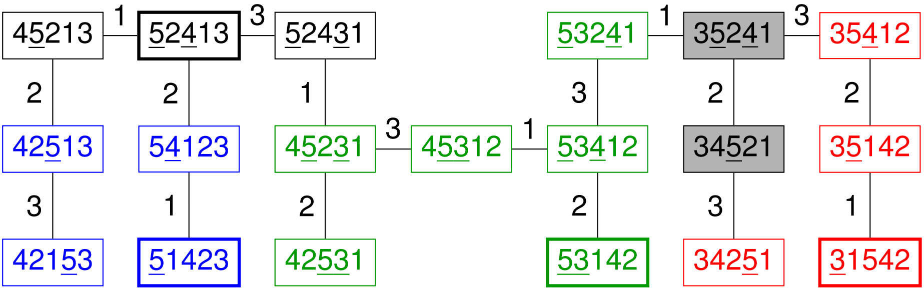
Example: $\mathcal{P} = \mathcal{P}_{(5,4,2,1),6}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^2(s_{2211} + s_{3111})$



Example: $\mathcal{P} = \mathcal{P}_{(3,3,2,1),6}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^4(s_{33} + 2s_{42} + s_{51})$



Example: $\mathcal{P} = \mathcal{P}_{(3,1,1),5}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^3(s_{311} + s_{32} + 2s_{41})$

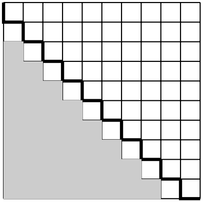


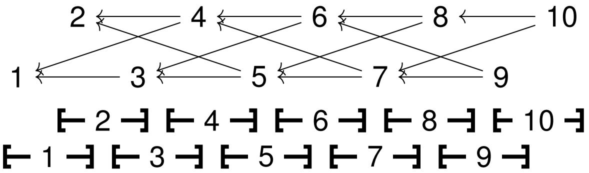
Outline

Ladder order

Definition

We say that $\mathcal{P}$ is a ladder order if it is isomorphic to $\mathcal{P}_{(n-2,n-3,\ldots,1),n}$ for some n.





Definition

We say that $X \subset [1, n]$ is a ladder in $\mathcal{P}$ if $\mathcal{P}|_X$ is a ladder order.

For example, any $[a, b] \subset [1, n]$ is a ladder in $\mathcal{P}_{(n-2,n-3,\ldots,1),n}$.

Climbing a ladder

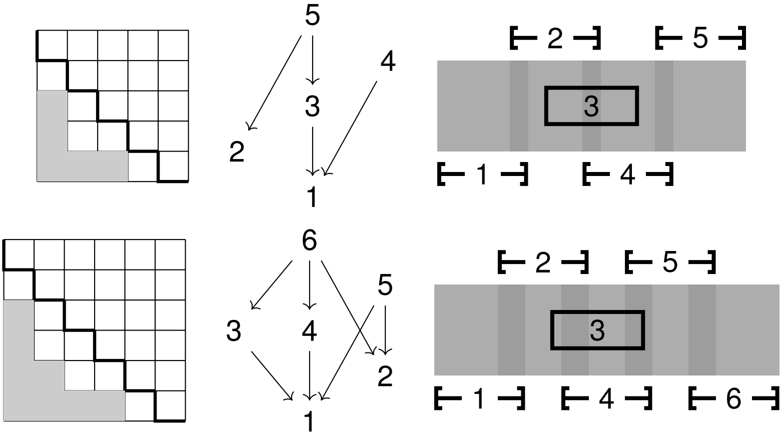
Definition

We say that someone is climbing a ladder in $\mathcal{P}$ or $\mathcal{P}$ is ladder-climbing if there exist $x, y_1, \ldots, y_k \in [1, n]$ such that

- $x \notin \{y_1, y_2, \ldots, y_k\}$,
- $\{y_1, y_2, \ldots, y_k\}$ is a ladder in $\mathcal{P}$, and
- $y_1 \leftarrow_{\mathcal{P}} x \leftarrow_{\mathcal{P}} y_k$.

In such a case, we also say that x is climbing a ladder in $\mathcal{P}$ or x is climbing (the ladder) $\{y_1, \ldots, y_k\}$ in $\mathcal{P}$. Otherwise we say that no one is climbing a ladder in $\mathcal{P}$ or $\mathcal{P}$ is not ladder-climbing.

Example: $\mathcal{P}_{(3,1,1),5}$ and $\mathcal{P}_{(4,2,1,1),6}$



Theorem

$\mathcal{P}$ is not ladder-climbing if and only if $\mathcal{P}$ avoids $\mathcal{P}_{(3,1,1),5}$ and $\mathcal{P}_{(4,2,1,1),6}$.

Main theorem

Theorem (Main theorem)

Suppose that $\mathcal{P}$ is a natural unit interval order that is not ladder-climbing. Then for any $\mathcal{P}$-Knuth equivalence class Γ, there exist $\lambda_1, \ldots, \lambda_k$ such that

1. $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^{|\,\text{f-inv}_{\mathcal{P}}(\Gamma)|}(s_{\lambda_1} + \cdots + s_{\lambda_k})$ *where* $|\,\text{f-inv}_{\mathcal{P}}(\Gamma)|$ *is* $|\,\text{f-inv}_{\mathcal{P}}(w)|$ *for any* $w \in \Gamma$*, and*
2. *the lengths of* $\lambda_1, \ldots, \lambda_k$ *are all equal.*

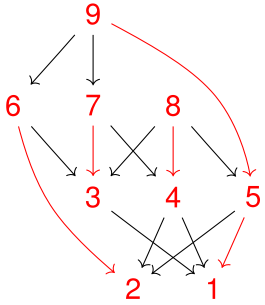
It is likely that the first part of the main theorem is valid for any natural unit interval order $\mathcal{P}$ albeit the second part is not true.

Outline

$\mathcal{P}$-tableaux

A tableau T is said to be a $\mathcal{P}$-tableau if

- the entries of T are exactly $1, 2, \ldots, n$ without repetition,
- the entries are increasing along columns with respect to $\mathcal{P}$ and
- the entries are nondecreasing along rows with respect to $\mathcal{P}$.



1	4	3	2
5	8	7	6
9			

Properties of $\mathcal{P}$-Robinson-Schensted algorithm

Theorem

Suppose that $\mathcal{P}$ is not ladder-climbing. Then there exists a bijection $\mathcal{P}\text{-RS} : \mathcal{S}_n \to \bigsqcup_{\lambda \vdash n} \mathcal{P}\text{-Tab}_\lambda \times \text{SYT}_\lambda$ such that:

(A) $\text{des}_{\mathcal{P}}(w) = \{n - x \mid x \in \text{des}(Q)\} = \text{des}(\text{evac}(Q))$.

(B) $w \sim_{\mathcal{P}} \text{read}(P)$ *(reading word of P).*

(C) *If $w' \in \mathcal{S}_n$ satisfies $\mathcal{P}\text{-RS}(w') = (P', Q')$ and $w \sim_{\mathcal{P}} w'$, then the first columns of P and P' are of the same length.*

Unlike the usual Robinson-Schensted algorithm, in general we do not have the same $\mathcal{P}$-tableaux for w, w' such that $w \sim_{\mathcal{P}} w'$.

Proof of the main theorem

Theorem (Main theorem)

Suppose that $\mathcal{P}$ is not ladder-climbing. Then for any $\mathcal{P}$-Knuth equivalence class Γ, there exist $\lambda_1, \ldots, \lambda_k$ such that

1. $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^{|\text{f-inv}_{\mathcal{P}}(\Gamma)|}(s_{\lambda_1} + \cdots + s_{\lambda_k})$ *where* $|\text{f-inv}_{\mathcal{P}}(\Gamma)|$ *is* $|\text{f-inv}_{\mathcal{P}}(w)|$ *for any* $w \in \Gamma$*, and*
2. *the lengths of* $\lambda_1, \ldots, \lambda_k$ *are all equal.*

Sketch of the proof.

Using (B) one can show that $\mathcal{P}$-RS restricts to a bijection $\Gamma \simeq \bigsqcup_{i=1}^{k} \{P_i\} \times \mathrm{SYT}_{\mathrm{sh}(P_i)}$ for certain $\mathcal{P}$-tableaux $P_1, \ldots, P_k$. Now by (A), it follows that $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^{|\text{f-inv}_{\mathcal{P}}(\Gamma)|} \sum_{i=1}^{n} s_{\mathrm{sh}(P_i)}$, which proves (1). The first columns of $P_1, \ldots, P_k$ are of the same length by (C), and thus (2) follows. □

Thank you!