Point-box incidences and logarithmic density of semilinear graphs

Abdul Basit
Iowa State University

Joint work with
Artem Chernikov, Sergei Starchenko, Terence Tao, and Chieu-Minh Tran
Incidences between points and rectangles

Given $n_1$ points and $n_2$ closed rectangles with axis-parallel sides in $\mathbb{R}^2$. If no $k$ rectangles have $k$ points in common, what is the maximum number of incidences?
Incidences between points and rectangles

Given \( n_1 \) points and \( n_2 \) closed rectangles with axis-parallel sides in \( \mathbb{R}^2 \). If no \( k \) rectangles have \( k \) points in common, what is the maximum number of incidences?

An incidence is a point-rectangle \((p, r)\) pair such that the point \( p \) lies in the rectangle \( r \).
Incidences between points and rectangles

Given $n_1$ points and $n_2$ closed rectangles with axis-parallel sides in $\mathbb{R}^2$. If no $k$ rectangles have $k$ points in common, what is the maximum number of incidences?

An incidence is a point-rectangle $(p, r)$ pair such that the point $p$ lies in the rectangle $r$.

A configuration where no two rectangles have two points in common and 9 incidences.
Incidences between points and rectangles

Given $n_1$ points and $n_2$ closed rectangles with axis-parallel sides in $\mathbb{R}^2$. If no $k$ rectangles have $k$ points in common, what is the maximum number of incidences?

An incidence is a point-rectangle $(p, r)$ pair such that the point $p$ lies in the rectangle $r$.

A configuration where no two rectangles have two points in common and 9 incidences.
Incidences between points and rectangles

Given $n_1$ points and $n_2$ closed rectangles with axis-parallel sides in $\mathbb{R}^2$. If no $k$ rectangles have $k$ points in common, what is the maximum number of incidences?

An incidence is a point-rectangle $(p, r)$ pair such that the point $p$ lies in the rectangle $r$.

Without the assumption that no $k$ boxes have $k$ points in common, there could be $n_1 \cdot n_2$ incidences.
Zarankiewicz’s problem

A question in extremal graph theory:
For $k \in \mathbb{N}$, let $K_{k,k}$ denote the complete bipartite graph with $k$ vertices in each block.

For fixed $k$, what is the maximum number of edges in a $K_{k,k}$-free bipartite graph $G = (V_1, V_2; E)$?
Zarankiewicz’s problem

A question in extremal graph theory:
For \( k \in \mathbb{N} \), let \( K_{k,k} \) denote the complete bipartite graph with \( k \) vertices in each block.

For fixed \( k \), what is the maximum number of edges in a \( K_{k,k} \)-free bipartite graph \( G = (V_1, V_2; E) \)?

Kövári–Sós–Turán ’54:
If \( G = (V_1, V_2; E) \) with \( |V_1| + |V_2| = n \) is \( K_{k,k} \)-free, then \( |E| \leq O_k \left( n^{2-1/k} \right) \).
Zarankiewicz’s problem

A question in extremal graph theory:
For $k \in \mathbb{N}$, let $K_{k,k}$ denote the complete bipartite graph with $k$ vertices in each block.

For fixed $k$, what is the maximum number of edges in a $K_{k,k}$-free bipartite graph $G = (V_1, V_2; E)$?

Kövári–Sós–Turán ’54:
If $G = (V_1, V_2; E)$ with $|V_1| + |V_2| = n$ is $K_{k,k}$-free, then $|E| \leq O_k(n^{2-1/k})$.

Known to be best possible for $k \leq 3$. Conjectured to be best possible for all $k \in \mathbb{N}$. 
Incidences between points and rectangles

Given \( n_1 \) points and \( n_2 \) axis-parallel rectangles in \( \mathbb{R}^2 \).

Let \( G = (V_1, V_2; E) \) the the incidence graph. That is:

- Let vertices in \( V_1 \) correspond to points, vertices in \( V_2 \) correspond to rectangles, and
- \( E = \{ (p, r) \in V_1 \times V_2 : \text{point } p \text{ is in rectangle } r \} \).
Incidences between points and rectangles

Given $n_1$ points and $n_2$ axis-parallel rectangles in $\mathbb{R}^2$.

Let $G = (V_1, V_2; E)$ the incidence graph. That is:

Let vertices in $V_1$ correspond to points, vertices in $V_2$ correspond to rectangles, and

$E = \{(p, r) \in V_1 \times V_2 : \text{point } p \text{ is in rectangle } r\}$.

No two rectangles have two points in common

No $K_{2,2}$
Incidences between points and rectangles

Given $n_1$ points and $n_2$ axis-parallel rectangles in $\mathbb{R}^2$.

Let $G = (V_1, V_2; E)$ the incidence graph. That is:
Let vertices in $V_1$ correspond to points, vertices in $V_2$ correspond to rectangles, and
$E = \{(p, r) \in V_1 \times V_2 : \text{point } p \text{ is in rectangle } r\}$.

If no $k$ rectangles have $k$ points in common, then $G$ is $K_{k,k}$-free. So by Kövári–Sós–Turán, the number of incidences is $O_k(n^{2 - 1/k})$ for each $k \in \mathbb{N}$. 
Geometric Incidence Problems

The bounds implied by Kövári–Sós–Turán can often be improved for incidence graphs of objects in $\mathbb{R}^d$. 
Geometric Incidence Problems

The bounds implied by Kövári–Sós–Turán can often be improved for incidence graphs of objects in $\mathbb{R}^d$.

Given $n_1$ points and $n_2$ lines in $\mathbb{R}^2$, what is the maximum number of incidences?
Geometric Incidence Problems

The bounds implied by Kövári–Sós–Turán can often be improved for incidence graphs of objects in $\mathbb{R}^d$.

Given $n_1$ points and $n_2$ lines in $\mathbb{R}^2$, what is the maximum number of incidences?

The point-line incidence graph does not contain a $K_{2,2}$, so by Kövári–Sós–Turán, number of incidences is $O(n^{3/2})$. 
Geometric Incidence Problems

The bounds implied by Kövári–Sós–Turán can often be improved for incidence graphs of objects in $\mathbb{R}^d$.

Given $n_1$ points and $n_2$ lines in $\mathbb{R}^2$, what is the maximum number of incidences?

The point-line incidence graph does not contain a $K_{2,2}$, so by Kövári–Sós–Turán, number of incidences is $O\left(n^{3/2}\right)$.

Szemerédi-Trotter ’83:
The number of incidences is $O\left(n^{4/3}\right)$.
Geometric Incidence Problems

The bounds implied by Kövári–Sós–Turán can often be improved for incidence graphs of objects in $\mathbb{R}^d$.

Given $n_1$ points and $n_2$ lines in $\mathbb{R}^2$, what is the maximum number of incidences?

The point-line incidence graph does not contain a $K_{2,2}$, so by Kövári–Sós–Turán, number of incidences is $O(n^{3/2})$.

Szemerédi-Trotter ’83:
The number of incidences is $O(n^{4/3})$.

This is optimal, i.e., there exist configurations of points and lines with $\Omega(n^{4/3})$ incidences.
Zarankiewicz’s problem for semialgebraic graphs

A graph $G = (V_1, V_2; E)$ is semialgebraic if $V_1 \subset \mathbb{R}^{d_1}$, $V_2 \subset \mathbb{R}^{d_2}$, and there exists a system of polynomial inequalities $\varphi(x, y)$ such that $E = \{(a, b) \in V_1 \times V_2 : \varphi(a, b)\}$.
Zarankiewicz’s problem for semialgebraic graphs

A graph $G = (V_1, V_2; E)$ is semialgebraic if $V_1 \subset \mathbb{R}^{d_1}$, $V_2 \subset \mathbb{R}^{d_2}$, and there exists a system of polynomial inequalities $\phi(x, y)$ such that $E = \{(a, b) \in V_1 \times V_2 : \phi(a, b)\}$.

Incidence graph of points and lines in $\mathbb{R}^2$ is semialgebraic. Lines correspond to points in $\mathbb{R}^2$, e.g., the line $b_1 x + b_2 y = 1$ corresponds to the point $(b_1, b_2)$. Then $E = \{(a, b) \in V_1 \times V_2 : a \cdot b = 1\}$. 
Zarankiewicz’s problem for semialgebraic graphs

A graph $G = (V_1, V_2; E)$ is semialgebraic if $V_1 \subset \mathbb{R}^{d_1}$, $V_2 \subset \mathbb{R}^{d_2}$, and there exists a system of polynomial inequalities $\varphi(x, y)$ such that $E = \{ (a, b) \in V_1 \times V_2 : \varphi(a, b) \}$.

Incidence graph of points and lines in $\mathbb{R}^2$ is semialgebraic. Lines correspond to points in $\mathbb{R}^2$, e.g., the line $b_1 x + b_2 y = 1$ corresponds to the point $(b_1, b_2)$. Then $E = \{ (a, b) \in V_1 \times V_2 : a \cdot b = 1 \}$.

Fox-Pach-Sheffer-Suk-Zahl ’12:

Let $G = (V_1, V_2; E)$ be a semialgebraic graph with $|V_1| + |V_2| = n$. If $G$ is $K_{k,k}$-free, then $|E| = O_{k, \varphi}(n^{2-c})$ where $0 < c < 1$ depends only on $d_1$ and $d_2$. 
Incidences between points and rectangles

Kövári–Sós–Turán implies the bound $O_k(n^{2-1/k})$. 
Incidences between points and rectangles

Kövári–Sós–Turán implies the bound $O_k(n^{2-1/k})$.

The point-rectangle incidence graph is semialgebraic. Identify axis-parallel rectangles with points in $\mathbb{R}^4$. That is, the coordinates of the bottom left endpoint combined with the coordinates of the top right endpoint to obtain a semialgebraic graph in $\mathbb{R}^2 \times \mathbb{R}^4$. 
Incidences between points and rectangles

Kövári–Sós–Turán implies the bound $O_k(n^{2-1/k})$.

The point-rectangle incidence graph is semialgebraic. Identify axis-parallel rectangles with points in $\mathbb{R}^4$. That is, the coordinates of the bottom left endpoint combined with the coordinates of the top right endpoint to obtain a semialgebraic graph in $\mathbb{R}^2 \times \mathbb{R}^4$.

Fox-Pach-Sheffer-Suk-Zahl implies the bound $O_{k,\varepsilon}(n^{10/7+\varepsilon})$. 
Incidences between points and rectangles

Kövári–Sós–Turán implies the bound $O_k\left(n^{2-1/k}\right)$.

The point-rectangle incidence graph is semialgebraic. Identify axis-parallel rectangles with points in $\mathbb{R}^4$. That is, the coordinates of the bottom left endpoint combined with the coordinates of the top right endpoint to obtain a semialgebraic graph in $\mathbb{R}^2 \times \mathbb{R}^4$.

Fox-Pach-Sheffer-Suk-Zahl implies the bound $O_{k,\varepsilon}\left(n^{10/7+\varepsilon}\right)$.

In the point-line incidence graph, $E$ is defined by the inner product, using addition and multiplication.

In the point-rectangle incidence graph, $E$ is defined using only ordering.

So there is some hope for better bounds!
Incidences between points and boxes

Theorem 1 (B.-Chernikov-Starchenko-Tao-Tran ’20):

(i) Given $n_1$ points and $n_2$ closed rectangles with axis-parallel sides in $\mathbb{R}^2$. If no $k$ rectangles have $k$ points in common, the number of incidences is $O_k(n \log^4 n)$. 
Incidences between points and boxes

Theorem 1 (B.-Chernikov-Starchenko-Tao-Tran ’20):

(i) Given $n_1$ points and $n_2$ closed rectangles with axis-parallel sides in $\mathbb{R}^2$. If no $k$ rectangles have $k$ points in common, the number of incidences is $O_k(n \log^4 n)$.

(ii) If the rectangles are dyadic, i.e., products of intervals of the form $[s2^t, (s + 1)2^t)$ for $s, t \in \mathbb{N}$, then the number of incidences is $O_k \left( n \frac{\log n}{\log \log n} \right)$. 
Incidences between points and boxes

Theorem 1 (B.-Chernikov-Starchenko-Tao-Tran ’20):

(i) Given $n_1$ points and $n_2$ closed rectangles with axis-parallel sides in $\mathbb{R}^2$. If no $k$ rectangles have $k$ points in common, the number of incidences is $O_k(n \log^4 n)$.

(ii) If the rectangles are dyadic, i.e., products of intervals of the form $[s2^t, (s + 1)2^t)$ for $s, t \in \mathbb{N}$, then the number of incidences is $O_k\left(n \frac{\log n}{\log \log n}\right)$.

(iii) For arbitrarily large $n$, there exist a set of $n$ points and $n$ dyadic rectangles, such that the incidence graph is $K_{2,2}$-free and the number of incidences is $\Omega\left(n \frac{\log n}{\log \log n}\right)$. 
Incidences between points and boxes

**Theorem 1 (B.-Chernikov-Starchenko-Tao-Tran ’20):**

(i) Given $n_1$ points and $n_2$ closed rectangles with axis-parallel sides in $\mathbb{R}^2$. If no $k$ rectangles have $k$ points in common, the number of incidences is $O_k(n \log^4 n)$.

(ii) If the rectangles are dyadic, i.e., products of intervals of the form $[s2^t, (s + 1)2^t)$ for $s, t \in \mathbb{N}$, then the number of incidences is $O_k\left(n \frac{\log n}{\log \log \log n}\right)$.

(iii) For arbitrarily large $n$, there exist a set of $n$ points and $n$ dyadic rectangles, such that the incidence graph is $K_{2,2}$-free and the number of incidences is $\Omega\left(n \frac{\log n}{\log \log \log n}\right)$.

**Tomon-Zakharov ’20:**
When $k = 2$, the bound in (i) can be improved to $O(n \log n)$. 

Zarankiewicz’s problem for semilinear graphs

A graph $G = (V_1, V_2; E)$ is semilinear if $V_1 \subset \mathbb{R}^{d_1}$, $V_2 \subset \mathbb{R}^{d_2}$, and there exists a system of linear inequalities $\varphi(x, y)$ such that $E = \{(a, b) \in V_1 \times V_2 : \varphi(a, b)\}$. 
Zarankiewicz’s problem for semilinear graphs

A graph $G = (V_1, V_2; E)$ is semilinear if $V_1 \subset \mathbb{R}^{d_1}$, $V_2 \subset \mathbb{R}^{d_2}$, and there exists a system of linear inequalities $\varphi(x, y)$ such that $E = \{(a, b) \in V_1 \times V_2 : \varphi(a, b)\}$.

Theorem 2 (B.-Chernikov-Starchenko-Tao-Tran ’20):
Let $G = (V_1, V_2; E)$ be a semilinear graph with $|V_1| + |V_2| = n$. If $G$ is $K_{k,k}$-free, then $|E| = O_{k,\varphi}(n \log^c n)$, where $c$ is the number of linear inequalities in the defining system.
Zarankiewicz’s problem for semilinear graphs

A graph $G = (V_1, V_2; E)$ is semilinear if $V_1 \subset \mathbb{R}^{d_1}$, $V_2 \subset \mathbb{R}^{d_2}$, and there exists a system of linear inequalities $\phi(x, y)$ such that $E = \{(a, b) \in V_1 \times V_2 : \phi(a, b)\}$.

Theorem 2 (B.-Chernikov-Starchenko-Tao-Tran ’20):
Let $G = (V_1, V_2; E)$ be a semilinear graph with $|V_1| + |V_2| = n$. If $G$ is $K_{k, k}$-free, then $|E| = O_{k, \phi}(n \log^c n)$, where $c$ is the number of linear inequalities in the defining system.

More generally:
Functions that are coordinate-wise monotone.
Any ordered division ring instead of $\mathbb{R}$. 
Proof of Theorem 2

Proof Idea:

Induction on number of linear equations $s$. Let $f_s(n)$ be the maximum number of edges in a $K_{k,k}$-free graph on $n$ vertices defined by $s$ linear equations.
Proof of Theorem 2

Proof Idea:

Induction on number of linear equations \( s \).
Let \( f_s(n) \) be the maximum number of edges in a \( K_{k,k} \)-free graph on \( n \) vertices defined by \( s \) linear equations.

Base Case: \( f_0(n) \leq kn \)

If \( s = 0 \), then \( G \) is the complete graph, so either \(|V_1| < k \) or \(|V_2| < k\), i.e., \(|E| \leq kn\).
Proof of Theorem 2

Proof Idea:

Induction on number of linear equations \( s \).
Let \( f_s(n) \) be the maximum number of edges in a \( K_{k,k} \)-free graph on \( n \) vertices defined by \( s \) linear equations.

Base Case: \( f_0(n) \leq kn \)
If \( s = 0 \), then \( G \) is the complete graph, so either \( |V_1| < k \) or \( |V_2| < k \), i.e., \( |E| \leq kn \).

Inductive Step: Enough to show \( f_s(n) \leq 2f_s\left(\lfloor \frac{n}{2} \rfloor \right) + f_{s-1}(n) \).
Use the order structure of \( \mathbb{R} \) to split up the graph and control incidences.

Suppose \( L \) is one of the defining inequalities. Can assume \( L \) has the form \( L_1(x) < L_2(y) \) with \( L_1 : V_1 \to \mathbb{R} \) and \( L_2 : V_2 \to \mathbb{R} \).
Proof of Theorem 2

$L$ has the form $L_1(x) < L_2(y)$. 
Proof of Theorem 2

$L$ has the form $L_1(x) < L_2(y)$.

$|E \cap (V_1^+ \times V_2^-)| = 0$
Proof of Theorem 2

$L$ has the form $L_1(x) < L_2(y)$.

- $|E \cap (V_1^+ \times V_2^-)| = 0$
- $|E \cap (V_1^- \times V_2^+)| \leq f_{s-1}(n)$
Proof of Theorem 2

$L$ has the form $L_1(x) < L_2(y)$.

- $|E \cap (V_1^+ \times V_2^-)| = 0$
- $|E \cap (V_1^- \times V_2^+)| \leq f_{s-1}(n)$
- $|E \cap (V_1^- \times V_2^-)| \leq f_s \left(\lfloor \frac{n}{2} \rfloor \right)$
- $|E \cap (V_1^+ \times V_2^+)| \leq f_s \left(\lfloor \frac{n}{2} \rfloor \right)$
Proof of Theorem 2

$L$ has the form $L_1(x) < L_2(y)$.

- $|E \cap (V_1^+ \times V_2^-)| = 0$
- $|E \cap (V_1^- \times V_2^+)| \leq f_{s-1}(n)$
- $|E \cap (V_1^- \times V_2^-)| \leq f_s\left(\lfloor \frac{n}{2} \rfloor \right)$
- $|E \cap (V_1^+ \times V_2^-)| \leq f_s\left(\lfloor \frac{n}{2} \rfloor \right)$
- $|E \cap (V_1^+ \times V_2^+)| \leq f_s\left(\lfloor \frac{n}{2} \rfloor \right)$

That is $f_s(n) \leq 2f_s\left(\lfloor \frac{n}{2} \rfloor \right) + f_{s-1}(n)$. 
Extensions to Hypergraphs

Erdős ’64:
Let $H = (V_1, V_2, \ldots, V_r, E)$ be a $r$-partite $r$-uniform hypergraph hypergraph with $|V_1| + \cdots + |V_r| = n$. If $H$ is $K_{k,\ldots,k}$-free, then $|E| = O_{r,k}\left(n^{r-\frac{1}{kr-1}}\right)$. 
Extensions to Hypergraphs

Erdős ’64:
Let \( H = (V_1, V_2, \ldots, V_r, E) \) be a \( r \)-partite \( r \)-uniform hypergraph hypergraph with \( |V_1| + \cdots + |V_r| = n \). If \( H \) is \( K_{k,\ldots,k} \)-free, then
\[
|E| = O_{r,k} \left( n^{r - \frac{1}{kr-1}} \right).
\]
Probabilistic lower bounds of the form \( |E| = \Omega_{r,k} \left( n^{r - \frac{c}{kr-1}} \right) \).
That is, the exponent can not be substantially improved.
Extensions to Hypergraphs

Erdős ’64:
Let $H = (V_1, V_2, \ldots, V_r, E)$ be a $r$-partite $r$-uniform hypergraph hypergraph with $|V_1| + \cdots + |V_r| = n$. If $H$ is $K_{k, \ldots, k}$-free, then $|E| = O_{r,k} \left( n^{r-\frac{1}{kr-1}} \right)$.

Probabilistic lower bounds of the form $|E| = \Omega_{r,k} \left( n^{r-\frac{c}{kr-1}} \right)$. That is, the exponent can not be substantially improved.

Do ’18:
Let $H = (V_1, V_2, \ldots, V_r, E)$ be a semialgebraic hypergraph with $|V_1| + \cdots + |V_r| = n$. If $H$ is $K_{k, \ldots, k}$-free, then $|E| = O_{r,k,\phi} \left( n^{r-c} \right)$ where $c$ depends only on $d_1, d_2, \ldots, d_r$. 
Extensions to Hypergraphs

Theorem 3 (B.-Chernikov-Starchenko-Tao-Tran ’20):
Let $H = (V_1, V_2, \ldots, V_r, E)$ be a semilinear hypergraph with $|V_1| + \cdots + |V_r| = n$. If $H$ is $K_{k,\ldots,k}$-free, then
$$|E| = O_{r,k,\varphi}(n^{r-1} \log^c n)$$
where $c$ depends only on $r$ and the number of defining inequalities.
Extensions to Hypergraphs

Theorem 3 (B.-Chernikov-Starchenko-Tao-Tran ’20):
Let $H = (V_1, V_2, \ldots, V_r, E)$ be a semilinear hypergraph with $|V_1| + \cdots + |V_r| = n$. If $H$ is $K_{k,\ldots,k}$-free, then $|E| = \mathcal{O}_{r,k,\varphi}(n^{r-1} \log^c n)$ where $c$ depends only on $r$ and the number of defining inequalities.

Proof Idea:
Double induction on uniformity $r$, and the number of inequalities $c$. 
Extensions to Hypergraphs

Theorem 3 (B.-Chernikov-Starchenko-Tao-Tran ’20):
Let $H = (V_1, V_2, \ldots, V_r, E)$ be a semilinear hypergraph with $|V_1| + \cdots + |V_r| = n$. If $H$ is $K_{k,\ldots,k}$-free, then $|E| = O_{r,k,\phi}(n^{r-1} \log^c n)$ where $c$ depends only on $r$ and the number of defining inequalities.

Proof Idea:
Double induction on uniformity $r$, and the number of inequalities $c$.

We also need a divide and conquer strategy. For $r = 2$, we reduced to graph with smaller $|V_1| + |V_2|$. For $r = 3$, we instead reduce to a graph with smaller $|V_1||V_2| + |V_2||V_3| + |V_3||V_1|$. 
Consequences of Theorem 2

Point-polytope incidences:
Given $n_1$ points and $n_2$ polytopes in $\mathbb{R}^d$ with faces in some fixed finite set of orientations, such that the incidence graph does not contain $K_{k,k}$, the number of incidences is $O(n^{1+\varepsilon})$, for any $\varepsilon > 0$. 
Consequences of Theorem 2

Point-polytope incidences:
Given $n_1$ points and $n_2$ polytopes in $\mathbb{R}^d$ with faces in some fixed finite set of orientations, such that the incidence graph does not contain $K_{k,k}$, the number of incidences is $O(n_1^{1+\varepsilon})$, for any $\varepsilon > 0$.

Proof:
Parametrize half-spaces with a fixed orientation by points in $\mathbb{R}^d$. A point being contained in a half-space is a linear inequality.
Consequences of Theorem 2

Point-polytope incidences:
Given $n_1$ points and $n_2$ polytopes in $\mathbb{R}^d$ with faces in some fixed finite set of orientations, such that the incidence graph does not contain $K_{k,k}$, the number of incidences is $O(n^{1+\varepsilon})$, for any $\varepsilon > 0$.

Proof:
Parametrize half-spaces with a fixed orientation by points in $\mathbb{R}^d$. A point being contained in a half-space is a linear inequality.

If there are $s$ orientations, then get a semilinear graph in $\mathbb{R}^d \times \mathbb{R}^{sd}$ with $s$ inequalities.

By Theorem 2, there are $O_{k,s,d}(n \log^s n)$ edges.
Consequences of Theorem 2

Question
Given \( n \) points in \( \mathbb{R}^2 \), what is the maximum number of pairs of points at distance one?
Consequences of Theorem 2

Question
Given $n$ points in $\mathbb{R}^2$, what is the maximum number of pairs of points at distance one?

Erdős unit distance conjecture:
In the $\ell_2$ norm, the number of unit distances is $O(n^{1+\varepsilon})$. 
Consequences of Theorem 2

**Question**
Given \( n \) points in \( \mathbb{R}^2 \), what is the maximum number pairs of points at distance one?

**Erdős unit distance conjecture:**
In the \( \ell_2 \) norm, the number of unit distances is \( O(n^{1+\varepsilon}) \).

For _polygonal norms_, it is possible to get \( \Omega(n^2) \).
Consequences of Theorem 2

Question
Given $n$ points in $\mathbb{R}^2$, what is the maximum number of pairs of points at distance one?

Erdős unit distance conjecture:
In the $\ell_2$ norm, the number of unit distances is $O(n^{1+\epsilon})$.

For polygonal norms, it is possible to get $\Omega(n^2)$.

Every blue-green pair is at distance one.
Consequences of Theorem 2

Unit distances in polygonal norms:
For any fixed $k$, if no $k$ points are at unit distance from any $k$ other points, then the number of unit distances is $O(n^{1+\varepsilon})$. 
Connections to Model Theory

Model theorists study a structure (e.g. $(\mathbb{Z}; +)$, $(\mathbb{C}; +, \times)$, etc) by considering the set of all first order sentences true in the structure, referred to as the theory of the structure.
Connections to Model Theory

Model theorists study a structure (e.g. \((\mathbb{Z}; +)\), \((\mathbb{C}; +, \times)\), etc) by considering the set of all first order sentences true in the structure, referred to as the theory of the structure.

Isomorphic structures have the same theory, but the converse is not true. In fact, given an infinite structure, there is at least one structure per infinite cardinality with the same theory.
Connections to Model Theory

Model theorists study a structure (e.g. \((\mathbb{Z}; +)\), \((\mathbb{C}; +, \times)\), etc) by considering the set of all first order sentences true in the structure, referred to as the theory of the structure.

Isomorphic structures have the same theory, but the converse is not true. In fact, given an infinite structure, there is at least one structure per infinite cardinality with the same theory.

E.g., any real closed field has the same theory as \(\mathbb{R}\).

- algebraic numbers
- hyperreal numbers
- computable numbers
Connections to Model Theory

What can be said when for some uncountable cardinality $\kappa$, there is exactly one structure (up to isomorphism) having the same theory as the given structure? Such structures are referred to as tame.
Connections to Model Theory

What can be said when for some uncountable cardinality $\kappa$, there is exactly one structure (up to isomorphism) having the same theory as the given structure? Such structures are referred to as tame.

One consequence of tameness is that graphs defined by formulas in such structures are not too complicated.
Connections to Model Theory

What can be said when for some uncountable cardinality $\kappa$, there is exactly one structure (up to isomorphism) having the same theory as the given structure? Such structures are referred to as tame.

One consequence of tameness is that graphs defined by formulas in such structures are not too complicated.

A formula $\varphi(x_1, \ldots, x_m)$ generalizes equations by allowing the quantifiers $\forall$ and $\exists$, where the $x_i$'s can take values in $M$. e.g. $\varphi(x, y) : x = y^2$ and $\varphi(x) : \exists y, x = y^2$

More generally, any algebraic equation (or a boolean combination of algebraic equations) is a formula.
Connections to Model Theory

What can be said when for some uncountable cardinality \( \kappa \), there is exactly one structure (up to isomorphism) having the same theory as the given structure? Such structures are referred to as tame.

One consequence of tameness is that graphs defined by formulas in such structures are not too complicated.

A formula \( \varphi(x_1, \ldots, x_m) \) generalizes equations by allowing the quantifiers \( \forall \) and \( \exists \), where the \( x_i \)'s can take values in \( M \).

E.g. \( \varphi(x, y) : x = y^2 \) and \( \varphi(x) : \exists y, x = y^2 \)

More generally, any algebraic equation (or a boolean combination of algebraic equations) is a formula.

We can think of \( \varphi(x, y) \) as a (possibly infinite) graph \( G = (M, M, E) \), where \( E = \{(a, b) \in M^{|x|} \times M^{|y|} : \varphi(a, b) \text{ is true}\} \).
Connections to Model Theory

Often tameness is related to combinatorial properties of the graphs of formulas, resulting in improved bounds for

- regularity lemma type statements
- Erdős-Hajnal problem
- Zarankiewicz’s problem
NIP Structures

If there is exactly one structure of each uncountable cardinality satisfying the same theory, then the graph of every formula \( \varphi(x, y) \) with \( x, y \in M \) has bounded VC-dimension.

More generally, structures where every formula has bounded VC-dimension are said to have the no independence property.
NIP Structures

If there is exactly one structure of each uncountable cardinality satisfying the same theory, then the graph of every formula $\varphi(x, y)$ with $x, y \in M$ has bounded VC-dimension.

More generally, structures where every formula has bounded VC-dimension are said to have the no independence property.

Examples:
real closed fields
the field of $p$-adic numbers
Algebraic closure of finite fields
NIP Structures

Fox-Pach-Sheffer-Suk-Zahl ’12:
Suppose $M$ is a structure, $\varphi(x; y)$ is a formula with VC-dimension $d$, $G = (V_1, V_2; E)$ is a $K_{k,k}$-free graph with $V_1 \subseteq M|x|$, $V_2 \subseteq M|y|$, and $E = \{(a, b) \in V_1 \times V_2 : \varphi(a, b)\}$. Then $|E| = O_k \left( n^{2-1/d} \right)$.

Janzer-Pohoata ’20:
If we also have that $k \geq d \geq 3$, $|E| = o \left( n^{2-1/d} \right)$. 
Distal Structures

A structure $M$ is distal if every formula $\varphi(x;y)$ is distal, i.e., it admits a definable cell decomposition.
Distal Structures

A structure $M$ is *distal* if every formula $\varphi(x; y)$ is distal, i.e., it admits a definable cell decomposition.

Chernikov-Galvin-Starchenko ’16:
Suppose $M$ is a structure, $\varphi(x; y)$ is a distal formula, $(V_1, V_2; E)$ is a $K_{k,k}$-free graph with $V_1 \subseteq M^{|x|}$, $V_2 \subseteq M^{|y|}$, and $E = \{(a, b) \in V_1 \times V_2 : \varphi(a, b)\}$. Then $|E| = O_k\left(n^{2-c}\right)$ where $c$ depends only on $\varphi$.

Generalizes Fox-Pach-Sheffer-Suk-Zahl.

Examples:
Real closed fields are distal.
Algebraic closure of finite fields are NIP but not distal.
Linear vs. nonlinear

The distinction between the point-rectangle incidence graph and the point-line incidence graph is a linear vs. nonlinear (or modular vs. nonmodular) distinction.

Introduced in an effort to answer the following question: What can be said when for each infinite cardinality $\kappa$, there is exactly one structure up to isomorphism having a given theory.
Linear vs. nonlinear

The distinction between the point-rectangle incidence graph and the point-line incidence graph is a linear vs. nonlinear (or modular vs. nonmodular) distinction.

Introduced in an effort to answer the following question: What can be said when for each infinite cardinality \( \kappa \), there is exactly one structure up to isomorphism having a given theory.

Examples:
Real closed fields are not modular.
Ordered group of real numbers is modular.
**Linear vs. nonlinear**

The distinction between the point-rectangle incidence graph and the point-line incidence graph is a linear vs. nonlinear (or modular vs. nonmodular) distinction.

Introduced in an effort to answer the following question: What can be said when for each infinite cardinality \( \kappa \), there is exactly one structure up to isomorphism having a given theory.

**Examples:**
Real closed fields are not modular.
Ordered group of real numbers is modular.

In this setting, Theoerm 2 is a statement about graphs definable in o-minimal modular structures.
Questions