Point-box incidences and logarithmic density of semilinear graphs

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Joint work with
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  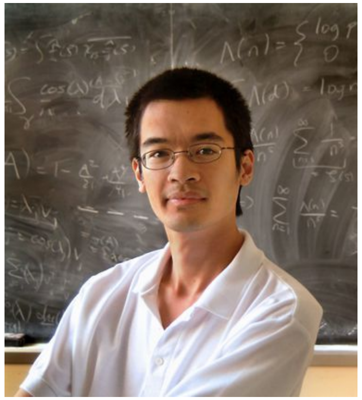 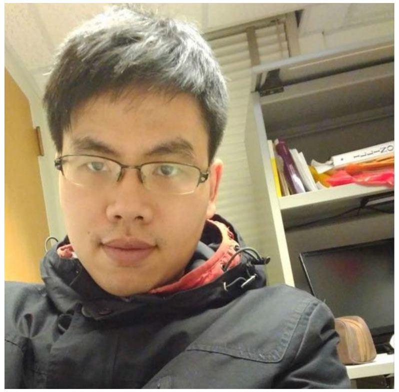

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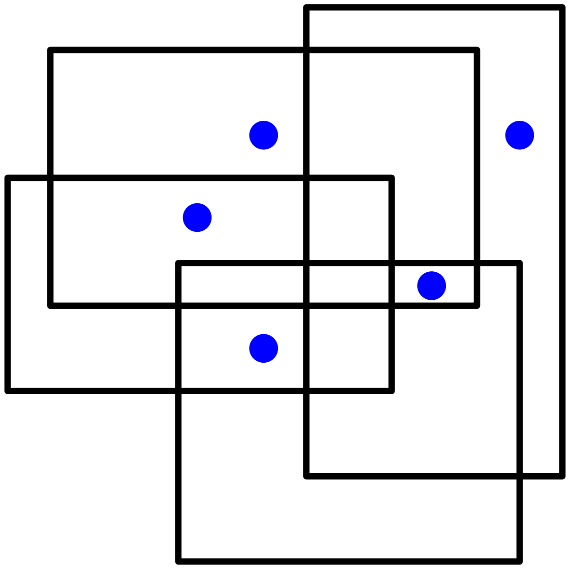

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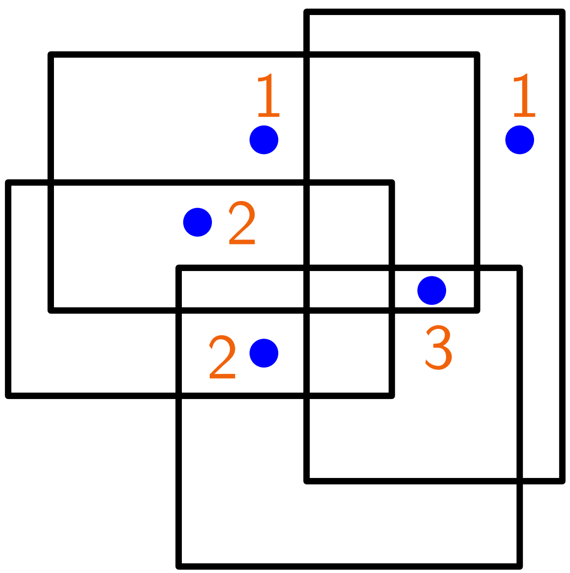

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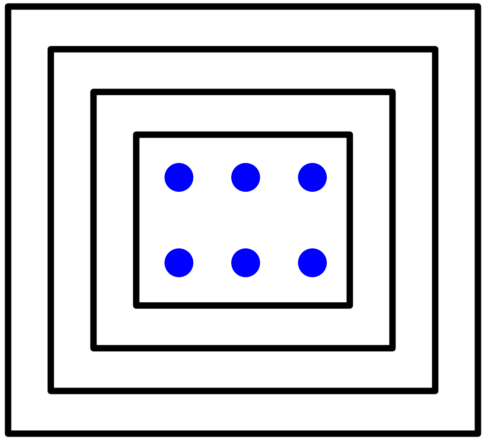

Without the assumption that no k boxes have k points in common, there could be $n_1 \cdot n_2$ incidences.

Zarankiewicz’s problem

A question in extremal graph theory:

For $k \in \mathbb{N}$, let $K_{k,k}$ denote the complete bipartite graph with k vertices in each block.

For fixed k, what is the maximum number of edges in a $K_{k,k}$-free bipartite graph $G = (V_1, V_2; E)$?

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Kövári–Sós–Turán '54:

If $G = (V_1, V_2; E)$ with $|V_1| + |V_2| = n$ is $K_{k,k}$-free, then $|E| \leq O_k\left(n^{2-1/k}\right)$.

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Known to be best possible for $k \le 3$.
Conjectured to be best possible for all $k \in \mathbb{N}$.

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Given n_1 points and n_2 axis-parallel rectangles in $\mathbb{R}^2$.

Let $G = (V_1, V_2; E)$ the the incidence graph. That is:
Let vertices in V_1 correspond to points, vertices in V_2 correspond to rectangles, and
$E = \{(p, r) \in V_1 \times V_2 : \text{ point } p \text{ is in rectangle } r\}$.

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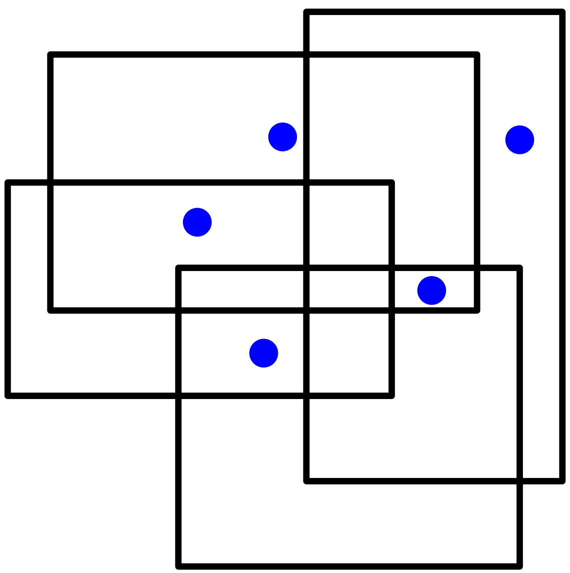

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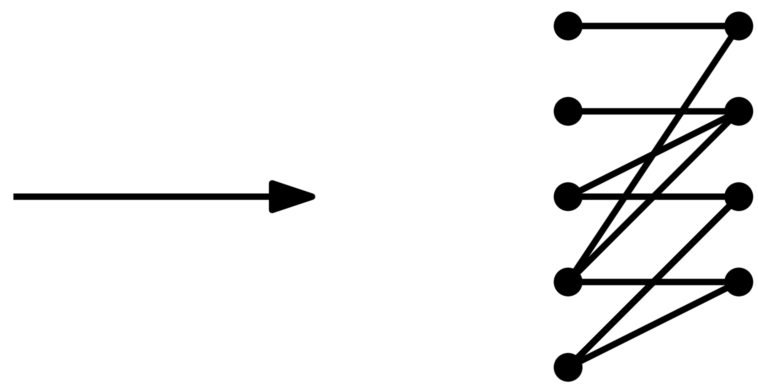
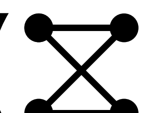


No $K_{2,2}$ ()

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If no k rectangles have k points in common, then G is $K_{k,k}$-free. So by Kővári–Sós–Turán, the number of incidences is $O_k\left(n^{2-1/k}\right)$ for each $k \in \mathbb{N}$.

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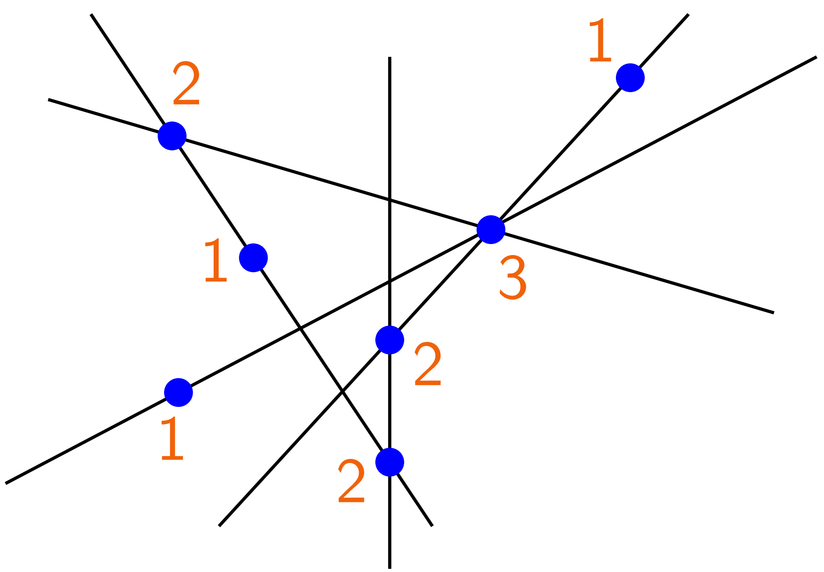

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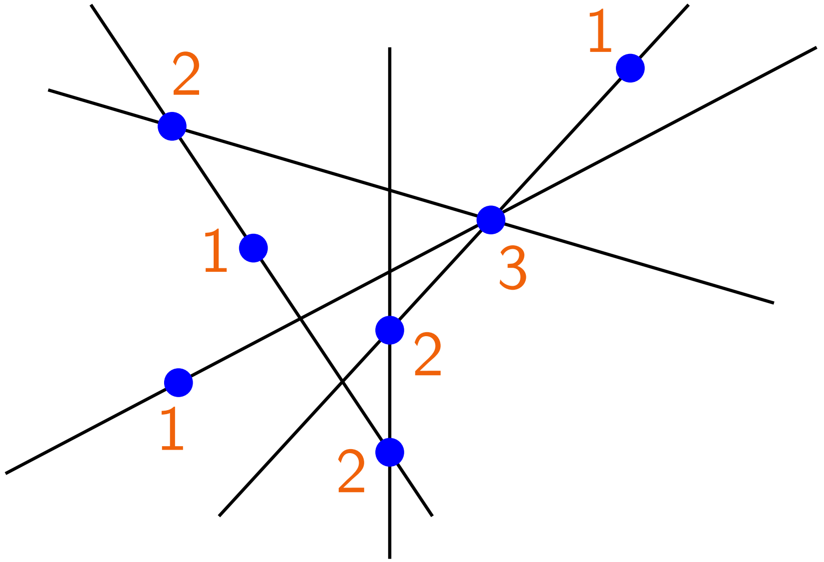

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The number of incidences is $O\left(n^{4/3}\right)$.

This is optimal, i.e., there exist configurations of points and lines with $\Omega\left(n^{4/3}\right)$ incidences.

Zarankiewicz's problem for semialgebraic graphs

A graph $G = (V_1, V_2; E)$ is semialgebraic if $V_1 \subset \mathbb{R}^{d_1}$, $V_2 \subset \mathbb{R}^{d_2}$, and there exists a system of *polynomial inequalities* $\varphi(x, y)$ such that $E = \{(a, b) \in V_1 \times V_2 : \varphi(a, b)\}$.

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Incidence graph of points and lines in $\mathbb{R}^2$ is semialgebraic.
Lines correspond to points in $\mathbb{R}^2$, e.g., the line $b_1 x + b_2 y = 1$ corresponds to the point (b_1, b_2).
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Fox-Pach-Sheffer-Suk-Zahl '12:
Let $G = (V_1, V_2; E)$ be a semialgebraic graph with $|V_1| + |V_2| = n$. If G is $K_{k,k}$-free, then $|E| = O_{k,\varphi}\left(n^{2-c}\right)$ where $0 < c < 1$ depends only on d_1 and d_2.

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The point-rectangle incidence graph is semialgebraic. Identify axis-parallel rectangles with points in $\mathbb{R}^4$. That is, the coordinates of the bottom left endpoint combined with the coordinates of the top right endpoint to obtain a semialgebraic graph in $\mathbb{R}^2 \times \mathbb{R}^4$.

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In the point-line incidence graph, E is defined by the inner product, using *addition and multiplication*.

In the point-rectangle incidence graph, E is defined using only *ordering*.

So there is some hope for better bounds!

Incidences between points and boxes

Theorem 1 (B.-Chernikov-Starchenko-Tao-Tran ’20):

(i) Given n_1 points and n_2 closed rectangles with axis-parallel sides in $\mathbb{R}^2$. If no k rectangles have k points in common, the number of incidences is $O_k\left(n \log^4 n\right)$.

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Tomon-Zakharov ’20:

When $k = 2$, the bound in (i) can be improved to $O(n \log n)$.

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Theorem 2 (B.-Chernikov-Starchenko-Tao-Tran '20):

Let $G = (V_1, V_2; E)$ be a semilinear graph with $|V_1| + |V_2| = n$. If G is $K_{k,k}$-free, then $|E| = O_{k,\varphi}(n \log^c n)$, where c is the number of linear inequalities in the defining system.

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More generally:

Functions that are coordinate-wise monotone.

Any ordered division ring instead of $\mathbb{R}$.

Proof of Theorem 2

Proof Idea:

Induction on number of linear equations s.

Let $f_s(n)$ be the maximum number of edges in a $K_{k,k}$-free graph on n vertices defined by s linear equations.

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Base Case: $f_0(n) \leq kn$

If $s = 0$, then G is the complete graph, so either $|V_1| < k$ or $|V_2| < k$, i.e., $|E| \leq kn$.

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Inductive Step: Enough to show $f_s(n) \leq 2f_s\left(\lfloor \frac{n}{2} \rfloor\right) + f_{s-1}(n)$.

Use the order structure of $\mathbb{R}$ to split up the graph and control incidences.

Suppose L is one of the defining inequalities. Can assume L has the form $L_1(x) < L_2(y)$ with $L_1 : V_1 \to \mathbb{R}$ and $L_2 : V_2 \to \mathbb{R}$.

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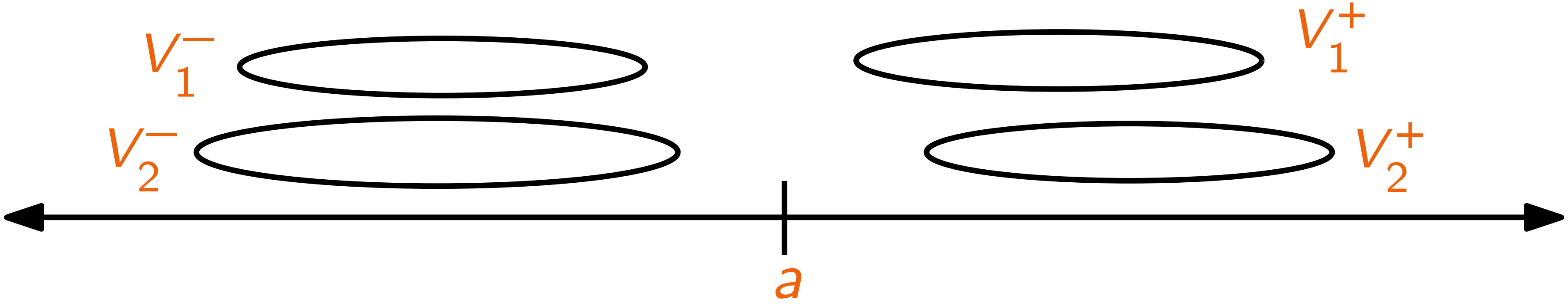

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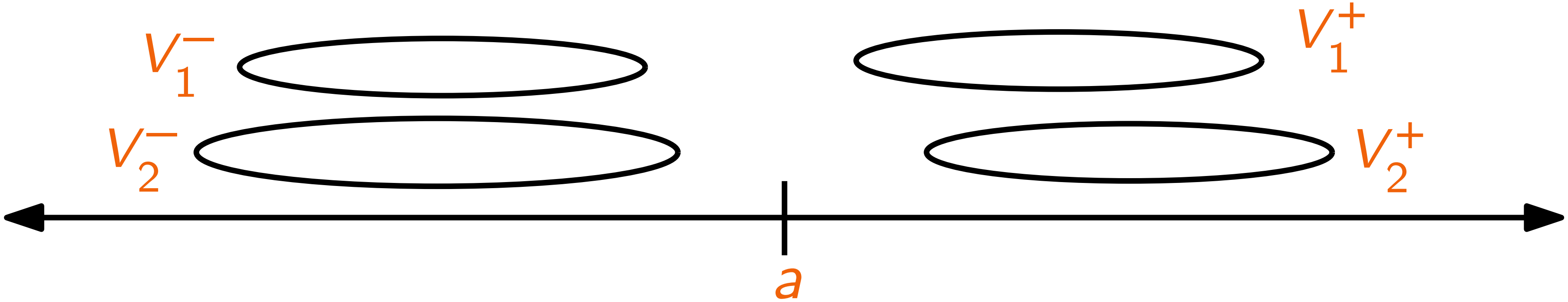

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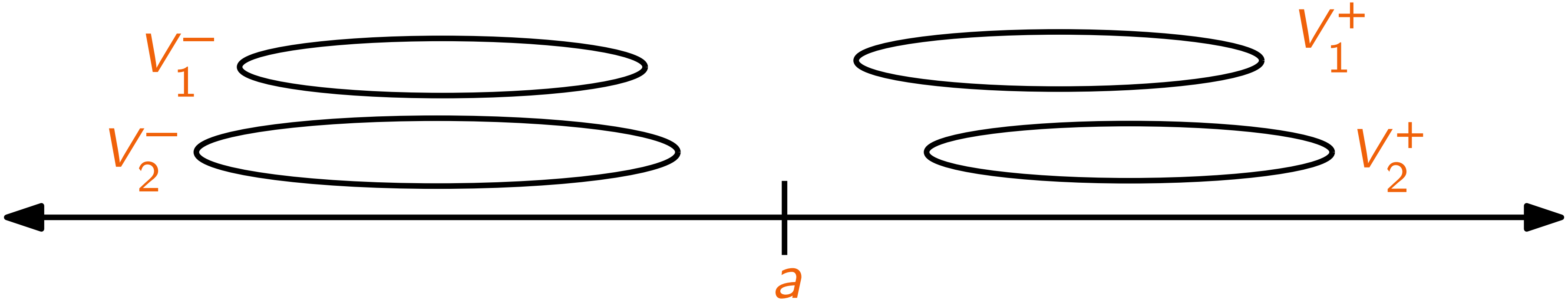

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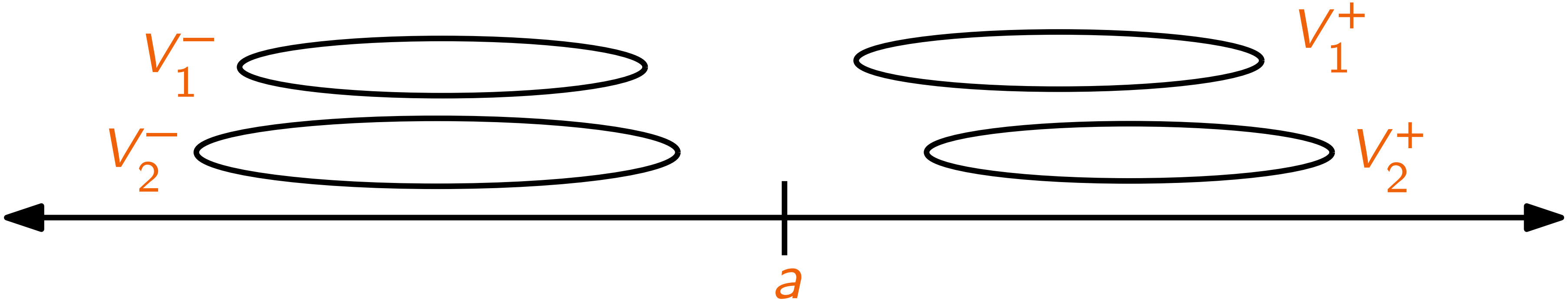

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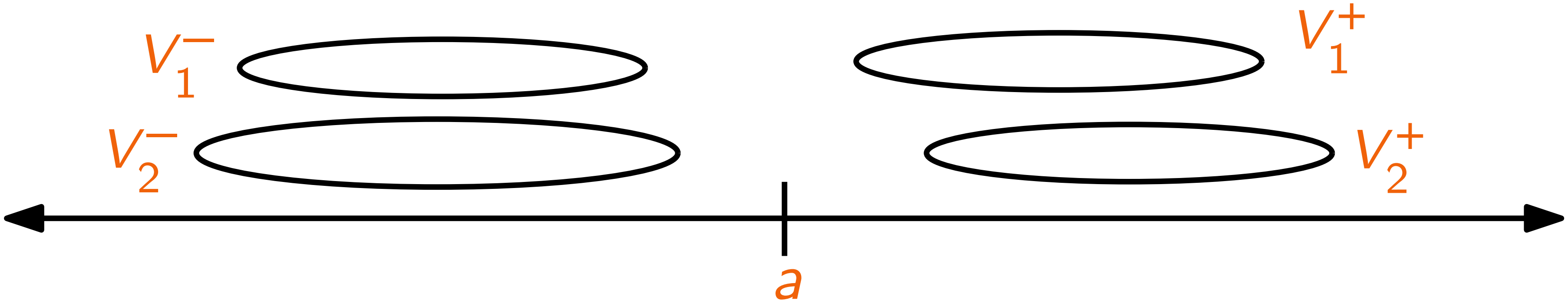

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That is $f_s(n) \leq 2f_s\left(\lfloor \frac{n}{2} \rfloor\right) + f_{s-1}(n)$.

Extensions to Hypergraphs

Erdős '64:

Let $H = (V_1, V_2, \ldots, V_r, E)$ be a r-partite r-uniform hypergraph hypergraph with $|V_1| + \cdots + |V_r| = n$. If H is $K_{k,\ldots,k}$-free, then $|E| = O_{r,k}\left(n^{r-\frac{1}{k^{r-1}}}\right)$.

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Do '18:

Let $H = (V_1, V_2, \ldots, V_r, E)$ be a semialgebraic hypergraph with $|V_1| + \cdots + |V_r| = n$. If H is $K_{k,\ldots,k}$-free, then $|E| = O_{r,k,\varphi}\left(n^{r-c}\right)$ where c depends only on $d_1, d_2, \ldots, d_r$.

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Theorem 3 (B.-Chernikov-Starchenko-Tao-Tran '20):

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We also need a divide and conquer strategy. For $r = 2$, we reduced to graph with smaller $|V_1| + |V_2|$. For $r = 3$, we instead reduce to a graph with smaller

$$|V_1||V_2| + |V_2||V_3| + |V_3||V_1|.$$

Consequences of Theorem 2

Point-polytope incidences:

Given n_1 points and n_2 polytopes in $\mathbb{R}^d$ with faces in some fixed finite set of orientations, such that the incidence graph does not contain $K_{k,k}$, the number of incidences is $O\left(n^{1+\varepsilon}\right)$, for any $\varepsilon > 0$.

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If there are s orientations, then get a semilinear graph in $\mathbb{R}^d \times \mathbb{R}^{sd}$ with s inequalities.

By Theorem 2, there are $O_{k,s,d}(n \log^s n)$ edges.

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Given n points in $\mathbb{R}^2$, what is the maximum number pairs of points at distance one?

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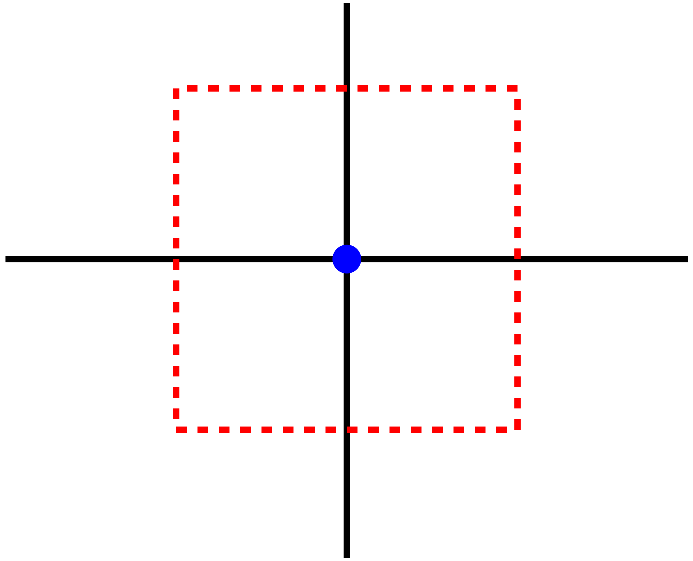

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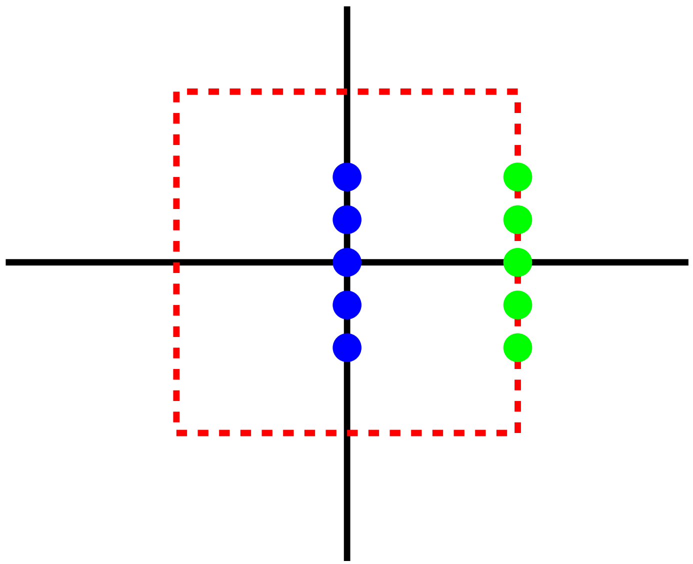
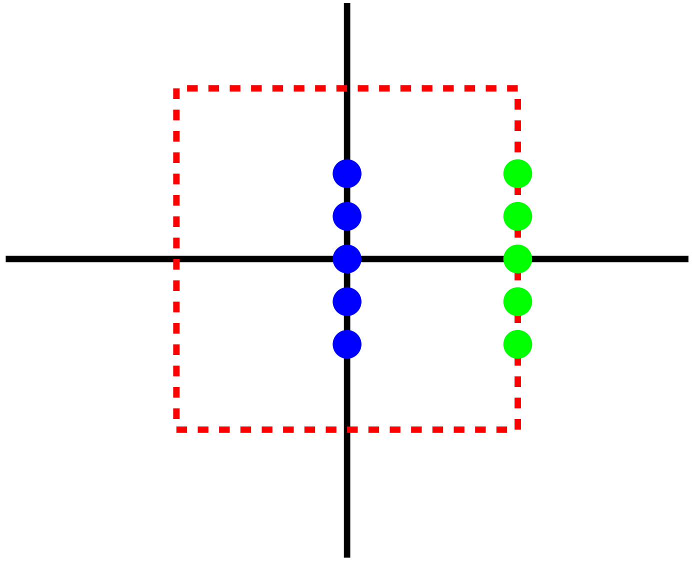


Every blue-green pair is at distance one

Consequences of Theorem 2

Unit distances in polygonal norms:

For any fixed k, if no k points are at unit distance from any k other points, then the number of unit distances is $O\left(n^{1+\varepsilon}\right)$.



Connections to Model Theory

Model theorists study a structure (e.g. $(\mathbb{Z}; +)$, $(\mathbb{C}; +, \times)$, etc) by considering the set of all first order sentences true in the structure, refered to as the theory of the structure.

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Isomorphic structures have the same theory, but the converse is not true. In fact, given an infinite structure, there is at least one structure per infinite cardinality with the same theory.

E.g., any *real closed field* has the same theory as $\mathbb{R}$.

- algebraic numbers
- hyperreal numbers
- computable numbers

Connections to Model Theory

What can be said when for some uncountable cardinality κ, there is exactly one structure (up to isomorphism) having the same theory as the given structure?
Such structures are referred to as tame.

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A formula $\varphi(x_1, \ldots, x_m)$ generalizes equations by allowing the quantifiers $\forall$ and $\exists$, where the x_i's can take values in M.
e.g. $\varphi(x, y) : x = y^2$ and $\varphi(x) : \exists y, x = y^2$
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More generally, any algebraic equation (or a boolean combination of algebraic equations) is a formula.

We can think of $\varphi(x, y)$ as a (possibly infinite) graph $G = (M, M, E)$, where $E = \left\{(a, b) \in M^{|x|} \times M^{|y|} : \varphi(a, b) \text{ is true}\right\}$.

Connections to Model Theory

Often tameness is related to combinatorial properties of the graphs of formulas, resulting in improved bounds for

- regularity lemma type statements
- Erdős-Hajnal problem
- Zarankiewicz's problem

NIP Structures

If there is exactly one structure of each uncountable cardinality satisfying the same theory, then the graph of every formula $\varphi(x, y)$ with $x, y \in M$ has bounded VC-dimension.

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Examples:

real closed fields

the field of p-adic numbers

Algebraic closure of finite fields

NIP Structures

Fox-Pach-Sheffer-Suk-Zahl '12:
Suppose M is a structure, $\varphi(x; y)$ is a formula with VC-dimension d, $G = (V_1, V_2; E)$ is a $K_{k,k}$-free graph with $V_1 \subseteq M^{|x|}$, $V_2 \subseteq M^{|y|}$, and $E = \{(a, b) \in V_1 \times V_2 : \varphi(a, b)\}$. Then $|E| = O_k\left(n^{2-1/d}\right)$.

Janzer-Pohoata '20:
If we also have that $k \geq d \geq 3$, $|E| = o\left(n^{2-1/d}\right)$.

Distal Structures

A structure M is distal if every formula $\varphi(x; y)$ is distal, i.e., it admits a definable cell decomposition.

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Chernikov-Galvin-Starchenko ’16:
Suppose M is a structure, $\varphi(x; y)$ is a distal formula, $(V_1, V_2; E)$ is a $K_{k,k}$-free graph with $V_1 \subseteq M^{|x|}$, $V_2 \subseteq M^{|y|}$, and $E = \{(a, b) \in V_1 \times V_2 : \varphi(a, b)\}$.
Then $|E| = O_k\left(n^{2-c}\right)$ where c depends only on φ.

Generalizes Fox-Pach-Sheffer-Suk-Zahl.

Examples:
Real closed fields are distal.
Algebraic closure of finite fields are NIP but not distal.

Linear vs. nonlinear

The distinction between the point-rectangle incidence graph and the point-line incidence graph is a linear vs. nonlinear (or modular vs. nonmodular) distinction.

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In this setting, Theoerm 2 is a statement about graphs definable in o-minimal modular structures

Questions