Positroids, knots, and $q, t$-Catalan numbers

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Abstract. We relate the mixed Hodge structure on the cohomology of open positroid varieties (in particular, their Betti numbers over $\mathbb{C}$ and point counts over $\mathbb{F}_q$) to Khovanov–Rozansky homology of the associated links. We deduce that the mixed Hodge polynomials of top-dimensional open positroid varieties are given by rational $q, t$-Catalan numbers. Via the curious Lefschetz property, this implies the $q, t$-symmetry and unimodality properties of rational $q, t$-Catalan numbers. We show that the $q, t$-symmetry phenomenon is a manifestation of Koszul duality for category $O$, and discuss relations with equivariant derived categories of flag varieties, and open Richardson varieties.

Keywords: Positroid varieties, $q, t$-Catalan numbers, HOMFLY polynomial, Khovanov–Rozansky homology, mixed Hodge structure, equivariant cohomology, Koszul duality.

1 Introduction

The Poincaré polynomial of the complex Grassmannian $\text{Gr}(k, n)$ is well known to be given by the Gaussian polynomial $\binom{n}{k}_q$. The number of points of $\text{Gr}(k, n)$ over a finite field $\mathbb{F}_q$ is given by the same polynomial. The reason these two polynomials coincide is that the mixed Hodge structure on the cohomology of $\text{Gr}(k, n)$ is pure. The situation is different when one considers the top-dimensional positroid variety $\Pi^\circ_{k,n} \subset \text{Gr}(k, n)$, introduced in [15] building on the results of [21]. The space $\Pi^\circ_{k,n}$ consists of row spans of full rank $k \times n$ matrices whose cyclically consecutive maximal minors are all nonzero. It turns out that the Poincaré polynomial and the point count of $\Pi^\circ_{k,n}$ are given by two different $q$-analogs of (rational) Catalan numbers: one of our main results is that when $\gcd(k, n) = 1$, the Poincaré polynomial of $\Pi^\circ_{k,n}$ is given by $\sum_{P \in \text{Dyck}_{k,n}} q^{\text{area}(P)}$ while the number of points of $\Pi^\circ_{k,n}$ over $\mathbb{F}_q$ equals $\frac{1}{|n|} \binom{n}{k}_q$ up to a simple factor. The mixed Hodge structure on $H^\bullet(\Pi^\circ_{k,n})$ is non-pure, and we show that its bigraded Poincaré polynomial $\mathcal{P}(\Pi^\circ_{k,n}; q, t)$ coincides with the rational $q, t$-Catalan number $C_{k,n-k}(q, t)$ introduced...
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in [7, 18]. Our proof proceeds via relating both sides to Khovanov–Rozansky knot homology [14]. Our results apply more generally to arbitrary positroid and Richardson varieties.

2 Positroid varieties and Catalan numbers

Rational $q, t$-Catalan numbers. Let $a$ and $b$ be coprime positive integers. The rational $q, t$-Catalan number $C_{a,b}(q, t) \in \mathbb{N}[q, t]$ was introduced by Loehr–Warrington [18], generalizing the work of Garsia–Haiman [7]. It is defined as follows:

$$C_{a,b}(q, t) := \sum_{P \in \text{Dyck}_{a,b}} q^{\text{area}(P)} t^{\text{dinv}(P)}, \quad (2.1)$$

where $\text{Dyck}_{a,b}$ is the set of lattice paths $P$ inside a rectangle of height $a$ and width $b$ that stay above the diagonal, $\text{area}(P)$ is the number of unit squares fully contained between $P$ and the diagonal, and $\text{dinv}(P)$ is the number of pairs $(h, v)$ satisfying the following conditions: $h$ is a horizontal step of $P$, $v$ is a vertical step of $P$ that appears to the right of $h$, and there exists a line of slope $a/b$ (parallel to the diagonal) intersecting both $h$ and $v$. For example, we have

$$C_{3,5}(q, t) = q^4 + q^3 t + q^2 t^2 + q^2 t^2 + qt^3 + qt^2 + t^4, \quad (2.2)$$

as shown in Figure 1.

Positroid varieties in the Grassmannian. The Grassmannian $\text{Gr}(k, n)$ is the space of linear $k$-dimensional subspaces of $\mathbb{C}^n$. Alternatively, it can be identified with the space of full rank $k \times n$ matrices modulo row operations. Building on Postnikov’s cell decomposition [21] of its totally nonnegative part, Knutson–Lam–Speyer [15] constructed a stratification $\text{Gr}(k, n) = \bigsqcup_{f \in S_{k,n}} \Pi_f$ of the Grassmannian into (open) positroid varieties. Roughly speaking, each positroid variety $\Pi_f$ corresponds to a permutation $f \in S_n$ such

\[\text{More precisely, positroid varieties are in bijection with decorated permutations, where a decoration of } f \text{ is an arbitrary coloring of fixed points of } f \text{ into black and white colors. The actual set } S_{k,n}) = \bigsqcup_{f \in S_{k,n}} \Pi_f \text{ of the Grassmannian into (open) positroid varieties. Roughly speaking, each positroid variety } \Pi_f \text{ corresponds to a permutation } f \in S_n \text{ such} \]

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\[\text{The most interesting special case for us occurs when } f \text{ is a single cycle, where the decoration is trivial.} \]
that \(#\{1 \leq i \leq n \mid f(i) < i\} = k\); the set of such permutations is denoted by \(S_{k,n}\). For each \(f \in S_{k,n}\), the space \(\Pi^o_f\) is a smooth algebraic variety. Two basic questions one can ask about such a space are: what is the number of points in \(\Pi^o_f(\mathbb{F}_q)\) over a finite field \(\mathbb{F}_q\) with \(q\) elements, and what is the cohomology of \(\Pi^o_f\) considered as a variety over \(\mathbb{C}\)?

These two questions turn out to be closely related to each other through the work of Deligne [3] on mixed Hodge structures, explored in the case of cluster varieties in [16]. Since the work of Scott [23], positroid varieties have been expected to admit a natural cluster algebra structure arising from Postnikov diagrams. We recently proved this conjecture building on the results of [17, 20, 24].

**Theorem 2.1** ([6]). The coordinate ring of each positroid variety \(\Pi^o_f\) is isomorphic to the associated cluster algebra.

This result allows one to study \(\Pi^o_f\) as a cluster variety, in which case Deligne’s mixed Hodge structure can be explored using the machinery developed by Lam–Speyer [16]. The mixed Hodge structure endows the cohomology \(H^\bullet(\Pi^o_f)\) of \(\Pi^o_f\) with a second grading, and the suitably renormalized Poincaré polynomial \(\mathcal{P}(\Pi^o_f; q, t)\) of this bigraded vector space answers both of the above questions simultaneously:

**Theorem 2.2** ([16, 6]). For each \(f \in S_{k,n}\), the bigraded Poincaré polynomial \(\mathcal{P}(\Pi^o_f; q, t) \in \mathbb{N}[q^{\frac{1}{2}}, t^{\frac{1}{2}}]\) satisfies the following properties:

1. **\(q, t\)-symmetry:** \(\mathcal{P}(\Pi^o_f; q, t) = \mathcal{P}(\Pi^o_f; t, q)\);
2. **\(q, t\)-unimodality:** for each \(d\), the coefficients of \(\mathcal{P}(\Pi^o_f; q, t)\) at \(q^d t^0, q^{d-1} t^1, \ldots, q^0 t^d\) form a unimodal sequence;
3. \(\mathcal{P}(\Pi^o_f; q^2, 1)\) equals the Poincaré polynomial of \(\Pi^o_f\) (considered as a variety over \(\mathbb{C}\));
4. \(q^{\frac{1}{2} \dim \Pi^o_f} \cdot \mathcal{P}(\Pi^o_f; q, t)|_{t = -q^{-\frac{1}{2}}}\) equals the point count \(#\Pi^o_f(\mathbb{F}_q)\).

The positroid stratification contains a unique open stratum, the top-dimensional positroid variety \(\Pi^o_{k,n} := \Pi^o_{f_{k,n}}\) corresponding to the permutation \(f_{k,n} \in S_{k,n}\) sending \(i \mapsto i + k\) modulo \(n\) for all \(1 \leq i \leq n\). It is given by

\[\Pi^o_{k,n} := \{\text{RowSpan}(A) \mid A \in \text{Mat}(k,n; \mathbb{C}) : \Delta_{1,2,...,k}(A), \Delta_{2,3,...,k+1}(A), \ldots, \Delta_{n,1,...,k-1}(A) \neq 0\}\].

Here \(\Delta_I(A)\) is the maximal minor of \(A\) with column set \(I \subset \{1, 2, \ldots, n\}\), \(|I| = k\). We are ready to state (the most important special case of) our main result.

**Theorem 2.3.** Assume that \(\gcd(k, n) = 1\). Then

\[\mathcal{P}(\Pi^o_{k,n}; q, t) = \left(q^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^{n-1} C_{k,n-k}(q, t).\]
The equality (2.3) arises as a conjecture from the works [26, 25] and we thank Vivek Shende for drawing our attention to the conjecture. We generalize Theorem 2.3 to all positroid varieties in Theorem 4.2 below.

Our proof of Theorem 2.3 involves a number of ingredients, including Khovanov–Rozansky knot homology and equivariant perverse sheaves. The point count specialization \((t^1 = -q^{-\frac{1}{2}})\) turns out to require less advanced machinery. Namely, let us denote \([n]_q := 1 + q + \cdots + q^{n-1} \), \([n]_q! := [1]_q[2]_q \cdots [n]_q\), and \([n]_q^F := \frac{[n]_q!}{[k]_q^1[n-k]_q!}\). We give an elementary proof of the following special case of Theorem 2.3.

**Proposition 2.4.** Assume that \(\gcd(k, n) = 1\). Then \(\#\Pi^C_{k,n}(\mathbb{F}_q) = (q - 1)^{n-1} \cdot \frac{1}{[n]_q^F [k]_q}\). In other words, the probability that a uniformly random \(k\)-dimensional subspace of \((\mathbb{F}_q)^n\) belongs to \(\Pi^C_{k,n}(\mathbb{F}_q)\) equals \(\frac{(q-1)^n}{q^n - 1}\).

**Remark 2.5.** Surprisingly, this probability \(\frac{(q-1)^n}{q^n - 1}\) does not depend on \(k\). We do not have a combinatorial explanation for this phenomenon.

Our proof proceeds by associating a link \(\hat{\beta}_f\) to each positroid variety \(\Pi^C_f\) (Section 3) and then comparing the point count \(\#\Pi^C_f(\mathbb{F}_q)\) to the HOMFLY polynomial of \(\hat{\beta}_f\). The HOMFLY polynomial is categorified by Khovanov–Rozansky knot homology, and our proof of Theorem 2.3 may be considered a “categorification” of Proposition 2.4.

**Remark 2.6.** After discovering the proof of Proposition 2.4 via the HOMFLY polynomial, we found that it can also be deduced from the results of [26, 25]. Our proof is new and yields a generalization (Theorem 3.4) of Proposition 2.4 to arbitrary open positroid varieties.

**Torus action.** The appearance of the extra factor \((q^\frac{1}{2} + t^\frac{1}{2})^{n-1}\) in (2.3), as well as the condition \(\gcd(k, n) = 1\) are neatly explained by the torus action on \(\text{Gr}(k, n)\). Let \(T \cong (\mathbb{C}^*)^{n-1}\) be the quotient of the group of diagonal \(n \times n\) matrices by the group of scalar matrices. The group \(T\) acts on \(\text{Gr}(k, n)\) by rescaling the columns of \(k \times n\) matrices. This action leaves each positroid variety \(\Pi^C_f\) invariant. We say that \(T\) acts freely on \(\Pi^C_f\) if all non-identity elements of \(T\) act on \(\Pi^C_f\) without fixed points. It is straightforward to check that the action of \(T\) on \(\Pi^C_f\) is free if and only if the permutation \(f\) is a single cycle. Note that \(f_{k,n}\) is a single cycle if and only if \(\gcd(k, n) = 1\). We will see later in Definition 3.2 that \(f\) is a single cycle precisely when the associated link \(\hat{\beta}_f\) has a single component, i.e., is a knot.

Let \(\text{ncyc}(f)\) denote the number of cycles of \(f \in S_n\), and let \(S_{k,n}^{\text{ncyc}=1} := \{f \in S_{k,n} \mid \text{ncyc}(f) = 1\}\). For \(f \in S_{k,n}^{\text{ncyc}=1}\), the quotient \(\Pi^C_f / T\) is again a smooth cluster variety, and Theorem 2.2 applies to it. The associated bigraded Poincaré polynomials are related as
Table 1: The mixed Hodge table recording the dimensions of $H^k(p,p)(\Pi^o_{3,8}/T)$ for the cluster algebra of type $E_8$; see [16, Table 5]. The dimensions agree with the coefficients of $C_{3,5}(q,t)$; see Example 2.9.

<table>
<thead>
<tr>
<th>$H^k$</th>
<th>$H^0$</th>
<th>$H^1$</th>
<th>$H^2$</th>
<th>$H^3$</th>
<th>$H^4$</th>
<th>$H^5$</th>
<th>$H^6$</th>
<th>$H^7$</th>
<th>$H^8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k - p = 0$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<tr>
<td>$k - p = 1$</td>
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</tbody>
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Theorem 2.3, we find

$$\mathcal{P}(\Pi_{k,n}^o/T; q, t) = C_{k,n-k}(q, t).$$

Combining this with Theorem 2.2 has consequences for $q, t$-Catalan numbers and positroid varieties which can be stated in an elementary way. Let us denote $d_{k,n} := (k-1)(n-k-1) = \dim(\Pi_{k,n}^o/T)$.

Corollary 2.7. Assume that $\gcd(k, n) = 1$. We have:

(i) $q, t$-symmetry: $C_{k,n-k}(q, t) = C_{k,n-k}(t, q)$;
(ii) $q, t$-unimodality: for each $d$, the coefficients of $C_{k,n-k}(q, t)$ at $q^d t^0, q^{d-1} t^1, \ldots, q^0 t^d$ form a unimodal sequence;
(iii) the Poincaré polynomial of $\Pi_{k,n}^o/T$ is given by

$$\sum_d q^d \dim H^{d_{k,n}-d}(\Pi_{k,n}^o/T) = C_{k,n-k}(q, 1) = \sum_{\eta \in \text{Dyck}_{k,n-k}} q^{\text{area}(\eta)};$$

(iv) the number of $\mathbb{F}_q$-points of $\Pi_{k,n}^o/T$ is given by

$$\#(\Pi_{k,n}^o/T)(\mathbb{F}_q) = \frac{1}{[n]_q} \left[ n \atop k \right]_q = q^{\frac{1}{2} d_{k,n}} \cdot C_{k,n-k}(q, 1/q).$$

Remark 2.8. When $a = n$ and $b = n+1$, $C_{a,b}(q, t)$ recovers the famous $q, t$-Catalan numbers $C_n(q, t)$ of Garsia and Haiman [7]. The fact that $C_n(q, t)$ is $q, t$-symmetric and $q, t$-unimodal follows from the results of Haiman [10, 9]. For arbitrary $a, b$, the $q, t$-symmetry property follows from the celebrated recent proof of the rational shuffle conjecture [19]. To our knowledge, $q, t$-unimodality of $C_{k,n-k}(q, t)$ is a new result.

Example 2.9. For $k = 3, n = 8$, the coordinate ring of $\Pi_{k,n}^o/T$ is a cluster algebra of type $E_8$ (with no frozen variables). The associated mixed Hodge table is given in Table 1; see [16, Table 5]. The grading conventions are chosen so that the first row contributes $q^4 + q^3 t^3 + \cdots$ to the Poincaré polynomial of $\Pi_{3,8}^o/T$.
For each element \( u \in S_n \), let \( \hat{\beta}(u) \) denote the corresponding braid, obtained by choosing a reduced word \( u = s_{i_1}s_{i_2}\cdots s_{i(\ell)} \) for \( u \) and then replacing each \( s_i \) with \( \sigma_i \).

**Definition 3.2.** For \( f \in S_{k,n} \) and \( v \leq w \in S_n \) as in Proposition 3.1, let \( \hat{\beta}_f := \hat{\beta}(w) \cdot \hat{\beta}(v)^{-1} \). We refer to the closure \( \hat{\beta}_f \) as the link associated to \( f \). See Figure 2 for an example.

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2If \( f \) has fixed points, the pair \((v,w)\) must be compatible with the decoration of \( f \).
Observe that \( \hat{\beta}_f \) is a knot (i.e., has one connected component) if and only if \( f \in S_{k,n}^{\text{ncyc}=1} \). We note that two other (more complicated) ways of assigning a Legendrian or a transverse link to a positroid variety have appeared recently in [25, 5].

The HOMFLY polynomial \( P(L) = P(L; a, z) \) of an (oriented) link \( L \) is defined by a skein relation \( aP(L_+) - a^{-1}P(L_-) = zP(L_0) \) and \( P(\bigcirc) = 1 \), where \( \bigcirc \) denotes the unknot and \( L_+, L_-, L_0 \) are three links whose planar diagrams locally differ as follows:

\[
\begin{array}{ccc}
\text{\( L_+ \)} & \text{\( L_- \)} & \text{\( L_0 \)} \\
\end{array}
\]

**Example 3.3.** For \( n = 2 \), we may take \( L_+ \) to be the closure of \( \sigma_1 \), in which case \( L_- \) is the closure of \( \sigma_1^{-1} \) and \( L_0 = \bigcirc \) is the 2-component unlink. Applying the skein relation, we find \( P(L_0) = \frac{a - a^{-1}}{z} \).

Surprisingly, the HOMFLY polynomial computes the number of \( \mathbb{F}_q \)-points of any positroid variety.

**Theorem 3.4.** For all \( f \in S_{k,n} \), let \( P_f^{\text{top}}(q) \) be obtained from the top \( a \)-degree term of \( P(\hat{\beta}_f; a, z) \) by substituting \( a := q^{\frac{1}{2}} \) and \( z := q^{\frac{1}{2}} - q^{-\frac{1}{2}} \). Then \( \# \Pi_f(\mathbb{F}_q) = (q - 1)^{n-1} \cdot P_f^{\text{top}}(q) \).

**Remark 3.5.** When \( \gcd(k, n) = 1 \), we have \( f_{k,n} \in S_{k,n}^{\text{ncyc}=1} \), and the associated knot \( \hat{\beta}_{f_{k,n}} \) is the \((k, n-k)-\text{torus knot}\); see Figure 2(right). The value of \( P(\hat{\beta}_{f_{k,n}}; a, z) \) was computed in [11], and its relationship with Catalan numbers was clarified in [8]. Thus Proposition 2.4 follows from Theorem 3.4 as a direct corollary.

**Example 3.6.** For \( k = 3, n = 8 \), one calculates (for instance, using Sage\(^3\)) that the top \( a \)-degree term of \( P(\hat{\beta}_{f_{k,n}}; a, z) \) equals \( \frac{z^8 + 3 z^6 + 21 z^4 + 21 z^2 + 7}{a^8} \). Substituting \( a := q^{\frac{1}{2}} \) and \( z := q^{\frac{1}{2}} - q^{-\frac{1}{2}} \), we get

\[
P_f^{\text{top}}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1 = q^4 \cdot C_{3,5}(q, 1/q).
\]

This agrees with (2.6) and Theorem 3.4.

**Links associated to Richardson varieties.** By Proposition 3.1, positroid varieties correspond to pairs \( v \leq w \) of permutations such that \( w \) is \( k \)-Grassmannian. There is a more general class of (open) Richardson varieties \( R_{v,w} \), indexed by all pairs \( v \leq w \in S_n \), and the majority of the above story generalizes to this setting. The varieties \( R_{v,w} \) form a stratification of the complete flag variety \( \text{Fl}_n(\mathbb{C}) \). For a permutation \( f \in S_{k,n} \) corresponding to a pair \( v \leq w \in S_n \) via Proposition 3.1, the projection map \( \text{Fl}_n(\mathbb{C}) \to \text{Gr}(k,n) \) restricts

to an isomorphism $\Pi_f^\circ \cong R_{v,w}^\circ$. Thus positroid varieties are special cases of Richardson varieties.

Now, let $G$ be a complex semisimple algebraic group of adjoint type, and choose a pair $B, B_\perp \subset G$ of opposite Borel subgroups. Let $T := B \cap B_\perp$ be the maximal torus and $W := N_G(T)/T$ the associated Weyl group. For the case $G = \text{SL}_n(C)$, we have $W = S_n$, the subgroups $B, B_\perp \subset G$ consist of upper and lower triangular matrices, and $T \cong (C^*)^{n-1}$ is the group of diagonal matrices modulo scalar matrices.\footnote{Note that $G$ is assumed to be of \textit{adjoint type}, thus in type $A$ we should have $G = \text{PGL}_n(C)$. However, we choose to work with $G = \text{SL}_n(C)$ for simplicity.} We have Bruhat decompositions $G = \bigsqcup_{w \in W} BwB = \bigsqcup_{v \in W} B_\perp vB$, and the intersection $BwB \cap B_\perp vB$ is nonempty if and only if $v \leq w$ in the Bruhat order on $W$. For $v \leq w$, we denote by $R_{v,w}^\circ := (BwB \cap B_\perp vB)/B$ an open Richardson variety inside the complete flag variety $G/B$. For $G = \text{SL}_n(C)$, $R_{v,w}^\circ$ is the subset of $G/B = \text{Fl}_n(C)$ obtained by specifying the dimensions of the intersections of a given flag with a given coordinate flag and its opposite coordinate flag.

In the case $G = \text{SL}_n(C)$, one can similarly associate a braid $\beta_{v,w} := \beta(w) \cdot \beta(v)^{-1}$ to any pair $v \leq w$ and consider its closure $\hat{\beta}_{v,w}$. We refer to the links of the form $\hat{\beta}_{v,w}$ as Richardson links. The point count $\#R_{v,w}^\circ(F_q)$ is given by the Kazhdan–Lusztig $R$-polynomial \cite{12, 13}, and both the statement and the proof of Theorem 3.4 generalize to this setting.

## 4 Main results

All of the above results are actually special cases of a single statement which applies to arbitrary Richardson varieties. This includes all positroid varieties $\Pi_f^\circ$ for $f \in S_{k,n}$, where $\text{ncyc}(f)$ can be arbitrary. As a warm up, we start with the non-equivariant version.

**Ordinary cohomology.** Let $\mathfrak{h} := \text{Lie}(T)$ be the Cartan subalgebra of $\text{Lie}(G)$ corresponding to $T$, and denote $R := C[\mathfrak{h}^*]$. For $G = \text{SL}_n(C)$, $R = C[y_1, \ldots, y_{n-1}]$ is the polynomial ring. Since $W$ is a Coxeter group, we can consider the category $\text{SBim}$ of Soergel bimodules. Each element $B \in \text{SBim}$ is a graded $R$-bimodule, and we will be interested in its $R$-invariants, which by definition form the zeroth Hochschild cohomology $HH^0(B)$ of $B$. Denote $HH^0_C(B) := HH^0(B) \otimes_R C$, where $C = R/(\mathfrak{h}^*)$ is an $R$-module on which $\mathfrak{h}^*$ acts by 0. While the functor $HH^0$ involves Soergel bimodules, the functor $HH^0_C$ involves Soergel modules instead.

To any element $w \in W$, Rouquier \cite{22} associates a cochain complex $F^\bullet(w)$ of Soergel bimodules. He also associates another complex $F^\bullet(w)^{-1}$ such that their tensor product $F^\bullet(w) \otimes_R F^\bullet(w)^{-1}$ is homotopic to the identity. For a braid $\beta_{v,w} = \beta(w) \cdot \beta(v)^{-1}$, we set $F^\bullet_{v,w} := F^\bullet(w) \otimes_R F^\bullet(v)^{-1}$. Applying the functor $HH^\circ_C$ to each term of this complex yields
a complex $HH^0_C(F^*_{v,w})$ of graded $R$-modules. Taking its cohomology $HHH^0_C(F^*_{v,w}) := H^*(HH^0_C(F^*_{v,w}))$, we get a bigraded $R$-module. We denote by $H^k(p)(HH^0_C(F^*_{v,w}))$ the polynomial degree $2p$ part of $H^k(HH^0_C(F^*_{v,w}))$. By convention, the elements of $h^* \subset R$ are assumed to have polynomial degree 2. On the other hand, let us denote by $H^k(p,p)(R^\circ_{v,w})$ the $(p,p)$ part of the mixed Hodge structure on $H^k(R^\circ_{v,w})$. See Table 1 for an example.

**Theorem 4.1.** For all $v \leq w \in W$ and $k, p \in \mathbb{Z}$, we have

$$\dim H^k(p,p)(R^\circ_{v,w}) = \dim H^{-k}(p)(HH^0_C(F^*_{v,w})).$$  \hspace{1cm} (4.1)

**Equivariant cohomology.** The spaces $HHH^0_C(F^*_{v,w})$ and $HHH^0_C(F^*_{v,w})$ are closely related. By Theorem 4.1, $HHH^0_C(F^*_{v,w})$ yields the cohomology of $R^\circ_{v,w}$. It turns out that $HHH^0_C(F^*_{v,w})$ yields the torus-equivariant cohomology of $R^\circ_{v,w}$.

The algebraic torus $T$ acts on each Richardson variety $R^\circ_{v,w}$, and thus we can consider its $T$-equivariant cohomology with compact support, denoted $H^*_T(R^\circ_{v,w})$). It is equipped with an action of the ring $H^*_T(pt) \cong R$. Similarly to the positroid case, $H^*_T(R^\circ_{v,w})$ admits a second grading via the mixed Hodge structure and is therefore a bigraded $R$-module.

**Theorem 4.2.** For all $v \leq w \in W$, we have an isomorphism of bigraded $R$-modules

$$H^*_T(R^\circ_{v,w}) \cong HHH^0_C(F^*_{v,w}).$$  \hspace{1cm} (4.2)

It restricts to a vector space isomorphism $H^\ell_{T,V}(R^\circ_{v,w}) \cong H^k(p)(HH^0_C(F^*_{v,w}))$ for each $k, p \in \mathbb{Z}$, where $\ell_{v,w} = \ell(w) - \ell(v) = \dim R^\circ_{v,w}$.

**Koszul duality and $q,t$-symmetry.** One can encode the dimensions of bigraded components of $HHH^0_C(F^*_{v,w})$, resp., $HHH^0_C(F^*_{v,w})$ in a two-variable polynomial $P_{KR}(\beta; q, t)$, resp., $P_{KR}(\beta; q, t)$.\(^5\) For any $f \in S_{k,n}^{\text{cyc}}$, the positroid variety $\Pi_f^s / T$ is a cluster variety [6], so the polynomial $P_{KR}(\beta_f; q, t)$ satisfies the properties (i)–(iv) listed in Theorem 2.2 by the results of [16]. In particular, it is $q,t$-symmetric and $q,t$-unimodal.

Richardson varieties are not yet known to admit cluster structures (see [17]), in particular, it does not follow from Theorem 2.2 that $P_{KR}(\hat{\beta}_{v,w}; q, t)$ is $q,t$-symmetric for arbitrary $v \leq w \in S_n$. We show that the $q,t$-symmetry phenomenon for such links is a manifestation of Koszul duality for mixed perverse sheaves [1, 2].

**Theorem 4.3** (Koszul duality). For any $v \leq w \in S_n$, we have

$$P_{KR}(\beta_{v,w}; q, t) = P_{KR}(\hat{\beta}_{v,w}; t, q).$$

If $\hat{\beta}_{v,w}$ is a knot then it follows that $P_{KR}(\hat{\beta}_{v,w}; q, t) = P_{KR}(\hat{\beta}_{v,w}; t, q)$. This gives a new proof of the $q,t$-symmetry of $C_{k,n-k}(q,t)$ for $\gcd(k,n) = 1$.

\(^5\)The polynomial $P_{KR}(\beta; q, t)$ the top a-degree coefficient of Khovanov–Rozansky homology [14] of $\hat{\beta}$. 

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*Positroids, knots, and $q,t$-Catalan numbers*
Figure 3: The sets Deo_{f,k,n}^{\text{max}} and Dyck_{k,n-k} have the same cardinality by (5.1). Compare with Figure 1.

5 Catalan numbers associated to positroid varieties

An important combinatorial consequence of our results is an embedding of rational q,t-Catalan numbers C_{k,n-k}(q,t) into a family of q,t-polynomials P(\Pi_f^\circ/T; q,t) ∈ \mathbb{N}[q^{\frac{1}{2}}, t^{\frac{1}{2}}] indexed by permutations f ∈ S_{k,n}^{\text{ncyc}} (all of which are q,t-symmetric and q,t-unimodal).

Definition 5.1. For f ∈ S_{k,n}^{\text{ncyc}} = 1, define the f-Catalan number C_f ∈ \mathbb{Z} as the q = 1 specialization of the point count polynomial #(\Pi_f^\circ/T)(F_q).

Let us focus on the case f = f_{k,n} with gcd(k,n) = 1. We will show that C_{f_{k,n}} = C_{k,n-k}(1,1) = #Dyck_{k,(n-k)} counts certain pipe dreams inside a k × (n − k) rectangle. This interpretation extends to arbitrary f ∈ S_{k,n} in a straightforward fashion.

Definition 5.2. Let gcd(k,n) = 1. A maximal f_{k,n}-Deogram (short for Deodhar diagram) is a way of placing n − 1 elbows in a k × (n − k) rectangle and filling the rest with crossings so that (i) the resulting permutation obtained by following the paths is the identity, and (ii) the following distinguished condition [4] is satisfied: if any two paths have crossed an odd number of times, they cannot form an elbow. See Figure 3 for an example.

Denote the set of maximal f_{k,n}-Deograms by Deo_{f_{k,n}}^{\text{max}}. It follows by combining our results with [4] that C_{f_{k,n}} equals the number of maximal f_{k,n}-Deograms:

C_{f_{k,n}} = \# \text{Deo}_{f_{k,n}}^{\text{max}}.  \tag{5.1}

An analogous result holds for arbitrary f ∈ S_{k,n}. It would be interesting to give a bijective proof of (5.1).

Problem 5.3. Find a bijection between Deo_{f_{k,n}}^{\text{max}} and Dyck_{k,(n-k)} for the case gcd(k,n) = 1.

For the case n = 2k + 1 of the standard Catalan numbers, the maximal f_{k,n}-Deograms are easily seen (exercise) to be in bijection with non-crossing alternating trees on n + 1 vertices. A recursive proof of (5.1) for the case n = dk ± 1 (d ≥ 2) was found by David Speyer. We were able to find a recursive proof of (5.1) for arbitrary k,n. The problem of finding a bijective proof remains open.
Acknowledgements

We are grateful to David Speyer for his contributions to this project at the early stages. We also thank Pramod Achar, Sergey Fomin, Alexei Oblomkov, and Vivek Shende for stimulating discussions.

References


