BRAID VARIETY CLUSTER STRUCTURES, II: GENERAL TYPE

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ABSTRACT. We show that braid varieties for any complex simple algebraic group G are cluster varieties. This includes open Richardson varieties inside the flag variety G/B.

1. Introduction

This is one of two papers concerned with the construction of cluster structures on braid varieties. In the present paper, we work in the setting of a general simple algebraic group G and construct cluster seeds using algebraic geometry. In the companion paper [GLSBS22], joint with David Speyer, we give an alternative proof in the special case G = SL_n, using the combinatorics of plabic graphs and surfaces. The current work is logically independent of [GLSBS22], which, however, ultimately produces the same cluster structure in the case G = SL_n.

Let G be a complex, simple, simply-connected algebraic group, B_± opposing Borel subgroups, U_± their unipotent radicals, H := B_+ ∩ B_- the torus, I the vertex set of the Dynkin diagram, W the Weyl group with simple generators s_i, i ∈ I, and denote by w the lift of w ∈ W to G as in (2.1). Let w_0 ∈ W denote the longest element and i → i^* the action of w_0 on I. Let α_i, α_i^∨, ω_i for i ∈ I denote the simple roots, simple coroots, fundamental weights, respectively, and let A = (α_i ∨ α_j^∨)_{i,j∈I} be the Cartan matrix given by a_ij := ⟨α_i, α_j^∨⟩. Denote d_i := 2/(α_i, α_i) so that d_i a_ij = d_j a_{ji}.

1.1. Double braid varieties. A double braid word β = i_1 i_2 ... i_m is a word in the alphabet ±1. For i ∈ I, we set (−i)^* := −i*. For i ∈ ±I, define

\[ s_i^± := \begin{cases} s_i, & \text{if } i > 0, \\ \text{id}, & \text{if } i < 0, \end{cases} \quad s_i^- := \begin{cases} \text{id}, & \text{if } i > 0, \\ s_i, & \text{if } i < 0. \end{cases} \]

A weighted flag is an element F = gU_+ ∈ G/U_+. Two weighted flags (F, F') are weakly w-related (resp., strictly w-related) if there exist g ∈ G and h ∈ H (resp., g ∈ G) such that (gF, gF') = (U_+, hwU_+) (resp., (gF, gF') = (U_+, wU_+)). We write this as \( F \sim w F' \) (resp., \( F \sim w' F' \)).

Suppose that the Demazure product of β is w_0; see (2.4). We consider tuples \( (X_0, Y_0) \) of weighted flags satisfying the relative position conditions

\[
\begin{align*}
X_0 & \xleftarrow{s_{i_1}^+} X_1 \xleftarrow{s_{i_2}^+} \cdots \xleftarrow{s_{i_m}^+} X_m \\
Y_0 & \xrightarrow{s_{i_1}^-} Y_1 \xrightarrow{s_{i_2}^-} \cdots \xrightarrow{s_{i_m}^-} Y_m.
\end{align*}
\]

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The group $G$ acts on such tuples by acting on each individual weighted flag, and this action is free (Remark 2.15). The double braid variety $\hat{R}_\beta$ is a complex, affine, irreducible variety defined as the quotient modulo the $G$-action of the configuration space of tuples $(X, J)$ satisfying (1.2).

Double braid varieties include open Richardson varieties [Rie98], [KLS14], [Lec16], open positroid varieties [KLS13], double Bott-Samelson cells [SW21], the strata in [WY07], and the braid varieties of [Mc19], [CGGS20]; see [GLSBS22] for further discussion. For each $\beta$, we construct a cluster seed $\Sigma_\beta$. Our main result settles conjectures of [Lec16], [CGGS21] and generalizes work of [BFZ05], [GL19], [Ing19], [SW21].

**Theorem 1.1.** The coordinate ring of $\hat{R}_\beta$ is isomorphic to the cluster algebra $A_\beta = A(\Sigma_\beta)$.

It would be interesting to compare our construction and the cluster-categorical approach of [GLS06], [BIRS09], [Lec16], [Mén22], [CK22], as was done in type A in [SSB22].

**Remark 1.2.** At the final stages of completing our construction, we learned that a cluster structure for braid varieties was independently announced in a recent preprint [CGGG+22]. We thank the authors of [CGGG+22] for updating us on their progress. It would be interesting to understand the relation between our approach and their Legendrian-geometric viewpoint.

One application of Theorem 1.1 is that a curious Lefschetz theorem (see [HRV08], [LS16], [GLSBS22], Theorem 10.1), and [GL20], Theorem 1.5) holds for double braid varieties; see Theorem 6.8. In the case of open Richardson varieties, this implies that the doubly graded extension group $\text{Ext}_G(M_w, M_v)$ of two Verma modules in category $O$ satisfies curious Lefschetz; cf. [GL20], Section 1.11.

**1.2. The seed.** We give an informal summary of our construction; see Section 2 for details and [GLSBS22], [Gal23] for examples. We introduce an open dense algebraic torus $T_\beta \subset \hat{R}_\beta$ called the Deodhar torus, so named for its relation to the Deodhar decomposition of Richardson varieties [Deo85], [MR04]. It is defined by requiring the weighted flags $X_c, Y_c$ to be weakly $w_c$-related, where $w_c \in W$ is maximal possible subject to (1.2) (for each $c = 0, 1, ..., m$). The complement $\hat{R}_\beta \setminus T_\beta$ is a union of irreducible mutable Deodhar hypersurfaces $\{V_c | c \in J_\beta^\text{mut}\}$. We define a partial compactification of $\hat{R}_\beta$ so that the complement of $T_\beta$ in it also includes frozen Deodhar hypersurfaces $\{V_c | c \in J_\beta^\text{fro}\}$. We let $J_\beta := J_\beta^\text{fro} \sqcup J_\beta^\text{mut}$. The following definition, suggested by David Speyer, is key to our approach.

**Proposition-Definition 1.3.** For $c \in J_\beta$, define the cluster variable $x_c$ to be the unique character of $T_\beta$ that vanishes to order one on $V_c$ and has neither a pole nor a zero on $V_e$ for $e \in J_\beta \setminus \{c\}$. We denote the cluster by $x_\beta = \{x_c | c \in J_\beta\}$.

We show that the cluster variables form a basis of the character lattice of $T_\beta$, and that they extend to regular functions on $\hat{R}_\beta$. We consider a 2-form $\omega_\beta$ on $\hat{R}_\beta$, defined in terms of certain generalized minors as a sum of local contributions for each letter of $\beta$. This is similar in spirit to [BFZ05], [FG06], [Ing19], [SW21]. We introduce integers $d = (d_i)_{i \in J_\beta}$ and expand $\omega_\beta$ in the basis of cluster variables:

$$\omega_\beta = \sum_{c, e \in J_\beta, c \leq e} d_c B_{ce} d\log x_c \wedge d\log x_e = \sum_{c, e \in J_\beta, c \leq e} d_c \tilde{B}_{ce} d\log x_c \wedge d\log x_e.$$  

The coefficients $\tilde{B}_{ce}$ define a $J_\beta \times J_\beta^\text{mut}$ integer matrix $\tilde{B} : = (\tilde{B}_{ce})$. The principal $J_\beta^\text{mut} \times J_\beta^\text{mut}$ part of the matrix $\tilde{B}$ is skew-symmetrizable, with symmetrizer $\text{diag}(d_i, c \in J_\beta^\text{mut})$. Therefore $\Sigma_\beta := (x_\beta, \tilde{B})$ is a seed of a cluster algebra $A(\Sigma_\beta)$ of geometric type. The content of Theorem 1.1 is that $A(\Sigma_\beta) = C[\hat{R}_\beta]$.

**1.3. Overview of the proof.** The proof that $\Sigma_\beta$ provides a cluster structure for $\hat{R}_\beta$ is obtained in two steps. First, we develop a notion of deletion-contraction for cluster seeds in Section 3.3. We apply this in the case when $\beta = ii\beta'$ starts with a repeated letter; the braids $i\beta'$ and $\beta'$ correspond to deletion and contraction, respectively. Using deletion-contraction, we deduce that if Theorem 1.1 holds for $i\beta'$ and $\beta'$ then it holds for $\beta$. 

Second, in Section 2.2, we describe moves $\beta \sim \beta'$ on double braid words that induce natural isomorphisms $\hat{R}_\beta \cong \hat{R}_{\beta'}$. In Theorem 4.2, we show that the isomorphisms $\hat{R}_\beta \cong \hat{R}_{\beta'}$ give rise to a sequence of mutations connecting the corresponding seeds $\Sigma_\beta$ and $\Sigma_{\beta'}$. Theorem 1.1 then follows by induction. In Section 4, we prove Theorem 1.2 in the simply-laced case (i.e., for $G$ of type $A$, $D$, $E$); the seeds $\Sigma_\beta$ and $\Sigma_{\beta'}$ either coincide or are related by a single mutation. The proof of Theorem 4.2 in the multiply-laced case is achieved via folding in Sections 5 and 6; the seeds $\Sigma_\beta$ and $\Sigma_{\beta'}$ are related by a sequence of mutations. This generalizes a result of Fock and Goncharov [FG06, Theorem 3.5]. Finally, in Section 7, we give an algorithm, implemented in [Gal23], for computing our seeds using only root-system combinatorics.

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2. Deodhar geometry

We discuss the geometry of the double braid variety $\hat{R}_\beta$ with the goal of defining a cluster seed on it. The ingredients of a cluster seed were outlined in Section 1.2. In Section 2.3, we construct a Deodhar torus $T_\beta \subset \hat{R}_\beta$. In Section 2.7, we introduce a family $x_\beta = \{x_c\}_{c \in J_\beta}$ of cluster variables and show that they are regular functions on $\hat{R}_\beta$. Finally, in Section 2.8, we introduce a 2-form $\omega_\beta$ on $T_\beta$ from which the $\hat{B}$-matrix can be extracted via (1.3).

2.1. Background. For each $i \in I$, we fix a group homomorphism

$$\phi_i : \text{SL}_2 \to G, \quad \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) \mapsto x_i(t), \quad \left( \begin{array}{cc} 1 & 0 \\ t & 1 \end{array} \right) \mapsto y_i(t),$$

where $x_i(t), y_i(t)$ are the exponentiated Chevalley generators. The data $(H, B_+, B_-, x_i, y_i; i \in I)$ is a pinning of $G$; see [Lus94, Section 1.1].

Let $\Phi$ be the root system of $G$, with positive roots $\Phi^+$ corresponding to $B_+$. Let $X^*(H) := \text{Hom}(H, \mathbb{C}^*)$ be the character lattice of $H$ and $X_*(H) := \text{Hom}(\mathbb{C}^*, H)$ be the cocharacter lattice of $H$. Let $\{\alpha_i\}_{i \in I} \subset X^*(H)$ (resp., $\{\alpha_i^\vee\}_{i \in I} \subset X_*(H)$, $\{\omega_i\}_{i \in I} \subset X^*(H)$) be the simple roots (resp., simple coroots, fundamental weights) of $\Phi^+$. We have a natural pairing $\langle \cdot, \cdot \rangle : X^*(H) \times X_*(H) \to \mathbb{Z}$ satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ and $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$, where $A = (a_{ij})_{i,j \in I}$ is the Cartan matrix of $G$.

Let the Weyl group $W$ have simple generators $\{s_i\}_{i \in I}$, length function $\ell(\cdot)$, and identity $id \in W$. For $i \in I$, we set

$$s_i = \tilde{s}_i : = \phi_i \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \quad s_i^{-1} = \tilde{s}_i^\prime : = \phi_i \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right).$$

For a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$, where $l = \ell(w)$, we set

$$w = \overline{w} := s_{i_1} s_{i_2} \cdots s_{i_l}, \quad \overline{w} := s_{i_l} s_{i_{l-1}} \cdots s_{i_1}.$$

The resulting product does not depend on the choice of the reduced expression. For $u \in W$ and $h \in H$, we set $u \cdot h := uh \overline{w}^{-1} = \overline{u} h \overline{w}^{-1} = \overline{u} h \overline{w}^{-1}$. We also consider elements

$$z_i(t) := \phi_i \left( \begin{array}{cc} t & 1 \\ 0 & -1 \end{array} \right) = x_i(t) \tilde{s}_i, \quad \tilde{z}_i(t) := \phi_i \left( \begin{array}{cc} t & -1 \\ 1 & 0 \end{array} \right) = x_i(-t) \tilde{s}_i^{-1}.$$

For each $w \in W$, it is well known that the multiplication map gives rise to an isomorphism

$$(\overline{w}^{-1} U_+ \omega \cap U_-) \times (\overline{w}^{-1} U_+ \omega \cap U_+) \cong \overline{w}^{-1} U_+ \overline{w}.$$

2.2. Weighted flags. Recall from Section 1.1 that a weighted flag is an element $F = gu_+ \in G/U_+$. Associated to a weighted flag $F$ is the flag $\pi(F) = gB_+$, the image of $F$ in $G/B_+$.

The following elementary facts can be found in e.g. [SW21, Appendix]; see also [GLSBS22, Section 6.2].

Lemma 2.1. Let $F, F', F''$ be weighted flags. Suppose $v, w \in W$ with $\ell(vw) = \ell(v) + \ell(w)$.

1. $F \overset{id}{\rightarrow} F'$ if and only if $F = F'$.
Lemma 2.2. Suppose $F \xrightarrow{g} F'$ and say $F = gU_+$. Then there exists a unique $t \in \mathbb{C}$ such that $F' = g_z(t)U_+$. Similarly, if $F' = g'U_+$, there exists a unique $t' \in \mathbb{C}$ such that $F = g'Z(t')U_+$. The maps $(g,F') \mapsto t$ and $(g',F') \mapsto t'$ are regular on the appropriate subvarieties of $G \times G/U_+$.

Lemma 2.3. Suppose $F \xrightarrow{g} gU_+ \xrightarrow{g_z} g_{z,t}U_+$ and $F \xrightarrow{g} gZ(t)U_+$. If $\nu_\ast > v$, then $w = v\sigma_i$ for all $t \in \mathbb{C}$. If $wT < v$, then there exists $t \in \mathbb{C}$ such that $w = v\sigma_i$ for $t = t'$ and $w = v$ for $t$ in $C \setminus \{t\}$.

2.3. Subexpressions and the Deodhar torus. Fix $\beta = i_1i_2 \ldots i_m \in (\pm I)^m$. Write $[m] = \{1,2,\ldots,m\}$ and $[0,m] = \{0,1,\ldots,m\}$. Recall the notation $s_i^+ = \cdots$ from (1.1).

Given two elements $u, v \in W$ with $u \leq v$ in the Bruhat order, we write $\max(u,v) := v$ and $\min(u,v) := u$. We also define $u \triangleright s_i := \max(u,s_i)$. The Demazure product of $\beta$ is defined by

$$\pi(\beta) := s_{i_1}^+ \cdots s_{i_m}^+ \bigcirc s_{i_{m-1}}^- \cdots s_{i_1}^- \bigcirc s_{i_1}^- \cdots s_{i_m}^- \in W.$$ 

From now on, we assume that $\pi(\beta) = w_0$.

Definition 2.4. A $w_m$-subexpression of $\beta$ is a $w_m$-sequence $\u = (u_0u_1\ldots u_m) \in W^{m+1}$ such that $u_0 = 1$, $u_m = w_0$, and such that for each $c \in [m]$, we have either $u_{c-1} = u_c$ or $u_{c-1} = s_i^-u_c s_i^+$. Since $\pi(\beta) = w_0$ there exists a unique "rightmost" subexpression, called the positive distinguished subexpression. It is given by $u_m := w_0$ and $u_{c-1} := \min(u_c, s_i^-u_c s_i^+)$ for all $c = m, m-1, \ldots, 1$.

We also define $u_c := u_0u_c$ for $c \in [0,m]$ and $w = w_0u := (u_0u_1\ldots u_m)$. Note that $w_0 = w_0$. We set $\beta' := \{c \in [m] | u_c = u_{c-1}\}$. We refer to the indices in $\beta'$ as solid crossings and to the indices in $[m] \setminus \beta'$ as hollow crossings. We denote $d(\beta) := m - \ell(w_0) = |\beta'|$.

Definition 2.5. The Deodhar torus $T_\beta \subset \hat{R}_\beta$ consists of all tuples $(X_*, Y_*)$ satisfying

$$X_c \xleftarrow{w_c} Y_c \quad \text{for } c \in [0,m].$$

2.4. Torus-valued functions. Given $(X_*, Y_*) \in \hat{R}_\beta$, let $Z_c := Y_c^{-1}X_c \in U_+ \setminus G/U_+$. Abusing notation, we use double cosets $Z_c \in U_+ \setminus G/U_+$ interchangeably with their representatives in $G$. For $(X_*, Y_*) \in T_\beta$, $Z_c$ belongs to the Bruhat cell $X_{w_c} := B^+_1w_cB^+_1 = U_+w_cH_{U_+}$ of $G$. There exist unique elements $h_c^+, h_c^- \in H$ satisfying

$$Z_c \in U_+ \hat{w}_c h_c^+ U_+, \quad Z_c \in U_+ \hat{w}_c h_c^- U_+, \quad \text{thus,} \quad h_c^+ = \hat{w}_c h_c^+.$$ 

The third statement follows from the first two since $w_c = \hat{w}_c \hat{w}_c^-$ and $w_c \cdot h_c^+ = \hat{w}_c h_c^+ \hat{w}_c^-$.

The map $(X_*, Y_*) \mapsto h_\alpha^+ = \alpha^+_{|\beta|}$ is a rational $H$-valued function on $\hat{R}_\beta$, regular on $T_\beta$.

Lemma 2.6. There exist rational functions $(t_c)_{c \in \beta}$ on $\hat{R}_\beta$ such that for $c \in [m]$,

$$h_{c-1}^+ = \begin{cases} s_i^+ \cdot h_c^+ & \text{if } c \text{ is hollow, } i_c \in I; \\ \alpha_{i_c}^+(t_c) h_c^+ & \text{if } c \text{ is solid, } i_c \in I; \end{cases}$$

$$h_{c-1}^- = \begin{cases} s_i^- \cdot h_c^- & \text{if } c \text{ is hollow, } i_c \in -I; \\ \alpha_{i_c}^-(t_c) h_c^- & \text{if } c \text{ is solid, } i_c \in -I. \end{cases}$$

Proof. Fix $(X_*, Y_*) \in T_\beta$. For $c \in \beta$, define $t_c$ to be such that if $Z_c = \hat{w}_c h_\alpha^+ = \overline{w_c} h_\alpha^- \overline{w_c}$ then $Z_{c-1} = Z_c s_{i_c}(t_c)$ if $i_c \in I$ and $Z_{c-1} = \overline{z_{i_c}(t_c)}^{-1} Z_c$ if $i_c \in -I$; see Lemma 2.2. The following identities in $G$ can be checked inside $SL_2$:

$$x_i(t) s_i = y_i(1/t) \alpha_i^+(t) x_i(1/t) \quad \text{and} \quad s_i x_i(t) = x_i(-1/t) \alpha_i^+(t) y_i(1/t).$$

Suppose that $i_c \in I$. We have $Z_{c-1} = \hat{w}_c h_\alpha^+ x_i(t_c) s_{i_c}$. If $c$ is hollow then $\hat{w}_c h^- x_i(t_c) \in U_+ \hat{w}_c h^-$, and thus $Z_{c-1} \in U_+ \hat{w}_c s_{i_c}(s_i^- h_c^-)$. This implies that $h_{c-1}^+ = s_i^+ \cdot h_c^+$. If $c$ is solid then we use the first identity in (2.8) together with $\alpha_{i_c}^+(1/t) \overline{w_c} = \overline{w_c} \alpha_{i_c}^+(t_c)$ to write $Z_{c-1} = \hat{w}_c h_\alpha^+ y_i(1/t) \alpha_{i_c}^+(t_c) x_i(-1/t)$. Since $\hat{w}_c h_\alpha^+ y_i(t) \in U_+ \hat{w}_c h_\alpha^+$, we see that $Z_{c-1} \in U_+ \hat{w}_c h_\alpha^+ \alpha_{i_c}^+(t_c) U_+$, which implies that $h_{c-1}^- = h_c^+ \alpha_{i_c}^+(t_c)$.

The case when $i_c \in -I$ is handled similarly. When $c$ is solid, we use the second identity in (2.8) together with $\alpha_{i_c}^+(1/t) \overline{w_c} = \overline{w_c} \alpha_{i_c}^+(t_c)$; see [FPZ99, Equation (1.2)].

\[\square\]
Corollary 2.7. Suppose $c$ is hollow. If $i_c \in I$ then $h_{c-1}^-=h_c^-$, and if $i_c \in -I$ then $h_{c-1}^+=h_c^+$.

Corollary 2.8. The Deodhar torus $T_\beta \subset \hat{R}_\beta$ is isomorphic to an algebraic torus of dimension $d(\beta)$, and the functions $(t_c)_{c \in J_\beta}$ form a basis of the character lattice of $T_\beta$.

Proof. For $c \in J_\beta$, the function $t_c$ is regular on $T_\beta$ by Lemma 2.6. With notation as in the proof of Lemma 2.6, we have $Z_c = \tilde{w}_c h_c^+=\overline{w}_c h_c^-\overline{w}_c$ and $Z_{c-1} = Z_c i_c = Z_c i_c^+(t_c)$ if $i_c \in I$ and $Z_{c-1} = Z_c i_c^+(t_c)^{-1} Z_c$ if $i_c \in -I$. It follows that $t_c \neq 0$ if and only if $Z_{c-1} = \hat{X}_{w_c} = \hat{X}_{w_c}$. (Thus, $t^* = 0$ in the notation of Lemma 2.3.) The fact that the map $T_\beta \to (\mathbb{C}^\times)^{J_\beta}$, $(X_c, Y_c) \to (t_c)_{c \in J_\beta}$ is an isomorphism follows from (2.7). \hfill \Box

2.5. Grid and chamber minors. Recall that $A = (a_{ij})_{i,j \in I}$, $a_{ij} = (\alpha_i, \alpha_j^\vee)$ is the Cartan matrix, and that $d_{i}a_{ij} = d_{j}a_{ji}$. For $i,j \in \pm I$, we define $a_{ij} = 0$ if $i,j$ have different signs, and $a_{ij} = a_{(-i)(-j)}$ otherwise. Also set $d_{-i} := d_i$ for $i \in I$.

Following [FZ99, Definition 1.4], for $i \in I$ and $v,w \in W$, we have a generalized minor $\Delta_{v\omega_i,w\omega_i} : G \to \mathbb{C}$. It is a regular function satisfying
\[
\Delta_{\omega_i,\omega_i}(y-x y_+) = \Delta_{\omega_i,\omega_i}(x) \quad \text{for all} \quad (y-x, y y_+) \in U_- \times G \times U_+;
\]
\[
\Delta_{v\omega_i,w\omega_i}(x) = \Delta_{\omega_i,\omega_i}(v^{-1} x w) = \Delta_{\omega_i,\omega_i}(\tilde{v}^{-1} x \tilde{w});
\]
see [FZ99, Section 1.4]. For $h \in H$, we have $\Delta_{\omega_i,\omega_i}(h) = h^{\omega_i}$. For $x \in G$, we also have [FZ99, Equation (2.14)]
\[
\Delta_{\omega_i,\omega_i}(x h) = \Delta_{\omega_i,\omega_i}(h x) = h^{\omega_i} \Delta_{\omega_i,\omega_i}(x).
\]

Definition 2.9. For $c \in [0, m]$ and $k \in I$, we define the grid minors
\[
\Delta_{c,k}(X_c, Y_c) = \Delta_{w_c \omega_i, w_c \omega_i}(Z_c) \quad \text{and} \quad \Delta_{c,-k}(X_c, Y_c) = \Delta_{w_c \omega_i, w_c \omega_i}(Z_c).
\]

The chamber minors are defined as $\Delta_c := \Delta_{c-1,k}$, for $c \in [m]$.

Recall that we also view $Z_c \in U_+ \setminus G/U_+$ as an element of $G$, and that for $(X_c, Y_c) \in T_\beta$, we have $Z_c \in \hat{X}_{w_c}$ for all $c \in [0, m]$. Thus, for $(X_c, Y_c) \in \hat{R}_\beta$, $Z_c$ belongs to the closure $\hat{X}_{w_c}$ of $\hat{X}_{w_c}$ inside $G$. The next result implies that the grid minors restricted to $\hat{X}_{w_c}$ (and therefore to $X_{w_c}$) are invariant under the $U_- \times U_+$-action and thus are well-defined on $\hat{R}_\beta$.

Lemma 2.10. For $c \in [0, m]$, $k \in I$, and $(X_c, Y_c) \in T_\beta$, we have
\[
\Delta_{c,k}(X_c, Y_c) = (h^{\omega_i})^{w_c k} \quad \text{and} \quad \Delta_{c,-k}(X_c, Y_c) = (h^{\omega_i})^{w_c k}.
\]

Proof. Write $Z_c = y_{-i} \tilde{w}_c h_c^+ y_{i}^+$ for $y_{-i} y_{i}^+ \in U_+$. We have
\[
\Delta_{c,k}(X_c, Y_c) = \Delta_{w_c \omega_i, w_c \omega_i}(Z_c) = \Delta_{w_c \omega_i, w_c \omega_i}(w_c^{-1} y_{-i} \tilde{w}_c h_c^+ y_{i}^+) = \Delta_{w_c \omega_i, w_c \omega_i}(\tilde{w}_c^{-1} y_{-i} \tilde{w}_c h_c^+).
\]

Factoring $\tilde{w}^{-1} y_{-i} \tilde{w}_c = b_- b_+$ for $(b_- b_+) \in U_- \times U_+$ using (2.3), and using $(h_c^+)^{-1} b_+ h_c^+ \in U_+$, we get the first identity in (2.13). The proof of the second identity is similar. \hfill \Box

Combining Lemma 2.10 with (2.7), we get the following.

Corollary 2.11. If $c$ is solid and $k \in \pm I$ has the same sign as $i_c$ then
\[
\Delta_{c-1,k} = \begin{cases} t_c \Delta_{c,k}, & \text{if } k = i_c; \\ \Delta_{c,k}, & \text{if } k \neq i_c. \end{cases}
\]

Proposition 2.12.

(1) The grid minors are characters of $T_\beta$.

(2) The solid chamber minors $(\Delta_c)_{c \in J_\beta}$ form a basis of the character lattice of $T_\beta$.

Proof. We relate the parameters $(t_c)_{c \in J_\beta}$ from Corollary 2.8 to the grid and chamber minors by combining (2.7) with (2.13). Suppose that $i_c \in I$ and let $k \in I$. If $k \neq i_c$ then $\Delta_{c-1,k} = \Delta_{c,k}$ by Corollary 2.11. If $c$ is hollow then
\[
\Delta_{c-1,i_c} = (h_i^+)^{\omega_i} = (s_{i_c} h_i^+)^{\omega_i} = (h_i^+)^{s_{i_c} \omega_i c}.
\]
Expand \( s_{i_c} \omega_{i_c} = \omega_{i_c} - \alpha_{i_c} \) in the basis of fundamental weights using \( \alpha_{i_c} = \sum_{j \in I} a_{i,j} \omega_j \). This gives
\[
\Delta_{c-1,i} \Delta_{c,i} \prod_{j \neq i} \Delta_{c,j}^{a_{ij}} = 1, \quad \text{if } c \text{ is hollow and } i := i_c,
\]
which holds for \( i_c \in \{ \pm 1 \} \). If \( c \) is solid, Corollary 2.11 yields \( \Delta_{c-1,i_c} = t_c \Delta_{c,i_c} \). Thus
\begin{enumerate}[(i)]
\item For each solid \( c \in J_\beta \), \( t_c = \Delta_{c-1,i_c} / \Delta_{c,i_c} \) is a ratio of two grid minors.
\item For each solid \( c \in J_\beta \), the grid minors \( (\Delta_{c-1,j})_{j \in \{ \pm 1 \}} \) are Laurent monomials in the grid minors \( (\Delta_{c,k})_{k \in \{ \pm 1 \}} \) and the chamber minor \( \Delta_c = \Delta_{c-1,i_c} \).
\item For each hollow \( c \in [m] \setminus J_\beta \), the grid minors \( (\Delta_{c-1,j})_{j \in \{ \pm 1 \}} \) are Laurent monomials in the grid minors \( (\Delta_{c,k})_{k \in \{ \pm 1 \}} \).
\item Every grid minor \( \Delta_{c,j} \) is a Laurent monomial in the solid chamber minors \( (\Delta_c)_{c \in J_\beta} \).
\end{enumerate}
We have already shown (i)–(iii), and (iv) follows from (ii)–(iii). This implies the result.

2.6. Almost positive sequences and Deodhar hypersurfaces. Recall that we have
\[ u_{c-1} = \min(u_c s_{i_c} u_c s_{i_c}^+) \quad \text{and} \quad w_{c-1} = \max(w_c s_{i_c} w_c s_{i_c}+) \]
for all \( c \in [m] \).

Definition 2.13. Let \( e \in J_\beta \). Let \( u^{(e)}_{c-1} := u_{c-1} \), and for \( c = m, m-1, \ldots, 1 \), define
\[ u^{(e)}_c := \begin{cases} 
\max(u^{(e)}_{c-1}, s_{i_c} u^{(e)}_{c-1} s_{i_c}^+), & \text{if } c = e, \\
\min(u^{(e)}_{c-1}, s_{i_c} u^{(e)}_{c-1} s_{i_c}^+), & \text{otherwise.}
\end{cases} \]
We call the sequence \( u^{(e)} := (u^{(e)}_0, \ldots, u^{(e)}_m) \) the \( (e) \)-almost positive sequence. We set \( w^{(e)}_c := w_c u^{(e)}_c \) for all \( c \in [0, m] \), and write \( w^{(e)} := (w^{(e)}_0, \ldots, w^{(e)}_m) \).

Definition 2.14. We say that \( e \in J_\beta \) is mutable if \( u^{(e)}_0 = 1 \). Otherwise, \( e \) is frozen. We let \( J_\beta^{\text{mut}} \) (resp., \( J_\beta^{\text{fro}} \)) denote the set of mutable (resp., frozen) indices.

We define \( \check{\mathcal{Y}}_\beta := \{(X_*, Y_*) \mid \text{satisfying (1.2)}\} \subset (G/U_+)^{[0,m]} \times (G/U_+)^{[0,m]} \). Removing the condition \( X_0 \equiv Y_0 \) yields a partial compactification \( \check{\mathcal{Y}}_\beta \) of \( \check{\mathcal{Y}}_\beta \).

Remark 2.15. We defined \( \check{R}_\beta \) as the quotient of \( \check{\mathcal{Y}}_\beta \) by the diagonal \( G \)-action. This action is free since the \( G \)-action on pairs \( (X_0, Y_0) \) which are weakly \( w_c \)-related is free; cf. [GLSBS22] Proposition 6.8. The \( G \)-action on \( \check{\mathcal{Y}}_\beta \) is no longer free.

Remark 2.16. We can parameterize the variety \( \mathcal{Y}_\beta \) as follows. We choose an arbitrary weighted flag \( X_m = Y_m \), and then for \( c = m, m-1, \ldots, 1 \), assuming \( (X_c, Y_c) = (g_c U_+, g_c^t U_+) \), we set
\[
(X_{c-1}, Y_{c-1}) := \begin{cases} 
(g_{c} z_{i_c} (t'_c) U_+ + g_{c}^t U_+), & \text{if } i_c > 0, \\
(g_{c} U_+ + g_{c}^t z_{i_c} (t'_c) U_+), & \text{if } i_c < 0,
\end{cases}
\]
for arbitrary parameters \( t' := (t'_1, t'_2, \ldots, t'_m) \subset [0, m] \). For \( (X_*, Y_*) \) to be a point in \( \check{\mathcal{Y}}_{u, \beta} \), we require further that \( X_0 \equiv Y_0 \), which is an extra condition on the parameters \( (t', X_m = Y_m) \).

Lemma 2.17. For each \( c \in [0, m] \) and \( k \in \{ \pm 1 \} \), the grid minor \( \Delta_{c,k} \) gives rise to a \( G \)-invariant regular function on \( \check{\mathcal{Y}}_\beta \) and \( \check{\mathcal{Y}}_{u, \beta} \). These regular functions are compatible with the quotient map \( \check{\mathcal{Y}}_{u, \beta} \to \check{R}_{u, \beta} \) and the inclusion map \( \check{\mathcal{Y}}_{u, \beta} \to \mathcal{Y}_{u, \beta} \).

Definition 2.18. Let \( e \in J_\beta \). Define the Deodhar hypersurface \( \check{V}_e \subset \mathcal{Y}_\beta \) to be the closure of the locus satisfying
\[
X_e \equiv Y_e \quad \text{for all } c \in [0, m].
\]
It follows that an index \( e \in J_\beta \) is mutable (resp., frozen) if and only if \( \check{V}_e \subset \check{\mathcal{Y}}_{\beta} \) (resp., \( \check{V}_e \cap \check{\mathcal{Y}}_{\beta} = \emptyset \)). If \( e \) is mutable, then the \( G \)-action on \( \check{V}_e \) is free and \( V_e = \check{V}_e / G \) is a subvariety of \( \check{R}_\beta \). We let \( \check{T}_\beta \) denote the preimage of \( T_\beta \) under the quotient map \( \check{\mathcal{Y}}_\beta \to \check{R}_\beta \).
Proposition 2.19. The closed subset $\mathcal{Y}_\beta \setminus \tilde{T}_\beta$ is a union of the Deodhar hypersurfaces $\tilde{V}_e$ for $e \in J_\beta$. Each $\tilde{V}_e$ is irreducible and has codimension one in $\mathcal{Y}_\beta$.

Proof. We first show that the closed subset $\mathcal{Y}_\beta \setminus \tilde{T}_\beta$ is a union of the Deodhar hypersurfaces $\tilde{V}_e$ for $e \in J_\beta$. Applying (2.16), we see that the conditions (2.17) cut out an iterated fiber bundle over $G/U_+ \times G/U_+$, where each fiber is either $\mathbb{C}$, $\mathbb{C}^*$, or (in the case of the crossing $c = e$) a point. It follows that $\tilde{V}_e$ is an irreducible subvariety of $\mathcal{Y}_\beta$ of codimension one.

Let $(X_\bullet, Y_\bullet) \in \mathcal{Y}_\beta \setminus \tilde{T}_\beta$. Then (2.5) must fail. Let $e \in \{0, m\}$ be the largest index such that we do not have $X_e \subseteq Y_e$. Let $\tilde{V}_e \subseteq \mathcal{Y}_\beta$ be the locus of points where (2.5) holds for $c > e$ and fails for $c = e$. By Lemma 2.13 and Definition 2.13, an open dense subset $\tilde{V}_e'$ of $\tilde{V}_e$ consists of points satisfying (2.17). Thus, $\tilde{V}_e' \subseteq \tilde{V}_e$, and therefore $(X_\bullet, Y_\bullet) \in \tilde{V}_e$. We have shown that $\mathcal{Y}_\beta \setminus \tilde{T}_\beta = \bigcup_{e \in J_\beta} \tilde{V}_e$. □

2.7. Cluster variables. The irreducible components of $\mathcal{Y}_\beta \setminus \tilde{T}_\beta$ are the Deodhar hypersurfaces $\tilde{V}_e$, $e \in J_\beta$. For a grid minor $\Delta_{c,k}$ and $e \in J_\beta$, we denote by $\text{ord}_{V_e} \Delta_{c,k} \in \mathbb{Z}$ the order of vanishing of $\Delta_{c,k}$ on the hypersurface $\tilde{V}_e$; cf. Lemma 2.17. Since $\Delta_{c,k}$ is regular on $\mathcal{Y}_\beta$, we have that $\text{ord}_{V_e} \Delta_{c,k} \geq 0$.

We have the following basic unipotentially property.

Proposition 2.20. For $e \in J_\beta$ solid, $c \in \{0, m\}$, and $k \in \pm I$, we have

$$\text{ord}_{V_e} \Delta_{c,k} = \begin{cases} 0, & \text{if } e \leq c; \\ 1, & \text{if } (c, k) = (e - 1, i_e), \text{ i.e., } \Delta_{c,k} = \Delta_e. \end{cases}$$

Proof. Suppose that $e \leq c$. Let $(X_\bullet, Y_\bullet)$ be a generic point in $\tilde{V}_e$. Then we have $X_c \subseteq Y_c$, and thus $\Delta_{c,k}(X_\bullet, Y_\bullet) \neq 0$. It follows that $\text{ord}_{V_e} \Delta_{c,k} = 0$ when $e \leq c$.

Suppose now that $(c, k) = (e - 1, i_e)$. We may parameterize $\mathcal{Y}_\beta$ using parameters $(\Psi', X_m = Y_m)$ as in (2.16). The union $\tilde{T}_\beta \cup \tilde{V}_c$ contains an open dense subset of tuples $(X_\bullet, Y_\bullet)$ satisfying $X_c \subseteq Y_c$ for all $c' \leq e$. Choose one such tuple $(X_\bullet, Y_\bullet)$ and note that $X_e \subseteq Y_e$.

Assume that $i_e \in I$. Thus, we have $Z_e \subseteq U_+ \tilde{w}_e h U_+$ for some $h \in H$. The proof of Lemma 2.10 implies that $\tilde{w}_e Z_e \subseteq U_+ h U_+$. Let us write $\tilde{w}_e^{-1} Z_e = y (-h) y_+$ for $(y, y_+) \in U_- \times U_+$. Since $e \in J_\beta$ is solid, we have $w_{e-1} = w_e$. Setting $t = t'$, we find

$$\Delta_{e-1,k}(X_\bullet, Y_\bullet) = \Delta_{\tilde{w}_e^{-1} \omega_k \omega_k}(Z_{e-1}(t)) = \Delta_{\omega_k \omega_k}(y (-h) y_+ z_k(t)).$$

Recall that $z_k(t) = \tilde{s}_k(t) \tilde{s}_k$. Let $\Psi := \Phi^+ \setminus \{\alpha_k\}$ and let $U_+ (\Psi) := (\tilde{s}_k^{-1} U_+) \cap U_+$. The corresponding root subgroup; see [Hum75 Theorem 26.3]. We have $x_k(-t) U_+ (\Psi) x_k(t) \subseteq U_+ (\Psi)$ by [Hum75 Lemma 32.5]. Next, we have $\tilde{s}_k^{-1} U_+ (\Psi) \subseteq U_+ (\Psi)$, since $s_k$ permutes $\Psi$. Using (2.3), we can factorize $y_+ = x_k(p) y'_p$ for some $p \in \mathbb{C}$ and $y'_p \in U_+$. We therefore get $y'_p x_k(t) \tilde{s}_k = x_k(t) \tilde{s}_k U_+$. Using (2.9), we get

$$\Delta_{e-1,k}(X_\bullet, Y_\bullet) = \Delta_{\omega_k \omega_k}(y (-h) k(p) y'_p x_k(t) \tilde{s}_k) = \Delta_{\omega_k \omega_k}(h x_k(p + t) \tilde{s}_k).$$

It is clear that if $p + t = 0$ then $\Delta_{e-1,k}(X_\bullet, Y_\bullet) = 0$. If $p + t \neq 0$, applying the first identity in (2.8) to $x_k(p + t) \tilde{s}_k$ and using (2.11), we find

$$\Delta_{e-1,k}(X_\bullet, Y_\bullet) = (p + t) \Delta_{\omega_k \omega_k} (h).$$

Thus (2.19) holds regardless of whether $p + t = 0$, and $p + t = 0$ if and only if the condition $X_{e-1} \subseteq Y_{e-1}$ fails, i.e., $(X_\bullet, Y_\bullet) \in \tilde{V}_e$. By (2.19), since $\Delta_{\omega_k \omega_k}(h) \neq 0$, we have $p + t = 0$ if and only if $\Delta_{e-1,k}(X_\bullet, Y_\bullet) = 0$. Since $\Delta_{e-1,k}$ is of degree 1 in $t$, we find that $\text{ord}_{V_e} \Delta_{e-1,k} \leq 1$. On the other hand, we have shown that $\Delta_{e-1,k}$ vanishes on $\tilde{V}_e$, so $\text{ord}_{V_e} \Delta_{e-1,k} \geq 1$. □

The integers $\text{ord}_{V_e} \Delta_{c,k}$ are nonnegative. Our next result shows that whether $\text{ord}_{V_e} \Delta_{c,k}$ is zero or positive is determined by the almost positive subexpression $\Pi^{(c)}$. The stronger result that $\text{ord}_{V_e} \Delta_{c,k} \in \{0, 1\}$ holds when $G = \text{SL}_n$ [GLSBS22 Proposition 7.10]. The precise value of $\text{ord}_{V_e} \Delta_{c,k}$ is for $G$ of arbitrary type is given in Section 7.
Proposition 2.21. For all $c \in [0,m]$, $e \in J_\beta$, and $k \in I$, we have
\[
\text{ord}_{V_e} \Delta_{e,k} = 0 \iff u_c \omega_k = u_c^{(e)} \omega_k \quad \text{and} \quad \text{ord}_{V_e} \Delta_{c,-k} = 0 \iff u_c^{-1} \omega_k = (u_c^{(e)})^{-1} \omega_k.
\]
Proof. Let $\leq$ denote the Bruhat order on $W$. Comparing Definitions 2.4 and 2.13 we see that $u_c \leq u_c^{(e)}$ and $w_c \geq w_c^{(e)}$ for all $c \in [0,m]$. Thus $w_c \omega_k \geq w_c^{(e)} \omega_k$ for all $c \in [0,m]$ and $k \in I$.

Let $\tilde{V}_e' \subset \tilde{V}_e$ be the open dense subset of points satisfying (2.17). For $(X_\bullet, Y_\bullet) \in \tilde{V}_e'$, we have $Z_c \in \tilde{X}_{w_c^{(e)}} \subset X_{w_c}$ for all $c \in [0,m]$, because $w_c^{(e)} \leq w_c$. Recall that $\Delta_{c,k}(X_\bullet, Y_\bullet) = \Delta_{w_c \omega_k, \omega_k}(Z_c)$. It is well known that the function $\Delta_{w_c \omega_k, \omega_k}(Z_c)$ when restricted to $Z_c \in X_{w_c}$ does not vanish at $Z_c \in \tilde{X}_{w_c^{(e)}}$ if and only if $w_c^{(e)} \omega_k = w_c \omega_k$; this can be shown by an analog of the proof of [FZ99, Proposition 2.4]. Similarly, we consider $\Delta_{c,-k}(X_\bullet, Y_\bullet) = \Delta_{w_c \omega_k, u_c^{-1} \omega_k}(Z_c)$ and observe that this function does not vanish at $Z_c \in \tilde{X}_{w_c^{(e)}} \subset X_{w_c}$ if and only if $u_c^{-1} \omega_k = (u_c^{(e)})^{-1} \omega_k$. \hfill \Box

Corollary 2.22. The $J_\beta \times J_\beta$ matrix $M_\beta = (\text{ord}_{V_e} \Delta_{c,e})_{c,e \in J_\beta}$ is upper unitriangular.

Inverting the matrix $M_\beta$, we arrive at the following definition, which is crucial for our analysis; cf. Proposition-Definition 1.3. Recall from Proposition 2.12 that a character on $T_\beta$ is just a Laurent monomial in the solid chamber minors $\{\Delta_e\}_{e \in J_\beta}$.

Definition 2.23. For $c \in J_\beta$, the cluster variable $x_c$ is the unique character of $T_\beta$ satisfying
\[
\text{ord}_{V_e} x_c = \begin{cases} 
1, & \text{if } c = e, \\
0, & \text{otherwise,}
\end{cases} \text{ for all } e \in J_\beta.
\]

We denote the cluster by $x_\beta = \{x_c\}_{c \in J_\beta}$.

Corollary 2.24.
\begin{enumerate}
\item For $c \in J_\beta$, the character $x_c \in X^*(T_\beta)$ extends to a regular function on $\hat{R}_\beta$, $\hat{V}_\beta$, $Y_\beta$.
\item For $c \in J_\beta^{\text{reg}}$, the frozen cluster variable $x_c$ is invertible in $\mathbb{C}[\hat{R}_\beta]$ and $\mathbb{C}[\hat{V}_\beta]$. \hfill \Box
\item The cluster variables in $x_\beta$ are irreducible and algebraically independent.
\end{enumerate}

Proof. For $c \in J_\beta$, $x_c$ is a rational function on $Y_\beta$ which does not have a pole on $\hat{T}_\beta$ or on $\tilde{V}_e$ for all $e \in J_\beta$. This implies that $x_c$ is regular. By Proposition 2.19, $x_c$ is irreducible. Since $x_\beta = \{x_c\}_{c \in J_\beta}$ is a basis of the character lattice of $T_\beta$, we see that the cluster variables are algebraically independent. Finally, for $c \in J_\beta^{\text{reg}}$, the function $1/x_c$ is regular on $Y_\beta \setminus \tilde{V}_e$, and as we mentioned after Definition 2.18, we have $\tilde{V}_e \cap \hat{Y}_\beta = \emptyset$. \hfill \Box

Let us denote $\text{ord}(\beta; e, c, k) := \text{ord}_{V_e} \Delta_{e,c,k}$. Recall from Proposition 2.20 that $\text{ord}(\beta; e, c, k)$ can only be nonzero when $e > c$. The next result follows from the parametrization (2.16) and will be used later in the proof.

Lemma 2.25. The integer $\text{ord}_{V_e} \Delta_{e,c,k}$ only depends on $i_{c+1}, \ldots, i_m$. That is, suppose that $\beta = i_1 i_2 \cdots i_m$ and $\beta' = i'_1 i'_2 \cdots i'_m$ are two double braid words in the alphabet $\pm I$ such that for some $c \in [m]$ and $c' \in [m']$ (with $m - c = m' - c'$), we have $i_{c+1}, \ldots, i_m = i'_{c+1}, \ldots, i'_{m'}$. Then we have
\[
\text{ord}(\beta; e, c, k) = \text{ord}(\beta'; e', c', k)
\]
for all $k \in \pm I$, $e > c$, and $e' > c'$ such that $m - e = m' - e'$. \hfill \Box

2.8. A two-form on the braid variety. We start by introducing a family of 1-forms on $T_\beta$. For $i,j \in \pm I$, recall that $a_{ij} = 0$ if $i,j$ have different signs, and $a_{ij} = a_{(-i)(-j)}$ otherwise, and that $d_i := d_{[i]}$.

For each $c \in [0,m]$ and $i \in \pm I$, we set
\[
L_{c,i} := \frac{1}{2} \sum_{k \in \pm I} a_{ik} \text{dlog} \Delta_{c,k}.
\]
Consider the following 2-forms on $T_\beta$:

\[(2.21) \quad \omega_{\beta,c} := \text{sign}(i) d_i L_{c-1,i} \wedge L_{c,i} \quad \text{for} \ c \in [m], \ i := i_c, \quad \text{and} \quad \omega_{\beta} := \sum_{c \in [m]} \omega_{\beta,c}.\]

Since $T_\beta$ is open dense in $\hat{R}_\beta$, the forms $\omega_{\beta}$ and $\omega_{\beta,c}$ are rational 2-forms on $\hat{R}_\beta$. Though it is not apparent from the above formula, it will follow from our main result that $\omega_{\beta}$ is actually a regular 2-form on $\hat{R}_\beta$.

Recall that $(2.15)$ holds for $i_c \in \pm I$. Taking $d\log$ of both sides of $(2.15)$, we get

\[(2.22) \quad L_{c-1,i_c} + L_{c,i_c} = 0 \quad \text{if } c \text{ is hollow.}\]

Thus, $\omega_{\beta,c} = 0$ for all $c \in [m] \setminus J_\beta$, which implies the following result.

**Corollary 2.26.** We have $\omega_{\beta} = \sum_{c \in J_\beta} \omega_{\beta,c}$.

3. **Cluster algebras**

Cluster algebras were discovered by Fomin and Zelevinsky \cite{FZ02}. We consider skew-symmetrizable cluster algebras of geometric type, relying on formalism similar to \cite{FG09}.

3.1. **Background.**

**Definition 3.1.** A rank $n$ and dimension $n+m$ (abstract) seed is a quadruple $\Sigma = (T, x, d, \omega)$, where

1. $T$ is a complex algebraic torus of dimension $n+m$,
2. $x = (x_1, \ldots, x_{n+m})$ is an ordered basis of $X^*(T)$, where $x_1, \ldots, x_n$ (resp., $x_{n+1}, \ldots, x_{n+m}$) are mutable (resp., frozen) variables,
3. $d = (d_1, \ldots, d_{n+m})$ is a collection of positive integers,
4. $\omega$ is a 2-form on $T$ of the form

\[(3.1) \quad \omega = \sum_{i} d_j \tilde{B}_{ij} d\log x_i \wedge d\log x_j = \sum_{i < j} d_i \tilde{B}_{ij} d\log x_j \wedge d\log x_i,\]

where $\tilde{B}_{ij} = 0$ for $i \in [n+m]$ and $\tilde{B}_{ij} \in \mathbb{Q}$ for all $i, j \in [n+m]$.

We say that $\Sigma$ is integral if $\tilde{B}_{ij} \in \mathbb{Z}$ for all $i \in [n+m]$ and $j \in [n]$.

The matrix $\tilde{B} = (\tilde{B}_{ij})_{i,j \in [n+m] \times [n]}$ is the usual $[n+m] \times n$ extended exchange matrix in the theory of cluster algebras. We have $d_j \tilde{B}_{ij} = -d_i \tilde{B}_{ji}$ for $i, j \in [n+m]$; in particular, the top $n \times n$ principal part $B$ of $\tilde{B}$ is skew-symmetrizable.

**Definition 3.2.** Let $\Sigma = (T, x, d, \omega)$ be a seed and $k$ a mutable index. We say that $\Sigma$ is integral at $k$ if $\tilde{B}_{jk} \in \mathbb{Z}$ for all $j \in [n+m]$. In this case, we define

\[(3.2) \quad x'_k := \prod_{B_{jk} > 0} x_j^{\tilde{B}_{jk}} + \prod_{B_{jk} < 0} x_j^{-\tilde{B}_{jk}}.\]

The mutation of $\Sigma$ in the direction $k$ is the seed $\mu_k(\Sigma) = (T', x', d, \omega')$ where $T'$ is the algebraic torus with basis of characters $x' = (x_1, \ldots, x'_k, \ldots, x_{n+m})$ and the 2-form $\omega'$ on $T'$ is the pullback of $\omega$ via the natural rational map $T' \to T$; see \cite{FG09}.

For the rest of this subsection, we assume that all seeds are integral. Following \cite{LS16} Section 5.1], a seed $\Sigma$ is really full rank if the columns of $\tilde{B}$ span $\mathbb{Z}^n$ over $\mathbb{Z}$. We will prove this for the seeds $\Sigma_\beta$ from Section 2 in Corollary 6.7.

Let $X$ be an irreducible complex algebraic variety of dimension $n+m$. A seed on $X$ is an abstract seed $\Sigma = (T, x, d, \omega)$ together with an identification $T \subset X$ of $T$ with an open dense subset of $X$. The inclusion $T \to X$ induces an identification of the field $\mathbb{C}(X)$ of rational functions on $X$ with the field of rational functions $\mathbb{C}(x) := \mathbb{C}(x_1, \ldots, x_{n+m})$ in the initial cluster variables $x$. In practice, we abuse notation and write $x$ for a tuple of elements in $\mathbb{C}(X)$.
The cluster algebra $\mathcal{A}(\Sigma)$ is the subring of $\mathbb{C}[x]$ generated by all cluster variables together with inverses of frozen variables. We let $\mathcal{X}(\Sigma) := \text{Spec}(\mathcal{A}(\Sigma))$ denote the cluster variety. We say that $(X, \Sigma)$ is a cluster variety if $X$ is an affine variety and the coordinate ring $\mathbb{C}[\mathcal{X}]$ is identified with $\mathcal{A}(\Sigma)$ under the identification $\mathbb{C}(X) \cong \mathbb{C}(x)$.

We will need the following property of cluster variables.

**Proposition 3.3** ([GLS13 Theorem 3.1]). Each cluster variable is an irreducible element of $\mathcal{A}(\Sigma)$.

**Definition 3.4.** Let $\Sigma$ be an abstract seed of rank $n$ and dimension $n+m$, and let $I \subseteq [n]$. The freezing of $\Sigma$ at $I$, denoted $\Sigma^I$, is the seed obtained from $\Sigma$ by declaring the variables $\{x_c\}_{c \in I}$ to be frozen. It is a seed of rank $n-|I|$ and dimension $n+m$. For $k \in [n]$, we denote $\Sigma^{\{k\}} := \Sigma^{|I|}$.

To a seed $\Sigma$ we associate a directed graph $\tilde{\Gamma}(\Sigma)$ with vertex set $[n+m]$ and an arrow $i \to j$ whenever $\tilde{B}_{ij} > 0$. We let $\Gamma := \Gamma(\Sigma)$ be the mutable part of $\tilde{\Gamma}(\Sigma)$, i.e., the induced subgraph of $\tilde{\Gamma}(\Sigma)$ with vertex set $[n]$. We say that a mutable index $s \in [n]$ is a sink if it has no outgoing arrows in $\Gamma$. Let $N^\text{in}_s(\Gamma)$ denote the set of vertices of $\Gamma$ having an arrow to $s$, and denote $\tilde{N}^\text{in}_s(\Gamma) := N^\text{in}_s(\Gamma) \cup \{s\}$. The following definition is a variation of locally-acyclic seeds [Mul13] and Louise seeds [LS16]; see also [GL22, Section 5.4 and Remark 5.14].

**Definition 3.5.** The class of sink-recurrent seeds is defined recursively as follows.

- Any seed $\Sigma$ such that $\Gamma(\Sigma)$ has no arrows is sink-recurrent.
- Any seed that is mutation equivalent to a sink-recurrent seed is sink-recurrent.
- Suppose that $\Sigma$ is a seed with a sink $s \in [n]$ such that the seeds $\Sigma^{\{s\}}$ and $\Sigma^{\tilde{N}^\text{in}_s(\Gamma)}$ are sink-recurrent. Then $\Sigma$ is sink-recurrent.

The upper cluster algebra [BFZ05] $U(\Sigma) \subseteq \mathbb{C}[x^{\pm 1}]$ is the intersection $\mathbb{C}[x^{\pm 1}] \cap \bigcap_{k \in [n]} \mathbb{C}[\mu_k(x)^{\pm 1}]$.

**Proposition 3.6.** Suppose that $\Sigma$ is a sink-recurrent seed. Then $\mathcal{A}(\Sigma) = U(\Sigma)$.

**Proof.** It follows from induction and [Mul13 Lemma 5.3] that sink-recurrent seeds are locally acyclic in the sense of [Mul13 Mul14]. By [Mul14 Theorem 2], we have $\mathcal{A}(\Sigma) = U(\Sigma)$.

### 3.2. Quasi-equivalence.

**Definition 3.7.** Two seeds $\Sigma = (T, \mathbf{x}, \mathbf{d}, \omega)$ and $\hat{\Sigma} = (\hat{T}, \hat{\mathbf{x}}, \hat{\mathbf{d}}, \hat{\omega})$ of rank $n$ and dimension $n+m$ are quasi-equivalent, denoted $\Sigma \sim \hat{\Sigma}$, if the following conditions are satisfied:

- $T = \hat{T}, \mathbf{d} = \hat{\mathbf{d}}, \omega = \hat{\omega}$;
- the sublattice of $X^*(T)$ spanned by the frozen variables $x_{n+1}, \ldots, x_{n+m}$ coincides with the sublattice spanned by $\hat{x}_{n+1}, \ldots, \hat{x}_{n+m}$;
- for each $k \in [n]$, we have $\hat{x}_k = x_k M_k$, where $M_k$ is a Laurent monomial in $x_{n+1}, \ldots, x_{n+m}$.

It is easy to see that if $\Sigma$ is integral and $\Sigma \sim \hat{\Sigma}$ then $\hat{\Sigma}$ is integral. The following is also straightforward to check.

**Lemma 3.8.** If $\Sigma$ and $\hat{\Sigma}$ are quasi-equivalent seeds then $\mu_k(\Sigma) \sim \mu_k(\hat{\Sigma})$ for all mutable $k$.

**Corollary 3.9.** Suppose two seeds $\Sigma, \hat{\Sigma}$ are quasi-equivalent. Then they define the same cluster algebra $\mathcal{A}(\Sigma) = \mathcal{A}(\hat{\Sigma}) \subset \mathbb{C}(T)$.

**Proof.** It follows from Lemma 3.8 that each cluster variable in $\mathcal{A}(\Sigma)$ differs from the corresponding cluster variable in $\mathcal{A}(\hat{\Sigma})$ by a factor equal to a Laurent monomial in the frozen variables.
3.3. Deletion-contraction. We give an inductive criterion for a pair \((X, \Sigma)\) to be a sink-recurrent cluster variety. In Section 4.8, we will apply this criterion to the seeds \(\Sigma_3\) constructed in Section 2. See [GL22, Corollary 5.15] for a different application suggesting our nomenclature.

Assumption 3.10. Throughout this section, we let \(\Sigma = (T, x, d, \omega)\) be an abstract seed of rank \(n\) and dimension \(n+m\). Let \(\Gamma := \Gamma(\Sigma)\). We assume that \(\Sigma\) is sink-recurrent, with sink \(s\) in \(\Gamma\) such that \(\Sigma\) is integral at \(s\). Further, we assume that there exists a frozen index \(j\) such that \(\tilde{B}_{js} = \pm 1\) and \(\tilde{B}_{fj} = 0\) for \(j \in [n] \setminus \{s\}\). Suppose that the exchange relation for \(x_s\) in \(\Sigma\) is given by \(x_s x'_s = M_1 + x_f M_2\) for some monomials \(M_1, M_2\) in \(\{x_j\}_{j \in [n+m] \setminus \{s,f\}}\).

Definition 3.11. Suppose that \(s\) has \(q := |N_s^\text{in}(\Gamma)|\) mutable neighbors. The contraction \(\Sigma/\s = (T/\s, x/\s, d/\s, \omega/\s)\) is a seed of rank \(n-q-1\) and dimension \(n+m-2\) defined as follows.

1. \(x/\s\) is obtained from \(x\) by omitting \(x_s\) and \(x_f\) and declaring the indices in \(N_s^\text{in}(\Gamma)\) to be frozen.
2. \(T/\s\) is an algebraic torus with character lattice generated by \(x/\s\).
3. \(d/\s\) is obtained by restricting the sequence \(d\) to the set \([n+m] \setminus \{s,f\}\).
4. \(\omega/\s\) is obtained from \(\omega\) by writing it in the form \((3.1)\) and substituting \(\text{dlog} x_s := 0\) and \(\text{dlog} x_f := \text{dlog} M_1 - \text{dlog} M_2\).

The deletion \(\Sigma \setminus \s\) is the seed of rank \(n-1\) and dimension \(n+m\) obtained by declaring \(x_s\) to be frozen (cf. Definition 3.4).

Theorem 3.12 (Deletion-contraction recurrence). Let \(X\) be an affine, normal, irreducible, complex algebraic variety, and let \(\Sigma = (T, x, d, \omega)\) be a seed on \(X\) with a sink \(s\) satisfying Assumption 3.10. Assume that all cluster variables in \(x \cup \{x'_s\}\) are regular on \(X\). Define subvarieties \(W := \{x_s \neq 0\}\) and \(V := \{x_s = 0\}\) of \(X\). Suppose we have isomorphisms \(W \cong X(\Sigma/\s)\) and \(V \cong \text{Spec}(\mathbb{C}[x'_s]) \times X(\Sigma/\s) \cong \mathbb{C} \times X(\Sigma/\s)\) such that each cluster variable of \(\Sigma/\s\), resp. \(\Sigma/\s\), is the pullback of the same-named cluster variable of \(\Sigma\) under the inclusion \(\iota_W : W \hookrightarrow X\), resp. \(\iota_V : V \hookrightarrow X\), and the pullback of \(x'_s\) under \(\iota_V\) is the same-named function on the first factor of \(\text{Spec}(\mathbb{C}[x'_s]) \times X(\Sigma/\s) \cong V\). Then \((X, \Sigma)\) is a cluster variety.

Proof. First, \(\Sigma \setminus \s\) is integral by assumption. Since \(\Sigma\) is integral at \(s\), this implies \(\Sigma\) is also integral.

Let \(j \in [n] \setminus \{s\}\) be a mutable index. Clearly, the (pullback under \(\iota_W\) of the) exchange relation for \(x_j\) in \(\Sigma\) coincides with the exchange relation for \(x_j\) in \(\Sigma/\s\). Thus, the mutated variable \(x'_j\) is regular on \(W\). Next, assume that \(j \notin N_s^\text{in}(\Gamma)\). By assumption, \(j\) is not connected to \(s, f\) in \(\Gamma\), and thus the terms involving \(\text{dlog} x_j\) are unchanged when passing from \(\omega\) to \(\omega/\s\). Thus, the pullback of the exchange relation for \(x_j\) under \(\iota_V\) is still the exchange relation for \(x_j\) in \(\Sigma/\s\), and therefore the mutated variable \(x'_j\) is regular on \(V\). For \(j \in N_s^\text{in}(\Gamma)\), \(x'_j\) must also be regular on \(V\) since the pullback of \(x_j\) is a frozen variable. It follows that for all \(j \in [n] \setminus \{s\}\), the mutated variable \(x'_j\) is a regular function on \(X\) since it is regular on both \(V\) and \(W\). For \(j = s, x'_s\) is regular on \(X\) by assumption.

Next, we show that \(\mathbb{C}[X] \subset \mathcal{U}(\Sigma)\). This is equivalent to constructing inclusions \(T \hookrightarrow X\) and \(\mu_j(T) \hookrightarrow X\) for all \(j \in [n]\). Since \(\Sigma\) is a seed on \(X\), we have \(T \subset X\). For the tori \(\mu_j(T)\), we know that the subset \(X_j \subset X\) where the regular functions in \(\mu_j(x)\) are all non-vanishing is isomorphic to an algebraic torus \(\mu_j(T) \cong (\mathbb{C}^\times)^{n+m}\) via the map \(\phi_j : X_j \rightarrow \mu_j(T)\) sending \(y \in X_j\) to \(z := (x_1(y), ..., x_j(y), ..., x_{n+m}(y))\). If \(j \in [n] \setminus s\), then we have \(X_j \subset X\), and thus the statement follows since \(W\) is a cluster variety. So let \(j = s\). Consider the torus \(\mu_s(T) \cong (\mathbb{C}^\times)^{n+m}\). Let \(p := M_1 + x_f M_2 \in \mathbb{C}[X]\) be the exchange binomial for \(x_s\) (cf. Assumption 3.10). Since \(p\) does not involve \(x_s\) or \(x'_s\), we can also view \(p\) as a regular function on \(\mu_s(T)\) compatible with pullback under \(\phi_s\). Let \(z = (z_1, ..., z_{n+m}) \in \mu_s(T)\). Our goal is to show that \(z\) has a unique preimage under \(\phi_s\). Suppose first that \(p(z) \neq 0\). Then \(\phi_s^{-1}(z) \subset T\), and the result follows. Suppose now that \(p(z) = 0\). Then \(\phi_s^{-1}(z) \subset V\). Recall that \(V \cong \text{Spec}(\mathbb{C}[x'_s]) \times X(\Sigma/\s)\). Since \(x'_s = z_s\), the \(\text{Spec}(\mathbb{C}[x'_s])\)-coordinate of the preimage is uniquely determined by \(z\). The \(X(\Sigma/\s)\)-coordinate of the preimage is uniquely determined by \((z_i)_{i \in [n+m] \setminus \{s,f\}}\). We have shown that \(z\) has a unique preimage under \(\phi_s\), which completes the proof of the inclusion \(\mathbb{C}[X] \subset \mathcal{U}(\Sigma)\). The statement of the theorem now follows from Proposition 3.13 below.
Proposition 3.13. Let $X$ be an affine, normal, irreducible, complex algebraic variety, and let $\Sigma = (T, x, d, \omega)$ be an integral sink-recurrent seed on $X$. Suppose that $\mathbb{C}[X] \subset \mathcal{U}(\Sigma)$. Then $(X, \Sigma)$ is a cluster variety.

Proof. The inclusions $\mathbb{C}[X] \subset \mathbb{C}[x_1^{\pm 1}, \ldots, (x_j')^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ give tori $T$ of Theorem 3.12. By a standard argument, this implies that the complement of $T \cup \bigcup_{j \in [n]} X_j$ has codimension greater than or equal to two in $X$; see [Zel00, Section 3], [BFZ05, Proof of Theorem 2.10], or [GLSBS22, Lemmas 9.5–9.8]. Since $X$ is normal, we have $\mathbb{C}[X] = \mathbb{C}[T \cup \bigcup_{j \in [n]} X_j] = \mathcal{U}(\Sigma)$. By assumption, $\Sigma$ is sink-recurrent, and we are done by Proposition 3.6. □

Remark 3.14. If the seed $\Sigma \backslash s$ is really full rank then it follows from Assumption 3.10 that $\Sigma$ is really full rank. Indeed, row $f$ of the exchange matrix of $\Sigma$ contains a single nonzero entry equal to $\pm 1$ in column $s$. The exchange matrix of $\Sigma \backslash s$ is obtained from that of $\Sigma$ by removing column $s$. This implies that if $\Sigma \backslash s$ is really full rank then so is $\Sigma$.

Remark 3.15. The statement of Theorem 3.12 remains true if “sink-recurrent” is replaced with “locally acyclic.”

4. Double braid moves

In this section, we study natural isomorphisms between braid varieties corresponding to double braid moves, and determine the effect of these isomorphisms on seeds. Double braid moves are defined as follows:

(B1) $ij \leftrightarrow ji$ if $i, j \in \pm I$ have different signs;
(B2) $ij \leftrightarrow ji$ if $i, j \in \pm I$ have the same sign and $(s_{ij} s_{ji})^2 = 1$;
(B3) $iji\ldots \leftrightarrow ji j\ldots$ if $i, j \in \pm I$ have the same sign and $(s_{ij} s_{ji})^{m_{ij}} = 1$ with $m_{ij} \geq 3$;
(B4) $\beta_0 i \leftrightarrow \beta_0 (-i^*)$ for $i \in \pm I$ and $\beta_0 \in (\pm I)^{m-1}$;
(B5) $i \beta_0 \leftrightarrow (-i) \beta_0$ for $i \in \pm I$ and $\beta_0 \in (\pm I)^{m-1}$.

If double braid words $\beta$ and $\beta'$ are related by one of the moves (B1)–(B5), there is a natural isomorphism $\phi: \tilde{R}_\beta \sim \tilde{R}_{\beta'}$.

Definition 4.1. Suppose that $\beta$ and $\beta'$ are related by one of the moves (B1)–(B3). If this move involves indices $l, l+1, \ldots, r$, the isomorphism $\phi$ sends $(X_*, Y_*) \in \tilde{R}_\beta$ to the unique tuple $(X'_*, Y'_*) \in \tilde{R}_{\beta'}$ such that $X'_c = X_c$ and $Y'_c = Y_c$ for $0 \leq c < l$ or $r \leq c \leq m$. The remaining weighted flags $X'_1, \ldots, X'_{r-1}, Y'_1, \ldots, Y'_{r-1}$ are uniquely determined using Lemma 2.1.

For the moves (B4) and (B5), the isomorphism $\phi$ is described in Sections 4.6 and 4.7, respectively. The main result of this section is the following.

Theorem 4.2. Suppose that $\beta$ and $\beta'$ are related by one of the moves (B1)–(B5). If $(\tilde{R}_\beta, \Sigma_\beta)$ is a cluster variety then so is $(\tilde{R}_{\beta'}, \Sigma_{\beta'})$.

We then use Theorem 4.2 and Theorem 3.12 to prove Theorem 1.1; see Theorem 4.10 and Sections 4.9 and 6.3.

The proof of Theorem 4.2 will occupy Sections 4.6. Along the way, we will construct a seed $\Sigma' = (T', x', d', \omega')$ obtained from $\Sigma_\beta = (T, x, d, \omega)$ by one or several mutations, followed by a relabeling. We will show the following for moves (B1)–(B5):

(F) The 2-form is invariant: $\phi^* \omega_{\beta'} = \omega_{\beta}$.

(Q) Suppose that $(\tilde{R}_\beta, \Sigma_\beta)$ is a cluster variety. Then the seeds $\Sigma'$ and $\phi^* \Sigma_{\beta'}$ are quasi-equivalent.

Here, for a seed $\Sigma_{\beta'} = (T_{\beta'}, x_{\beta'}, d_{\beta'}, \omega_{\beta'})$, $\phi^* \Sigma_{\beta'} = (T^*, x^*, d^*, \omega^*)$ is an abstract seed on $\tilde{R}_\beta$ defined by $T^* := \phi^{-1}(T_{\beta'})$, $x^* := \phi^* x_{\beta'}$, $d^* := d_{\beta'}$, and $\omega^* := \phi^* \omega_{\beta'}$. Note that (Q) immediately implies Theorem 4.2.
**Definition 4.3.** A move $([B1],[B3])$ is solid if all indices involved are solid. For $i,j \in I$, the $([B1])$ move $(-i)j \leftrightarrow j(-i)$ on indices $c,c+1$ is special if $u,cj = sjjc$ and solid-special if it is both solid and special.

A $([B3])$ move with $m_{ij} > 3$ is long; all other moves are short. Finally, a move $([B1]),([B5])$ is a mutation move if it involves at least one cluster mutation; otherwise it is a non-mutation move.

**Remark 4.4.** As we will show in Section 4.1 a solid-special $([B1])$ move corresponds to a single mutation, at the rightmost index involved in the move. The move $([B3])$ involving $q$ solid indices corresponds to a sequence of $(\frac{q}{2})$ mutations on the rightmost $m_{ij} - 2$ indices involved in the move.

We will show $([F],[Q])$ for short moves directly. This will complete the proof of Theorem 4.2 in simply-laced types. We then use this and folding to show $([F],[Q])$ for long moves in Sections 5 and 6.

Throughout this section, we fix $\beta,\beta'$ related by a short move and thus an isomorphism $\phi: \hat{R}_{\beta'} \rightarrow \hat{R}_{\beta}$. For a rational function or a form $f$ on $\hat{R}_{\beta'}$, we use the shorthand $f^* := \phi f$.

**Remark 4.5.** If all indices involved in a move $([B1],[B3])$ are hollow, then the statements $([F],[Q])$ follow trivially; cf. Corollary 2.26.

### 4.1. Mutation move: $([B1])$, solid-special.

Consider the case of a solid-special move $([B1])$ on indices $c,c+1$. Since both indices are solid, we denote $u := u_{c-1} = u_c = u_{c+1}$ and $w := w_{c-1} = w_c = w_{c+1}$.

The indices $i,j \in I$ are of opposite signs; we assume that $i \in -I$ and $j \in I$ as the other case is similar. The solid-special condition yields

\[
(4.1) \quad u < s_{|i|}u = us_j \quad \text{and} \quad s_{|i|}w = ws_j < w.
\]

**Proposition 4.6 ([FZ99] Theorem 1.17).** We have

\[
(4.2) \quad \Delta_{c,j} \Delta_{c,j}^\ast = \Delta_{c+1,j} \Delta_{c-1,j} + \prod_{k \neq j} \Delta_{c,k}^{-a_{jk}}.
\]

**Proof.** We may choose $t,t' \in \mathbb{C}$ such that $Z_c = Z_{c+1}z(t)$, $Z_c^\ast = \tilde{z}_{|i|}(t')^{-1}Z_{c+1}$, and $Z_{c-1} = Z_{c+1}^\ast = \tilde{z}_{|i|}^\ast(t')^{-1}Z_{c+1} \tilde{z}_j(t) = \delta_{|i|}x_{|i|}(t)Z_{c+1} \tilde{x}_j(t)$. Let $Z := x_{|i|}(t)Z_{c+1} \tilde{x}_j(t)$. By [FZ99] Theorem 1.17, we have

\[
(4.3) \quad \Delta_{w_{|j|},s_jw_j}(Z)\Delta_{w_{|j|},s_jw_j}(Z) = \Delta_{w_{|j|},s_jw_j}(Z)\Delta_{w_{|j|},s_jw_j}(Z) + \prod_{k \neq j} \Delta_{w_{|j|},s_jw_j}(Z)^{-a_{jk}}.
\]

Using properties of generalized minors from Section 2.5 one can check that each term of (4.2) equals the corresponding term of (4.3). For example, we have

\[
\Delta_{c,j} = \Delta_{w_{|j|},s_jw_j}(Z_{c+1} \tilde{x}_j(t) \tilde{s}_j) = \Delta_{w_{|j|},s_jw_j}(\tilde{w}^{-1}Z_{c+1} \tilde{x}_j(t) \tilde{s}_j) = \Delta_{w_{|j|},s_jw_j}(\tilde{w}^{-1}Z \tilde{s}_j) = \Delta_{w_{|j|},s_jw_j}(Z),
\]

where we have used $\tilde{w}^{-1}x_{|i|} := U_\tilde{w}^{-1}$; cf. (2.9) and (4.1). For $\Delta_{c,k}^{-a_{jk}}$, $k \neq j$, we additionally used that $s_jw_k = \omega_k$.

We shall use the following analog of [GLSBS22] Lemma 8.10.

**Lemma 4.7.** For $e \in \{0,m\}$ and $-i,j \in I$ such that $u_es_j = s_{|i|}u_e$, we have

\[
(4.4) \quad \prod_{k \in I} \Delta_{e,k}^{a_{ik}} = \prod_{k \in I} \Delta_{e,k}^{\epsilon a_{ik}} \quad \text{and} \quad \epsilon_e,\epsilon_{e,i} = \epsilon_{L_e,i} = \epsilon_{L_e,j}, \quad \text{where} \quad \epsilon := \begin{cases} 1, & \text{if } u_e < u_es_j; \\ -1, & \text{if } u_e > u_es_j. \end{cases}
\]

**Proof.** We have $\alpha_j = \sum_{k \in I} a_{j,k} \omega_k$ and similarly for $\alpha_{|i|}$. The first identity in (4.4) therefore becomes $(h_e^-)^{a_{|i|}} = (h_e^-)^{\epsilon \alpha_{|i|}}$, which follows from the assumption $u_e \alpha_j = \epsilon \alpha_{|i|}$ together with $h_e^- = u_e \cdot h_e^+$; cf. (2.6).

Taking $\log$ of both sides, we obtain the second identity.

**Remark 4.8.** Equations (4.2) and (4.4) are true as stated in the case $i,-j \in I$ as well.

---

1Our Cartan matrix is the transpose of that of [FZ99]; see [FZ99] Equation (2.27).
Proof of [F] for (B1), solid-special. Only the terms $\omega_{\beta,c}$ and $\omega_{\beta,c+1}$ change when doing the move (B1). Noting that we must have $d_i = d_j$ and applying (4.4), we get
\[
\frac{1}{d_j} (\omega_\beta - \omega_\beta^*) = \frac{1}{d_j} (\omega_{\beta,c} + \omega_{\beta,c+1} - \omega_{\beta,c}^* - \omega_{\beta,c+1}^*) = -L_{c-1,i} \land L_{c,i} + L_{c,j} \land L_{c+1,j} - L_{c-1,i}^* \land L_{c,i}^* + L_{c,j}^* \land L_{c+1,j}^* = -L_{c-1,j} \land L_{c,j} + L_{c,j} \land L_{c+1,j} - L_{c-1,j}^* \land L_{c,j}^* + L_{c,j}^* \land L_{c+1,j}^* = (L_{c,j} + L_{c,j}^*) \land (L_{c-1,j} + L_{c+1,j}).
\]
For $e \in \{c-1,c+1\}$, let $M_e := \prod_{k \neq j} \Delta_{c,k}^{-a_{jk}}$. Thus, $M_e$ is the third term in (4.2). By (2.7), we have
\[
h_c^+ = \alpha_c^+(t_{c+1})h_c^+ + 1, \quad h_c^- = \alpha_c^-(t_c)h_c^-. \quad \text{This implies that } h_{c-1}^+ = \alpha_c^+(t_c(t_{c+1})h_{c+1}^- \quad \text{since } h_c^- = u \cdot h_c^+.
\]
Thus, we have $M := M_{c-1} = M_{c} = M_{c+1}$. Since $M_{c+1} = M_{c+1}$, we get $M = M_{c-1} = M_{c} = M_{c+1}$.

Set $A := \Delta_{c-1,j} \Delta_{c,j}^* + 1$. Then (4.2) gives $A = B + 1$. Thus $dA = dB$ and $dlogA \land dlogB = 0$. It remains to note that $dlogA = L_{c,j} + L_{c,j}^*$ while $dlogB = L_{c-1,j} + L_{c+1,j}^*$. □

Proof of (Q) for (B1), solid-special. We do not use the assumption that $(\hat{R}_\beta, \Sigma_\beta)$ is a cluster variety until the last paragraph of this proof. Let $x := x_{c+1}$ and $V := V_{c+1}$. Applying Propositions 2.20 and 2.21, we see that $x$ is mutable,
\[
\text{ord}_V \Delta_{c,j} = \text{ord}_V \Delta_{c,i} = 1, \quad \text{and } \text{ord}_V \Delta_{c,k} = 0 \quad \text{for } (e,k) \notin \{ (c,j),(c,i) \}.
\]
In particular, $dlogx$ appears in $\omega_\beta$ only in the terms $L_{c,j}$ and $L_{c,i}$ in $\omega_{\beta,c+1} = d_j L_{c,j} \land L_{c+1,j}$ and $\omega_{\beta,c} = -d_i L_{c-1,i} \land L_{c,i}$, respectively. Recall from (4.4) that we actually have $L_{c,j} = L_{c,i}$. We see from (4.5) that the coefficient of $dlogx$ in $L_{c,j} = L_{c,i}$ is equal to 1. Collecting the terms of $\omega_\beta$ involving $dlogx$ and using $d_i = d_j$, we get
\[
d_j dlogx \land (L_{c+1,j} + L_{c-1,i}) = d_j dlogx \wedge \left( dlog(\Delta_{c+1,j} \Delta_{c-1,j}) - dlog \prod_{k \neq j} \Delta_{c,k}^{-a_{jk}} \right).
\]
By Proposition 2.20, a cluster variable $x_e$ for $e \in J_\beta$ may appear on the right-hand side of (4.6) only for $e \geq c$. Moreover, we have already observed that $d_c = d_{c+1} = d_i = d_j = d_{c+1}$. Let us denote $p_e := \text{ord}_V (\Delta_{c+1,j} \Delta_{c-1,j})$ and $q_e := \text{ord}_V \prod_{k \neq j} \Delta_{c,k}^{-a_{jk}}$. Clearly, $p_e, q_e \geq 0$. We therefore see from (3.1) that for all $e \in J_\beta$, we have $B_{c+1} = q_e - p_e$. By Definition 2.14, the cluster variable $x$ is mutable. Thus, the mutated variable $x' := x_{c+1}$ satisfies
\[
x' = \prod_{e \in J_\beta \setminus \{c+1\}} x_e^{p_e - q_e} + \prod_{e \in J_\beta \setminus \{c+1\}} x_e^{q_e - p_e}.
\]
We have $V_e = V_e^*$ and $x_e = x_e^*$ for all $e \in J_\beta \setminus \{c+1\}$. Let $V^* := V_{c+1}$ and $x^* := x_{c+1}^*$. A generic point $(X_\bullet, Y_\bullet) \in V^*$ satisfies $X_{c-1} \leftarrow w_{c-1} Y_{c+1}$ and $X_{c+1} \leftarrow w_{c+1} Y_{c-1}$, while a generic point $(X_\bullet, Y_\bullet) \in V^*$ satisfies $X_{c+1} \leftarrow w_{c+1} Y_{c-1}$ and $X_{c-1} \leftarrow w_{c-1} Y_{c+1}$. Thus, $V \not\supset V^*$.

For $e \in J_\beta$, applying ord$_V$ to both sides of (4.2), we get
\[
\text{ord}_V \Delta_{c,j} + \text{ord}_V \Delta_{c,j}^* \geq \min(p_e, q_e).
\]
For $e = c + 1$, we have ord$_V \Delta_{c,j} = 1$, ord$_V \Delta_{c,j}^* = 0$ (since $V \not\supset V^*$), and $p_{c+1} = q_{c+1} = 0$. Similarly, ord$_V \Delta_{c,j}^* = 1$, ord$_V \Delta_{c,j} = 0$, and the order of vanishing of $\Delta_{c+1,j} \Delta_{c-1,j}^*$ and $\prod_{k \neq j} (\Delta_{c,k}^{-a_{jk}})$ at $V^*$ is zero.

Dividing both sides of (4.2) by $\prod_{e \in J_\beta \setminus \{c+1\}} x_e^{\min(p_e, q_e)}$, we get
\[
x' = \prod_{e \in J_\beta \setminus \{c+1\}} x_e^{r_e} = \prod_{e \in J_\beta \setminus \{c+1\}} x_e^{p_e - q_e} + \prod_{e \in J_\beta \setminus \{c+1\}} x_e^{q_e - p_e},
\]
where $r_e := \text{ord}_V \Delta_{c,j} + \text{ord}_V \Delta_{c,j}^* - \min(p_e, q_e) \geq 0$. By (4.7), we get
\[
x' = x^* \prod_{e \in J_\beta \setminus \{c+1\}} x_e^{r_e}.\]

Now, assume that \((\hat{\mathcal{R}}_\beta, \Sigma_\beta)\) is a cluster variety. We get from Proposition 3.3 that the mutated cluster variable \(x'\) is irreducible in \(\mathbb{C}[\hat{\mathcal{R}}_\beta]\). The function \(x^*\) vanishes on \(V^* \subset \hat{\mathcal{R}}_\beta\) and therefore is not a unit in \(\mathbb{C}[\hat{\mathcal{R}}_\beta]\). It follows that \(r_e = 0\) for all mutable \(e\), and thus the mutated seed \(\Sigma' := \mu_{c+1} \Sigma_\beta\) is quasi-equivalent to the pulled back seed \(\Sigma_{by}'\).

4.2. Non-mutation move: \((B1)\), not solid-special. We continue to assume that the move involves indices \(c, c+1\), and that \(i \in -I, j \in I\).

4.2.1. \((B1)\) special, non-solid. Suppose that at least one of the indices is hollow, and that the move is special. Then it follows that \(c+1\) is hollow and \(c\) is solid in both \(\beta\) and \(\beta'\). By (2.22), \(L_{c,j} = -L_{c+1,j}\) and \(L_{c,i} = -L_{c+1,i}^*\). Applying (4.4) with \(\epsilon = 1\) for \(e = c-1, c\) and \(\epsilon = -1\) for \(e = c+1\) and using \(d_i = d_j\), we obtain

\[
\frac{\omega_{\beta,c}}{d_j} = -L_{c-1,i} \wedge L_{c,i} = L_{c-1,i}^* \wedge L_{c+1,i}^* = -L_{c-1,j}^* \wedge L_{c+1,j}^* = L_{c-1,j} \wedge L_{c,j}^* = \frac{\omega_{\beta',c}}{d_i},
\]

which proves (F). The clusters \(x_\beta\) and \(x_{by}'\) are identical, which proves (Q).

4.2.2. \((B1)\) non-special. We start by introducing a formalism for working with the forms \(L_{e,k}\). Let \(\lambda := \sum_{k \in I} b_k \omega_k\) with \(b_k \in \mathbb{Q}\), and let \(h\) be an \(H\)-valued rational function on \(\hat{\mathcal{R}}_\beta\). We introduce a rational 1-form

\[
dlogh_\lambda := \sum_{k \in I} b_k d\log(h^{\omega_k}).
\]

It is clear that

\[
d\log h_{\lambda_1 + \lambda_2} = d\log h_{\lambda_1} + d\log h_{\lambda_2} \quad \text{and} \quad d\log(h_1 h_2)_{\lambda} = d\log h_{\lambda_1} + d\log h_{\lambda_2}.
\]

For \(e \in [0,m]\) and \(k \in I\), Lemma 2.10 gives

\[
L_{e,k} = d\log(h_e^-)^{\omega_k}/2 = d\log(h_e^-)^{u_{\alpha_k}/2}, \quad L_{e,-k} = d\log(h_e^+)^{u_{\alpha_k}/2} = d\log(h_e^+)^{u_{\alpha_k}/2}.
\]

Finally, suppose that \(h_1 = h_2 \alpha_k(t)\). Then we have

\[
d\log h_1 = d\log h_2^\alpha + \langle \lambda, \alpha_{\beta'_y} \rangle d\log t.
\]

Proof of (F) and (Q) for \((B1)\) non-special. Suppose as before that the move involves indices \(c, c+1\), and that \(i \in -I, j \in I\). Assume first that both \(c, c+1\) are solid, and let \(u := u_{c-1} = u_{c} = u_{c+1}\). Let \(a := (u^{-1} \omega_{\alpha_{\beta'_y}}/2, \alpha_{\beta'_y})\) and \(a' := (u_{\alpha_{\beta'_y}(J)}/2, \alpha_{\beta'_y}(J))\). Using (1.11)–(1.12) and (2.7), we get

\[
L_{c,i} = L_{c+1,i} + d\log t_{c+1}, \quad L_{c,i}^* = L_{c+1,i}^* + d\log t_{c+1}^*, \quad L_{c,j} = L_{c+1,j} + d\log t_{c+1}, \quad L_{c,j}^* = L_{c+1,j}^* + d\log t_{c+1}^*.
\]

Since the move is non-special, the cooroots \(\alpha_{\beta'_y}\) and \(u^{-1} \omega_{\alpha_{\beta'_y}}/2\) are linearly independent, which implies \(t_{c+1}^* = t_{c+1} = t_c\). Note also that we have \(L_{c+1,i} = L_{c+1,i}^*\). Using (1.13)–(1.14) to express each 1-form \(L_{e,k}\) in terms of \(L_{c+1,i}\), \(L_{c+1,j}\), \(d\log t_{c+1}\), and \(d\log t_{c+1}^*\), we find

\[
\omega_{\beta,c} + \omega_{\beta,c+1} - \omega_{\beta',c} = (d_j a' - d_i a) d\log t_{c+1} + d\log t_{c+1}^*.
\]

Since \(d_j a' = d_i a\), we get that \(\omega_{\beta} = \omega_{by}'\). The clusters \(x_\beta\) and \(x_{by}'\) differ by a relabeling \(c \leftrightarrow c+1\).

Suppose now that one of \(c, c+1\) is hollow. For instance, let \(c \notin J_\beta\) and \(c+1 \notin J_\beta\). By Corollary 2.7, we have \(h_{c-1}^+ = h_{c-1}^+\), and thus \(L_{c,i} = -L_{c-1,i}^*\). Similarly, \(L_{c,i}^* = L_{c+1,i}^*\). Recall that \(L_{c+1}^* = L_{c+1,i}^*\). Thus, \(\omega_{\beta,c+1} = \omega_{\beta',c+1}\), and so \(\omega_{\beta} = \omega_{by}'\). The case where \(c \in J_\beta\) and \(c+1 \notin J_\beta\) is similar. The clusters \(x_\beta\) and \(x_{by}'\) differ by a relabeling \(c \leftrightarrow c+1\). For the case \(c+1 \notin J_\beta\), see Remark 4.5.

4.3. Non-mutation move: \((B2)\). Suppose that the move involves indices \(c, c+1\). We have \(\omega_{\beta,c} = \omega_{by}'\) and \(\omega_{\beta,c+1} = \omega_{by'}\), so \(\omega_{\beta} = \omega_{by}'\). The chamber minors satisfy \(\Delta_c = \Delta_{c+1}^*\) and \(\Delta_{c+1} = \Delta_c^*\). Thus, the clusters \(x_\beta\) and \(x_{by}'\) differ by a relabeling \(c \leftrightarrow c+1\). This shows (F) and (Q).
4.4. Mutation move: (B3) solid, short. We proceed analogously to the case of solid-special (B1) in Section 4.1. Suppose that the move $\beta \rightarrow \beta', iji \rightarrow jij$, involves indices $c-1, c, c+1$, and that all three indices are solid. Suppose in addition that $i, j \in I$; the case $i, j \in -I$ is similar.

Proposition 4.9 ([FZ99] Theorem 1.16(1)). We have
\begin{equation}
\Delta_{c,i} \Delta_{c,j}^* = \Delta_{c+1,i} \Delta_{c-2,j} + \Delta_{c-2,i} \Delta_{c+1,j}.
\end{equation}

Proof. We have $Z_{c-2} = Z_{c+1} z_i (t_1) z_j (t_2) z_i (t_3)$ for some $t_1, t_2, t_3 \in \mathbb{C}$. We have $z_i (t_1) z_j (t_2) z_i (t_3) = z_i (t_3) z_i (t_2) z_i (t_1)$ for $t_2 := t_1 t_3 - t_2$, which can be checked inside SL3. Thus, $Z_{c-1} = Z_{c+1} z_i (t_1) z_j (t_2)$, $Z_c = Z_{c+1} z_i (t_1)$, and $Z_c^* = Z_{c+1} z_j (t_3)$. Let $Z := Z_{c-2} (s_i s_j s_i)^{-1}$. Let $w := w_{c-1} = w_c = w_{c+1}$. By [FZ99] Theorem 1.16(1),
\begin{equation}
\Delta_{u, w_i} (Z) \Delta_{w, w_j} (Z) = \Delta_{w, w_i} (Z) \Delta_{w, w_j} (Z) + \Delta_{w, w_j} (Z) \Delta_{w, w_i} (Z).
\end{equation}

Similarly to the proof of Proposition 4.6, we observe that each term in (4.15) equals the corresponding term in (4.16).

4.5. Non-mutation move: (B3) non-solid, short. Suppose that at least one of the indices $c-1, c, c+1$ is hollow. By Remark 4.5, we may assume that there are either one or two hollow indices in \{c-1, c, c+1\}. Explicitly, underlining the hollow crossings, the possible moves are $ij\leftrightarrow ji\, j$ and $ijj \leftrightarrow jjj$ (or the moves obtained from these by swapping the roles of $i$ and $j$).

For $l \in \{i, j\}$ and $e \in J_{\beta'}$, let us denote
\begin{equation}
A_l := \text{dlog} \prod_{k \neq i, j} \Delta_{c+1,k}^{\beta}, \quad B_l := \text{dlog} \Delta_{c+1,l}^{\beta}, \quad \text{ and } \quad T_e := \text{dlog} t_e.
\end{equation}

Using (2.14) and (2.15), we can express the dlogs of grid minors $\Delta_{c,l}^{\beta}$ for $l \in \{i, j\}$ and $e \in \{c-1, c, c+1\}$ in the symbols (4.18). Using $\text{dlog} \Delta_{c-2, l}^{\beta} = \text{dlog} \Delta_{c-2, l}$ for $l \in \{i, j\}$, we express $T_e^{\beta'}$ in terms of $T_e$ for all indices $e \in \{c-1, c, c+1\}$ which are solid in $\beta'$. Thus, we can express the forms $\omega_{\beta, e}, \omega_{\beta', e} \in \{c-1, c, c+1\}$ in terms of the symbols (4.18). Using a straightforward computation, we check $\omega_{\beta} = \omega_{\beta'}$.

We observe using Corollary 2.11 that the clusters $x_{\beta}$ and $x_{\beta'}^{\ast}$ differ by a relabeling, which shows (Q).

4.6. Non-mutation move: (B4). Suppose that $i \in I$. The isomorphism $\phi : \hat{R}_\beta \sim \hat{R}_{\beta'}$ sending $(X_b, Y_b) \mapsto (X_{b'}^{\ast}, Y_{b'}^{\ast})$ is given by $X_{c-m} = X_{m}^{\ast} := X_{c-1}, Y_{c-m} = Y_{m}^{\ast} := Y_{c-1}$, and $(X_{c'}^{\ast}, Y_{c'}^{\ast}) := (X_{c}, Y_{c})$ for all $0 \leq c < m-1$. The last crossing in $\beta_0 i$ and $\beta_0 (-i^*)$ is always hollow, and thus the statements (F) and (Q) follow trivially.
4.7. **Non-mutation move:** (B5) Suppose that \( i \in I \). The isomorphism \( \phi : \hat{R}_\beta \cong \hat{R}_{\beta'} \) sending \((X_*, Y_*) \mapsto (X'_*, Y'_*) \) is defined as follows. For \( c \in [m] \), we set \((X'_c, Y'_c) := (X_c, Y_c) \) and \( X'_0 := X'_1 \). Note that \( Y_0 = Y_1 = Y'_1 \), so \( (X'_c, Y'_c) \in \hat{R}_{\beta'} \). We let \( Y'_0 \) be the unique weighted flag satisfying \( X'_0 \xrightarrow{w_{c+1}} Y'_0 \xrightarrow{\Delta} Y'_1 \). It follows that \( X'_0 \xrightarrow{w_{c+1}} Y'_0 \) and \( Y'_0 \xrightarrow{\Delta} Y'_1 \), so \((X'_c, Y'_c) \in \hat{R}_{\beta'} \). The inverse map is defined similarly: \( X_0 \xrightarrow{w_{c+1}} Y_0 \xrightarrow{\Delta} Y_1 \).

The statement (F) is trivial if the first crossing of \( \beta \) is hollow. If the first crossing of \( \beta \) is solid, we have
\[
X'_0 = X'_1 = X_1 \xrightarrow{w_{c+1}} X_0 \xrightarrow{\Delta} Y_0 \xrightarrow{\Delta} Y_1 = Y'_1.
\]
It follows that after acting on all these flags by some \( g \in G \), we can find \( t, t' \in \mathbb{C} \) and \( h \in H \) such that
\[
X'_0 = X'_1 = X_1 = w_0 \cdot s_i h \bar{z}_i(t) U_{11}, \quad X_0 = w_0 \cdot s_i h U_+ \quad \text{and} \quad Y_0 = Y_1 = z_i(t) U_{11}.
\]
Here, we have \( X_0 \xrightarrow{w_{c+1}} Y_0 \) and thus \( X_0 \xrightarrow{w_{c+1}} Y'_0 \), and we have used a representative \( w_0 s_i \) of \( w_0 s_i \) in \( NG(H) \).

Let us denote \( h_0 := h_0^+ = h_0^- \) and \( h_0^* := (h_0^+)^* = (h_0^-)^* \). We have \( Z_0 = Y_0^{-1} X_0 = z_i(t)^{-1} w_0 s_i h_0 \), and thus, proceeding in the proof of Lemma 2.6, we get \( h_0 = h \). Similarly, \( Z'_0 = (Y'_0)^{-1} X'_0 = w_0 s_i h \bar{z}_i(t) \), so \( h_0^* = s_i \cdot h \). Applying (4.11), we find
\[
L'_{0,-i} = \text{dlog}(s_i h \bar{z}_i(t))^{-1/2} = \text{dlog}(h_0)^{-1/2} = -L_{0,i}.
\]
Applying (4.4) for \( e = 0 \), we obtain \( L_{0,i} = L_{0,-i} \) and \( L_{1,i} = L_{1,-i} \). Recall that \( L_{1,i} = L_{1,i}^* \). Thus, we get \( \omega_{0,1} = \omega_{0,1}^* \), and therefore \( \omega_{1,1} = \omega_{1,1}^* \), finishing the proof of (F).

We now prove (Q). Let \( \beta = i \beta_0 \) and \( \beta' = (-i) \beta_0 \). If the first crossing is hollow, the claim is trivial. Suppose that the first crossing is solid. We have \( \Delta_{c,k} = \Delta_{c,k}^* \) for all \( c \geq 1 \) and \( k \in \mathbb{Z} \). Thus, \( x_c = x_c^* \) for all \( c \in J_\beta \) such that \( c > 1 \). Since \( h_0^* = s_i \cdot h_0 \), Lemma 2.10 implies that \( x_1^* = x_1^{-1} M \), where \( M \) is a Laurent monomial in the grid minors \( \Delta_{0,k} \) for \( k \neq i \) of the same sign as \( i \). It follows from Propositions 2.20 and 2.21 that \( M \) is a Laurent monomial in the frozen variables other than \( x_1 \). This shows (Q).

4.8. **Deletion-contraction for double braid varieties.** We apply the cluster algebraic results from Section 3.3 to the seeds \( \Sigma_\beta \).

**Theorem 4.10.** Let \( i \in I \) and consider a double braid word \( \beta = ii \beta' \) on positive letters. If \((\hat{R}_i \beta, \Sigma_i \beta') \) and \((\hat{R}_{\beta'}, \Sigma_{i \beta'}) \) are sink-recurrent cluster varieties, then \((\hat{R}_\beta, \Sigma_\beta) \) is a sink-recurrent cluster variety.

**Proof.** Suppose first that at least one of the first two crossings in \( \beta \) is hollow, in which case 1 must be solid and 2 must be hollow. Consider an arbitrary point \((X_*, Y_*) \in \hat{R}_\beta \). Since the letters in \( \beta \) are positive, we have \( Y_0 = Y_1 = \cdots = Y_m = X_m \). Since \( w_2 \leq w_0 s_i \) and \( w_0 = w_0 \), we must have \( X_1 \xrightarrow{w_0} X_m \) and \( X_2 \xrightarrow{w_0} X_m \). It follows that \( h_1^+ \) and \( h_2^+ \) are regular functions on \( \hat{R}_\beta \). Choose a representative \( Z_2 = w_0 h_2 z_i^{-1} \) as in (2.6), and let \( t, t' \in \mathbb{C} \) be such that \( Z_1 = Z_2 z_i(t) \) and \( Z_0 = Z_2 z_i(t) z_i(t') \). Thus, \( t, t' \) are regular functions on \( \hat{R}_\beta \). Proceeding as in the proof of Lemma 2.6, we find \( h_1^+ = h_2^- \) and \( h_2^+ = h_1^- \) are regular on \( \hat{R}_\beta \). It follows that \( \Delta_{0,1} = t' \Delta_{1,1} \). For any \( e \in J_\beta \) such that \( e > 1 \), the function \( x_e \) depends on \( Z_2, Z_3, \ldots, Z_m \) but does not depend on \( t, t' \). By Proposition 2.20, we have \( \Delta_{0,i} = x_1 M \) for some monomial \( M \) in \( \{x_e \}_{e>1} \). The Deodhar hypersurface \( V_1 \) is clearly given by the equation \( t' = 0 \). We conclude that \( x_1 = t' \). We thus have an isomorphism
\[
r : \hat{R}_\beta \cong \hat{R}_i \beta \times \mathbb{C}^*, \quad (X_*, Y_*) \mapsto ((X_1, \ldots, X_m, Y_1, \ldots, Y_m), x_1).
\]
Moreover, since \( \Delta_{1,i} = M \) involves only frozen variables, we see that 1 is connected to only frozen indices in \( \hat{R}_i \beta \). It follows that the principal parts of \( \Sigma_\beta \) and \( \Sigma_i \beta' \) agree, and therefore \((\hat{R}_\beta, \Sigma_\beta) \) is a sink-recurrent cluster variety. Moreover, if \( \Sigma_{i \beta'} \) is really full rank then so is \( \Sigma_\beta \).

Suppose now that the first two crossings are both solid. Our goal is to apply Theorem 3.12. First, we show that \( \Sigma_\beta \) is sink-recurrent. Let \( \Gamma := \Gamma(\Sigma_\beta) \). The seed \( \Sigma_{\beta, \beta} \) is obtained from \( \Sigma_i \beta' \) by adding an isolated frozen variable \( x_1 \), and \( \Sigma_{\beta, \beta} \) is obtained from \( \Sigma_{i \beta'} \) by adding isolated frozen variables \( x_1 \) and \( x_2 \), so both of these seeds are sink-recurrent. Further, in \( \Sigma_\beta \), the variable \( x_1 \) is frozen, \( \hat{B}_{12} = -1 \), and \( \hat{B}_{1c} = 0 \) for mutable \( c > 2 \). Next, by Corollary 2.11, the sum of terms of \( \omega_\beta \) involving \( x_2 \) is clearly
of the form \( d_i \text{dlog} x_2 \wedge \text{dlog} M \) for a Laurent monomial \( M \) in \( x \), and thus \( \Sigma_\beta \) is integral at 2. We have shown that \( \Sigma_\beta \) satisfies Assumption 3.10.

Next, we show that the mutated cluster variable \( x'_2 \) is regular on \( \hat{R}_\beta \). We apply the moves \( ii \beta' \) \( (B1) \), \((−i)i\beta' \) \( (B5) \) to \((−i)\beta' \). Denote \( \hat{x} := x_{−i}i\beta' \) and \( \hat{x} := x_{i} \). It follows from the argument in Section 4.1 that \( x'_2 \) is mutable in \( \Sigma_{−i}i\beta' \), and by \((4.9) \), its mutation \( x'_2 \) is regular on \( \hat{R}_{−i}i\beta' \), as it equals the pullback \( \hat{x}'_2 \) times a monomial in the other cluster variables in \( \hat{x} \) with nonnegative exponents. As explained in Section 4.7, the seeds \( \Sigma_\beta \) and \( \Sigma_{−i}i\beta' \) are quasi-equivalent. By Lemma 3.8 we find that the mutation \( x'_2 \) differs from \( x''_2 \) by a unit \( \text{cf. part (2) of Corollary 2.24} \), and thus \( x'_2 \) is regular on \( \hat{R}_\beta \).

Let \( W := \{ x_2 \neq 0 \} \) and \( V := V_2 = \{ x_2 = 0 \} \) be the open-closed covering of \( \hat{R}_\beta \) coming from \( x_2 \). Our final goal is to construct isomorphisms

\( W \cong \hat{R}_{i\beta'} \times C^* \cong \mathcal{X}(\Sigma_{i\beta'}) \) and \( V \cong \text{Spec}(C[x'_2]) \times \hat{R}_\beta \cong C \times \mathcal{X}(\Sigma_{\beta}) \)

satisfying the conditions of Theorem 3.12. By Proposition 2.21 for \( e \in J_\beta \), \( e > 2 \), we have \( \text{ord}_{V_1} \Delta_{1,i} = 0 \) if and only if \( \text{ord}_{V_2} \Delta_{0,i} = 0 \). Moreover, the same proposition implies \( \text{ord}_{V_2} \Delta_{0,i} = 0 \). It follows by Proposition 2.20 that \( \Delta_{1,i} \) is equal to \( x_2 \) times a monomial in the frozen variables, and that \( \Delta_{0,i} \) is equal to \( x_1 \) times a monomial in the same set of frozen variables. Since \( W \) is the complement of \( V_2 \), we see that \( (X_\bullet, Y_\bullet) \in W \) if and only if \( X_1 \cong x_2 = X_m \). Thus, \( h_1^+ \) is a regular function on \( W \). We choose a representative \( Z_1 = w_0 h_1^+ \) and let \( t \in C \) be such that \( Z_0 = Z_1z_1(t') \). Then we get \( t' = \Delta_{0,i} / \Delta_{1,i} = M/x_1/x_2 \), where \( M \) is a Laurent monomial in the frozen variables other than \( x_1 \). Similarly to \((4.19) \), we let \( r : W \to \hat{R}_{i\beta'} \times C^* \) be the map sending \((X_\bullet, Y_\bullet) \to ((X_1, ..., X_m, Y_1, ..., Y_m), M/x_1/x_2) \). By assumption, we have \( \hat{R}_{i\beta'} \cong \mathcal{X}(\Sigma_{i\beta'}) \). The frozen index 1 is only connected to other frozen indices in \( \hat{\Gamma}(\Sigma_\beta) \). Thus, the seed \( \Sigma_{i\beta} \) is obtained from \( \Sigma_{i\beta'} \) by adding an isolated frozen vertex, and therefore \( \mathcal{X}(\Sigma_{i\beta}) \cong \mathcal{X}(\Sigma_{i\beta'}) \times C \).

Adjusting the isolated frozen variable by a Laurent monomial in the other frozen variables, we see that the pullbacks of \( x_1, ..., x_{n+m} \) under the inclusion \( \mathcal{X}(\Sigma_{i\beta}) \cong W \to X \) are indeed the same-named cluster variables in \( \Sigma_{i\beta} \). This verifies the assumptions on \( tW \) in Theorem 3.12.

Now suppose that \( (X_\bullet, Y_\bullet) \in V \). We have \( X_0 \cong x_2 \), but not \( X_1 \cong x_2 \), so we must have \( X_1 \cong s_i x_2 \), and therefore \( X_2 \cong x_2 \). Consider the map \( r : V \to \hat{R}_{i\beta'} \times C \) sending \((X_\bullet, Y_\bullet) \to ((X_2, ..., X_m, Y_2, ..., Y_m), x'_2) \). We claim that this map is an isomorphism. To construct an inverse, we need to show how to recover \( X_0, X_1, Y_0, Y_1 \) from the image of \( r \). We have \( Y_0 = Y_1 = x_2 \). Also, \( X_1 \) is uniquely determined by \( Y_1 = Y_2 \) and \( X_2 \), since \( Y_1 \cong x_2 \). It remains to recover \( X_0 \). Note also that we can recover the cluster variables \( x_2 \), \( e > 2 \), as well as the mutated cluster variable \( x'_2 \), from the image of \( r \). Since \( (X_\bullet, Y_\bullet) \in V \), the frozen variable \( x_2 \) is also recovered from the exchange relation \( 0 = x_2 x'_2 = M_1 + x_1 M_1 \) for \( x_2 \).

In order to recover \( X_1 \), we apply moves \( ii \beta' \) \( (B1) \), \((−i)i\beta' \) \( (B5) \) to \((−i)\beta' \). As in Section 4.7, let \( Y'_0 \) be the unique weighted flag satisfying \( X_0 \equiv x_2 \equiv Y_0' \). Then \( \tilde{X}_2 = X_2 \), and \( \tilde{Y}_2 = Y_2 \), and \( (X_1, \tilde{X}_0, \tilde{Y}_0, \tilde{Y}_1) = (X_2, X_1, Y_0', Y_0) \). We have that \( Y_0' \) is uniquely determined by \( X_2, Y_2 \), and \( \tilde{x} \): if \( \tilde{x} = 0 \) then \( Y_0' \) is uniquely determined by \( Y_2 \equiv s_i x_2 \); otherwise, we have \( X_2 \equiv Y_0' \), and the values of \( \tilde{x} \) uniquely fixes the \( U_+ \times U_+ \)-double coset \( \tilde{Z}_1 := (Y_0')^{-1} X_2 \) which determines \( Y_0' \). The weighted flag \( X_0 \) is then uniquely determined by \( X_1 \equiv x_2 \equiv Y_0' \). It thus suffices to show that \( \tilde{x} \) is uniquely determined by the image of \( r \). For \( e \in J_\beta \), \( e > 2 \), we have \( \tilde{x}_e = x_2 \). Moreover, \( \tilde{x}_1 = M/x_1 \) for some monomial \( M \) in the frozen variables \( x_e \) other than \( x_1 \) (all of which must satisfy \( e > 2 \) since \( x_2 \) is mutable). Finally, by \((4.9) \), \( \tilde{x}_2 \) differs from \( x'_2 \) by a monomial in the cluster variables other than \( x_2 \). We are done with verifying the assumptions on \( tW \) in Theorem 3.12.

We have verified all conditions in Assumption 3.10 and Theorem 3.12. Thus, \( (\hat{R}_\beta, \Sigma_\beta) \) is a cluster variety. We have already shown that it is sink-recurrent. \( \square \)
4.9. **Proof of Theorem 1.1** for $G$ simply-laced. We proceed by induction on the number $m$ of indices in $\beta$. Recall that we always assume $\pi(\beta) = w_0$. The base case is $m = \ell(w_0)$, where all indices are hollow. The cluster algebra is $\mathcal{A}_\beta = \mathbb{C}$ and the braid variety $\tilde{R}_\beta$ is a point.

Suppose now that $m > \ell(w_0)$. Applying \([B1]\) and \([B4]\) we can assume that all letters of $\beta$ belong to $I$. Since $G$ is simply-laced, all braid moves are automatically short. Applying \([B2]\) \([B3]\) we may therefore transform $\beta$ into a braid word of the form $\beta_1 i \beta_2$ for some braid words $\beta_1, \beta_2$ and $i \in I$. We can also apply *conjugation moves* to $\beta$: if $\beta = j \beta_0$, the conjugation move consists of the moves

\[
\begin{align*}
\beta &\to j \beta_0 \to (−j) \beta_0 \to \cdots \to \beta_0 (−j) \to \beta_0 j^*. \\
\end{align*}
\]

Applying conjugation moves, we may further transform $\beta$ into the form $\beta' := i i \beta_0 \beta^*$, where $\beta^*$ is obtained from $\beta_1$ by applying the map $j \mapsto j^*$ to each letter. Applying Theorem 4.10 to $\beta'$, we find that $(\tilde{R}_{\beta'}, \Sigma_{\beta'})$ is a cluster variety. It follows from Theorem 4.2 (for short moves) and Corollary 3.9 that $(\tilde{R}_{\beta}, \Sigma_{\beta})$ is therefore also a cluster variety. \(\square\)

**Remark 4.11.** It follows from our proof that the seed $\Sigma_{\beta}$ is really full rank when $G$ is simply-laced. Indeed, this property is preserved under moves \([B1] [B5]\) and by Remark 3.14 is compatible with deletion-contraction.

Finally, we show that for $G$ simply-laced, double braid moves correspond to mutation equivalence.

**Proposition 4.12.** Suppose that $G$ is simply-laced and $\beta, \beta'$ are related by a braid move \([B1] [B4]\). The seeds $\Sigma_{\beta}, \Sigma_{\beta'}$ are mutation equivalent (up to relabeling cluster variables).

**Proof.** By Theorem 1.1 for simply-laced $G$, $(\tilde{R}_{\beta}, \Sigma_{\beta})$ is a cluster variety. By \([Q]\), there is a seed $\Sigma'$, which differs from $\Sigma_{\beta}$ by mutation and possibly relabeling, such that $\Sigma' \sim \Sigma_{\beta'}$. We claim that these seeds are actually identical. Indeed, choose a double braid word $\beta_0$ such that all cluster variables of $\beta, \beta'$ become mutable in $\tilde{\beta} := \beta_0 \beta, \tilde{\beta} := \beta_0 \beta'$; cf. Lemma 2.25. Let $\tilde{\Sigma}'$ be obtained from $\Sigma_{\beta}$ using the same mutations and relabeling by which $\Sigma'$ was produced from $\Sigma_{\beta}$. Now, $(\tilde{R}_{\beta}, \Sigma_{\beta})$ is also a cluster variety, so by \([Q]\), $\tilde{\Sigma}' \sim \Sigma_{\beta}$. Since all frozen variables of $\Sigma'$ are mutable in $\tilde{\Sigma}'$, it follows that the seeds $\Sigma'$ and $\Sigma_{\beta'}$ are identical. \(\square\)

5. **Folding**

We review some background on folding before completing the proof in Section 6.

5.1. **Pinnings.** Let $G$ be a complex, simple, simply-connected algebraic group. Choose a pinning $(H, B_+, B_−, x_i, y_i; i \in I)$. Then there exists an algebraic group $\tilde{G}$ of simply-laced type with pinning $(\tilde{H}, \tilde{B}_+, \tilde{B}_−, \tilde{x}_i, \tilde{y}_i; i' \in \tilde{I})$; see [Lus94] §1.6. We have a bijection $\sigma : \tilde{I} \to I$ which extends to an automorphism $\sigma : \tilde{G} \to \tilde{G}$, and a map $\iota : G \to \tilde{G}$ which yields algebraic group isomorphisms

\[
\begin{align*}
\iota : G &\to \tilde{G}^\sigma, \quad H \sim \tilde{H}^\sigma, \quad B_+ \sim (\tilde{B}_+)^\sigma, \quad U_+ \sim (\tilde{U}_+)^\sigma. \\
\end{align*}
\]

The maps $g B_+ \mapsto \iota(g) \tilde{B}_+$ and $g U_+ \mapsto \iota(g) \tilde{U}_+$ induce isomorphisms of varieties:

\[
\begin{align*}
\iota : G / B_+ &\sim \tilde{G} / \tilde{B}_+, \quad \tilde{G} / U_+ \sim (\tilde{G} / \tilde{U}_+)^\sigma. \\
\end{align*}
\]

For the first isomorphism, see [Lus94] §8.8. The surjectivity and injectivity of the second map follow from that of the first by a straightforward computation.

For an element $i \in I$, we denote by $i \subset \tilde{I}$ the associated $\sigma$-orbit, i.e., the orbit under the cyclic group generated by $\sigma$. We also let $−i := \{−i' \mid i' \in I \} \subset −I$. We let $\{\tilde{\alpha}_{i'} \mid i' \in \tilde{I}\}$, $\{\tilde{\alpha}_{i'} \mid i' \in I\}$, and $\{\tilde{\omega}_{i'} \mid i' \in \tilde{I}\}$ be the simple roots, simple coroots, and fundamental weights of the root system of $\tilde{G}$. Letting $\tilde{\alpha}_{i'} := \{\tilde{\alpha}_{i'}, \tilde{\alpha}_{i'}^\vee\}$ be the entries of the associated Cartan matrix (and setting $\tilde{\alpha}_{−(−i')} := \tilde{\alpha}_{i'}$ and $\tilde{\alpha}_{(±i') (±i')} := 0$ as before), we have

\[
\begin{align*}
d_i = |i| \quad \text{and} \quad a_{ij} = \sum_{j' \in j} \tilde{\alpha}_{i'} \tilde{\alpha}_{j'}^\vee \quad \text{for all } i, j \in \pm I \text{ and } i' \in i. \\
\end{align*}
\]
The Coxeter generators of the Weyl group \( \hat{W} \) of \( \hat{G} \) are denoted by \( \{ \tilde{s}_{i'} | i' \in \hat{I} \} \). Restricting \( \iota \) to the normalizer of \( H \), we get a group isomorphism

\[
\iota: W \xrightarrow{\sim} \hat{W}, \quad s_i \mapsto \prod_{i' \in i} \tilde{s}_{i'}.
\]

Here the order inside \( i \) is immaterial since the corresponding elements \( \tilde{s}_{i'} \) commute. It follows that the longest element \( w_0 \in W \) gets mapped under (5.3) to the longest element \( \tilde{w}_0 \) of \( \hat{W} \), because \( \sigma: \hat{W} \to \hat{W} \) preserves Coxeter length and therefore \( \tilde{w}_0 \in \hat{W}^0 \). The following result is immediate.

**Lemma 5.1.** Let \( B_1, B_2 \in G/U_+ \). If \( B_1 \xrightarrow{w} B_2 \) then \( \iota(B_1) \xrightarrow{\iota(w)} \iota(B_2) \). If \( B_1 \xrightarrow{w} B_2 \) then \( \iota(B_1) \xrightarrow{\iota(w)} \iota(B_2) \).

**5.2. Braid varieties.** Let \( \beta = i_1 i_2 \ldots i_m \in (\pm 1)^m \) be a double braid word. Let \( \tilde{\beta} = i'_1 i'_2 \ldots i'_{\tilde{m}} \in (\pm \tilde{I})^{\tilde{m}} \) be obtained by concatenating the letters in \( i_1, i_2, \ldots, i_m \) (choosing the order inside each \( i_c \) arbitrarily), where \( \tilde{m} := |i_1| + |i_2| + \cdots + |i_m| \). Let \( \lambda_{\tilde{\beta}}: [\tilde{m}] \to [m] \) denote the unique order-preserving map satisfying \( |\lambda_{\tilde{\beta}}^{-1}(c)| = |i_c| \) for all \( c \in [m] \). It is clear that an index \( c \in [m] \) is solid (resp., hollow) if and only if all indices in \( \lambda_{\tilde{\beta}}^{-1}(c) \) are solid (resp., hollow). In other words, the set \( \tilde{J}_{\tilde{\beta}} \) of solid crossings for \( \tilde{\beta} \) is given by

\[
\tilde{J}_{\tilde{\beta}} = \lambda_{\tilde{\beta}}^{-1}(J_{\beta}).
\]

Let \( \tilde{\mathcal{Y}}'_{\tilde{\beta}} \) be the variety of tuples \( (X_i, Y_i) \) of weighted flags in \( \hat{G}/\hat{U}_+ \) satisfying

\[
\begin{array}{ccc}
X_0 & \xleftarrow{\iota(s_{i_1}^+)} & X_1 \\
\downarrow{\tilde{w}_0} & & \downarrow{\iota(s_{i_2}^+)} \\
Y_0 & \xrightarrow{\iota(s_{i_1}^-)} & Y_1
\end{array}
\]

\[
\begin{array}{ccc}
& \cdots \cdots & \\
& \vdots & \\
& \cdots \cdots & \\
X_m & \xleftarrow{\iota(s_{i_m}^+)} & X_m \\
\downarrow{\tilde{w}_0} & & \downarrow{\iota(s_{i_m}^+)} \\
Y_m & \xrightarrow{\iota(s_{i_m}^-)} & Y_m
\end{array}
\]

Let \( \tilde{\mathcal{Y}}'_{\beta} \) be obtained by omitting the condition \( X_0 \xrightarrow{\tilde{w}_0} Y_0 \). Lemma 2.1 yields isomorphisms \( \tilde{\mathcal{Y}}'_{\tilde{\beta}} \cong \tilde{\mathcal{Y}}_{\beta} \) and \( \mathcal{Y}'_{\beta} \cong \mathcal{Y}_{\beta} \). Let \( \tilde{R}'_{\beta} \) be the quotient of \( \tilde{\mathcal{Y}}_{\beta}' \) by the free \( \hat{G} \)-action. Then \( \tilde{R}'_{\beta} \cong \tilde{R}_{\beta} \).

The map \( \sigma \) acts on the varieties \( \tilde{\mathcal{Y}}'_{\beta}, \mathcal{Y}'_{\beta} \), and \( \tilde{R}'_{\beta} \) termwise by acting on each \( \tilde{X}_c \) and \( \tilde{Y}_c \). Let \( T_{\beta}' \subset \tilde{R}'_{\beta} \) be the image of the Deodhar torus \( T_{\beta} \subset \tilde{R}_{\beta} \) under the isomorphism \( \tilde{R}_{\beta} \cong \tilde{R}'_{\beta} \). We have the following straightforward result.

**Proposition 5.2.** Applying \( \iota \) termwise yields isomorphisms

\[
\tilde{\mathcal{Y}}_{\beta} \xrightarrow{\sim} (\tilde{\mathcal{Y}}_{\beta})^\sigma, \quad \mathcal{Y}_{\beta} \xrightarrow{\sim} (\mathcal{Y}_{\beta})^\sigma, \quad \tilde{R}_{\beta} \xrightarrow{\sim} (\tilde{R}_{\beta})^\sigma, \quad \text{and} \quad T_{\beta} \xrightarrow{\sim} (T_{\beta})^\sigma.
\]

**5.3. Grid minors.** Recall that we have met the character and cocharacter lattices \( X^\ast(H) := \text{Hom}(H, \mathbb{C}^\times) \), \( X_\ast(H) := \text{Hom}(\mathbb{C}^\times, H) \). The map \( \iota: H \to \hat{H} \) induces a map \( \iota_\ast: X_\ast(H) \to X_\ast(\hat{H}) \) sending \( \alpha^\vee \mapsto \sum_{i \in I} \tilde{\alpha}_i^\vee \) for \( i \in I \), so that \( \iota(\alpha_i^\vee(t)) = \prod_{i' \in i} \tilde{\alpha}_{i'}^\vee(t) \) for \( t \in \mathbb{C}^\times \). It also induces a map \( \iota^\ast: X^\ast(\hat{H}) \to X^\ast(H) \) sending \( \tilde{\omega}_{i'} \mapsto \omega_i \) for all \( i \in I \) and \( i' \in \hat{i} \), so that \( \iota(h)^\omega_{i'} = h^{\omega_i} \) for \( h \in H \). It follows that for all \( g \in G, v, w \in W, i \in I, \) and \( i' \in \hat{i} \), we have

\[
\Delta_{\omega_i, w_{i'}}(g) = \Delta_{\iota(\omega_i), \iota(w_{i'})}(\iota(g)).
\]

Let \( (X_i, Y_i) \in \hat{R}_{\beta}' \). As usual, for \( c = 0, 1, \ldots, m \), we denote \( Z_c := Y_{c-1}X_c \). Let \( \tilde{u}_c := \iota(u_c) \) and \( \tilde{w}_c := \iota(w_c) \). For \( i' \in \hat{i} \), consider analogs of grid minors for \( \hat{R}_{\beta}' \).

\[
\Delta_{c,i'}(X_i, Y_i) = \Delta_{\tilde{u}_c, \tilde{w}_{i'}}(Z_c), \quad \Delta_{c,-i'}(X_i, Y_i) = \Delta_{\tilde{w}_c, \tilde{u}_{i'}}(Z_c).
\]

Comparing (5.6)–(5.7) to Definition 2.9, we find that the grid minors on \( \hat{R}_{\beta}' \) are pullbacks of the minors defined in (5.7): for \( c = 0, 1, \ldots, m, i \in \pm I \), and \( i' \in \hat{i} \), we have

\[
iota^\ast \, \Delta_{c,i'} = \Delta_{c,i}.
\]

Using Corollary 2.11, we obtain the following description of chamber minors on \( \hat{R}_{\beta}' \), which we denote by \( \tilde{\Delta}_{c', i' \in \tilde{J}_{\beta}} \); cf. (5.4).
Lemma 5.3. Let \( c' \in J_{\hat{\beta}} \) be a solid crossing for \( \hat{\beta} \). Set \( \iota' := \iota_{c'} \) and \( c := \lambda_\beta(c') \in J \). Then the isomorphism \( \hat{\beta} \hat{\beta} \cong \hat{\beta} \hat{\beta} \) sends the chamber minor \( \Delta_{c'} \) to \( \Delta'_{c'-1,\iota'} \), and we have \( \iota^\ast \Delta'_{c'-1,\iota'} = \Delta_c \).

5.4. 2-form. Our next goal is to show the following result.

Lemma 5.4. Let \( \omega'_{\beta} \) be the pullback of the 2-form \( \omega_\beta \) on \( \hat{\beta} \beta \) under the isomorphism \( \hat{\beta} \beta \cong \hat{\beta} \beta \). We have \( \iota^\ast \omega'_{\beta} = \omega_\beta \).

Proof. Recall from (2.20)–(2.21) that we have 1-forms \( L_{c,i} = \frac{1}{2} \sum_{k \in \pm I} a_{ik} d \log \Delta_{c,k} \) on \( \hat{\beta}_\beta \) for \( (c,i) \in [m] \times (\pm I) \), and that for \( c \in [m] \) and \( i = i_c \), we set \( \omega_c(\beta) := \text{sign}(i) d \xi_c L_{c-1,i} \wedge L_{c,i} \). For \( i' \in \pm I \), introduce a 1-form \( L'_{c,i} := \frac{1}{2} \sum_{j' \in \pm I} \hat{a}_{ij'} d \log \Delta'_{c,j'} \). By Corollary 2.11 we have

\[
\omega'_{\beta} = \sum_{c \in J_{\beta}} \text{sign}(i_c) \sum_{i' \in \iota_c} L'_{c-1,i',i'} \wedge L'_{c,i}.
\]

Next, applying (5.8) and (5.2), we see that for all \( c \in [m] \), \( i \in I \), and \( i' \in \iota \), we have

\[
\iota^\ast L'_{c,i} = \iota^\ast \left( \frac{1}{2} \sum_{j' \in \pm I} \hat{a}_{ij'} d \log \Delta'_{c,j'} \right) = \frac{1}{2} \sum_{j \in I} \left( \sum_{j' \in \pm I} \hat{a}_{ij'} \right) d \log \Delta_{c,j} = L_{c,i}.
\]

The result follows by combining (5.9)–(5.10) with (5.2).

5.5. Folding seeds. We briefly review the notion of folding seeds, following [FWZ16, Section 4.4], though translating into our conventions.

Definition 5.5. Let \( \tilde{\Sigma} = (\tilde{T}, \tilde{x}, \tilde{d}, \tilde{\omega}) \) be a seed with \( \tilde{d} = (1, \ldots, 1) \), with mutable indices \( \tilde{j}^{\text{mut}} \) and frozen indices \( \tilde{j}^{\text{fro}} \). Let \( \sigma \) be a bijection acting on \( \tilde{j} := \tilde{j}^{\text{mut}} \sqcup \tilde{j}^{\text{fro}} \). Let \( J \) be the set of \( \sigma \)-orbits, and for \( j \in J \), we denote the corresponding orbit by \( \tilde{j} \). An orbit is \emph{mutable} (resp., \emph{frozen}) if it consists entirely of mutable (resp., frozen) indices. The bijection \( \sigma \) also acts on the set of cluster variables by \( \sigma(\tilde{x}_{j'}) = \tilde{x}_{\sigma(j')} \). We call \( \tilde{\Sigma} \) \emph{weakly \( \sigma \)-admissible} if:

1. Every orbit is either mutable or frozen.
2. The 2-form \( \omega \) is invariant under the \( \sigma \)-action.
3. For all \( a', a'' \in \tilde{j}^{\text{mut}} \) in the same \( \sigma \)-orbit, \( \tilde{B}_{a'a''} = 0 \), where \( \tilde{B} \) is the exchange matrix of \( \tilde{\Sigma} \).

Part (1) implies a natural decomposition \( J = J^{\text{mut}} \sqcup J^{\text{fro}} \). The map \( \sigma \) also acts on the torus \( T \) by permuting coordinates. Notice that \( T^\sigma \) is isomorphic to \( (\mathbb{C}^\times)^{|J|} \). We denote by \( \iota : T^\sigma \hookrightarrow T \) the inclusion map.

Definition 5.6. Suppose \( \tilde{\Sigma} \) is weakly \( \sigma \)-admissible, with notation as in Definition 5.5. The \emph{folded seed} is a seed with index set \( J = J^{\text{mut}} \sqcup J^{\text{fro}} \), defined as \( \iota^\ast \tilde{\Sigma} = \Sigma := (T, x, d, \omega) \) where

- \( T = T^\sigma \);
- \( x = (x_j)_{j \in J} \), where for \( j \in J \), \( x_j := \iota^\ast \tilde{x}_{j'} \) for any \( j' \in \tilde{j} \);
- \( d_j = |\tilde{j}| \) for \( j \in J \);
- \( \omega = \iota^\ast \tilde{\omega} \).

Note that \( x_j \) is well-defined, since \( \iota^\ast \tilde{x}_{j'} = \iota^\ast \tilde{x}_{\sigma(j')} \) for all \( j' \in \tilde{j} \). The exchange matrix \( \tilde{B} \) of \( \tilde{\Sigma} \) is therefore written in terms of the exchange matrix \( \tilde{B} \) of \( \tilde{\Sigma} \) as

\[
\tilde{B}_{ab} = \sum_{a' \in a} \tilde{B}_{a'a}, \quad \text{where } b' \in b \text{ is arbitrary}.
\]

In particular, if \( \tilde{\Sigma} \) is integral then so is \( \Sigma \). For the rest of this subsection, we assume that \( \tilde{\Sigma} \) and \( \Sigma \) are integral.
For weakly $\sigma$-admissible $\Sigma$ and $j \in J^{\text{mut}}$, we denote by $\mu_j \hat{\Sigma} = \mu_j(\hat{\Sigma})$ the result of mutating $\hat{\Sigma}$ once at each index in $j$, and call $\mu_j$ an orbit-mutation. Note that $\mu_j \hat{\Sigma}$ does not depend on the order of mutation. The seed $\mu_j \hat{\Sigma}$ may not be weakly $\sigma$-admissible; we introduce the following notion to avoid such mutations.

**Definition 5.7.** Let $\hat{\Sigma}$ be a weakly $\sigma$-admissible seed and $j \in J^{\text{mut}}$. We call $\mu_j$ quasi-admissible if for all $k \in J^{\text{mut}}$, we have $B_{k'j'} B_{k''j'} \geq 0$ for all $k', k'' \in k$ and $j', j'' \in j$.

The name “quasi-admissible” is justified by the following proposition.

**Proposition 5.8.** Let $\hat{\Sigma}$ be a weakly $\sigma$-admissible seed and $j \in J^{\text{mut}}$. If $\mu_j$ is quasi-admissible, then $\mu_j(\hat{\Sigma})$ is weakly $\sigma$-admissible and

$$\mu_j(A) \sim B(\mu_j(\hat{\Sigma})),\]$$

where $A$ is an inclusion of the associated tori.

**Proof.** We use the notation of Definitions 5.5 and 5.6. In particular, let $\Sigma := \hat{\Sigma}$ be the folded seed on index set $J$. Since the exchange matrix $\hat{B}$ of $\Sigma$ is skew-symmetric, it is equivalent to a quiver $\hat{Q}$; we will use the two interchangeably.

It is clear that $\mu_j \hat{\Sigma}$ satisfies condition (1) of Definition 5.5. Because there are no arrows between vertices in $j$, mutating at all vertices of $j$ shows that the exchange matrix of $\mu_j \hat{\Sigma}$ satisfies $(\mu_j \hat{B})_{a'b'} = (\mu_j \hat{B})_{a' \sigma(b')} \sigma(a') \sigma(b')$ for $a', b' \in \hat{J}$. The assumption that $\mu_j$ is quasi-admissible implies that for $k \in J^{\text{mut}}$, $(\mu_j \hat{B})_{k'k''} = 0$ for all $k', k'' \in k$. Thus, $\mu_j \hat{\Sigma}$ is weakly $\sigma$-admissible.

Let $\Sigma_1 := \mu_j(\hat{\Sigma})$ and $\Sigma_2 := \mu_j(\hat{\Sigma})$. Let $y$ and $z$ be the clusters in $\Sigma_1$ and $\Sigma_2$, respectively. For $k \neq j$, $y_k = z_k$ because both are equal to the cluster variable $x_k$ of $\Sigma$.

To analyze the relationship between $y_j$ and $z_j$, we need the following notions. Let $a, k \in J$ and choose $a' \in \hat{a}$, $k' \in \hat{k}$. We call a path $a' \to k' \to a''$ in $\hat{Q}$ a bad path if $a', a''$ are in the same orbit; condition (2) of Definition 5.7 implies that no bad path in $\hat{Q}$ begins or ends in a mutable orbit. Let $P_{k'}$ be a maximal (by inclusion) collection of arrow-disjoint bad paths with middle vertex $k'$.

In $\hat{\Sigma}$, for $j' \in j'$, the mutation $\hat{x}_{j'}$ of $\hat{x}_{j'}$ is defined by the exchange relation

$$\hat{x}_{j'} \hat{x}_{j'} = M' N' + M'' N'', \quad \text{where}$$

$$N' := \prod_{(a' \to j' \to a'')} P_{k'}, \quad x_{j'}, \quad N'' := \prod_{(a' \to j' \to a'') \in P_{k'}} x_{a''},$$

and $M', M''$ are the appropriate monomials in the cluster variables of $\hat{\Sigma}$. Notice that if $\hat{x}_{a'}$ appears in $M'$ and $\hat{x}_{a''}$ appears in $M''$ for $a' \in \hat{a}, b' \in \hat{b}$, then $a \neq b$, by the maximality of $P_{k'}$. Notice also that by assumption, $N'$ and $N''$ are monomials in the frozen variables. We set $N := \hat{\Sigma}(N') = \hat{\Sigma}(N'')$. Using (5.11), we have

$$t^* \hat{x}_{j'} = N^* x_{j'} + x_{j'}^* M' + x_{j'}^* M'', \quad \text{and} \quad \mu_j(x_{j'}) = x_{j'}^* M' + x_{j'}^* M''.$$

This shows that the tori and the lattices spanned by the frozen of $\Sigma_1, \Sigma_2$ agree, and that cluster variables differ by Laurent monomials in frozens. The multipliers $d$ of both seeds are the same by definition. The 2-forms of the two seeds agree by the functoriality of pullbacks. \hfill $\square$

## 6. Proof of Theorem 4.2 for Long Braid Moves

Fix multiply-laced braid words $\beta, \beta'$ related by a long braid move (B3) so that

$$\beta = \beta_1 \underbrace{i_j i_j \ldots}_{m_{ij} \text{ letters}} \beta_2 = \beta_1 \sigma \beta_2 \quad \text{and} \quad \beta' = \beta_1 \underbrace{i_j i_j \ldots}_{m_{ij} \text{ letters}} \beta_2 = \beta_1 \sigma \beta_2.$$

By Definition 4.1 we have an isomorphism $\phi: \hat{R}_\beta \sim \hat{R}_{\beta'}$. The goal of this section is to show (F) (Q), and thus Theorem 4.2 for the geometrically defined seeds $\Sigma_{\beta}, \phi^* \Sigma_{\beta'}$ and a particular mutation $\Sigma'$ of
We focus on the first quasi-equivalence. By Lemma 6.2, it suffices to show that for each 
\[ \text{ord} \] 
where the seeds \( \dot{\Sigma} \) also fix \( \tilde{\pi} \) in Table 1(a–b). In our tables, we only list the restriction of \( \sigma \) would like to show that \( \Sigma \) are uniquely determined; cf. Definition 4.1. We have the equality \( \dot{\phi} = \phi \circ \alpha \). We have 
\[ \text{ord} \] 
Lemma 6.1. We have the equality \( \dot{\phi} = \phi \circ \alpha \). We have 
\[ \text{ord} \] 
Lemma 6.2. We have \( \omega_\beta = \phi \circ \omega_{\beta'} \); that is, \( \{ \mathcal{F} \} \) holds for \( \beta, \beta' \).

Proof. We have \( \omega_\beta = \ell ; \omega_\beta = \phi \circ \omega_{\beta'} = \phi \circ \omega_{\beta'} \), where we have used Lemma 5.4, (F) for \( \tilde{\beta}, \tilde{\beta}' \), Lemma 6.1, and Lemma 6.4 again (in that order).

6.2. Proof of \( \{ \mathcal{Q} \} \) for long braid moves. We continue to use the notation established earlier in this section. Without loss of generality, we assume that either \( \delta = i j j \) (in the case when \( a_i, a_j \) form a root subsystem of type \( B_2 \) or \( C_2 \), where \( |i| = 2 \) and \( |j| = 1 \), or \( \delta = 12121 \) (in the case of \( G = G_2 \), where \( |1| = 3 \) and \( |2| = 1 \).

The words \( \delta, \delta' \) involve indices \( r+1, \ldots, r+p \), and for convenience, we decrease all indices by \( r \) so that \( \delta, \delta' \) are supported on \( 1, \ldots, p \). Similarly, we assume that \( \delta, \delta' \) involve indices \( 1, \ldots, p \). We define the seed
\[ \Sigma' := \pi_{\text{fold}} \circ \mu_{\text{fold}}(\Sigma), \]
where \( \pi_{\text{fold}} \) is a permutation and \( \mu_{\text{fold}} \) is a sequence of mutations involving \( 1, \ldots, p \). We list \( \pi_{\text{fold}}, \mu_{\text{fold}} \) in Table 1(a–b). In our tables, we only list the restriction of \( \pi_{\text{fold}} \) to the solid crossings in \( 1, \ldots, p \). We would like to show that \( \Sigma' \) and \( \phi \circ \Sigma_{\beta'} \) are quasi-equivalent. To do so, we will eventually fold the seeds \( \Sigma_{\tilde{\beta}} \) and \( \Sigma_{\beta'} \), and then establish a chain of quasi-equivalences involving the folded seeds, \( \Sigma' \) and \( \phi \circ \Sigma_{\beta'} \).

As a first step, we fix a particular sequence \( S \) of braid moves (B3) between the lifts \( \tilde{\beta}, \tilde{\beta}' \), and thus also fix \( \tilde{\phi} \). By Theorems 1.1 and 4.2, there is a corresponding mutation sequence \( \mu_{\text{braid}} \) and relabeling \( \pi_{\text{braid}} \) such that
\[ \pi_{\text{braid}} \circ \mu_{\text{braid}}(\tilde{\Sigma}_{\tilde{\beta}}) = \tilde{\phi} \circ \tilde{\Sigma}_{\tilde{\beta'}}, \]
where the seeds \( \tilde{\Sigma}_{\tilde{\beta}}, \tilde{\Sigma}_{\tilde{\beta}'} \) are the seeds denoted \( \Sigma_{\tilde{\beta}}, \Sigma_{\tilde{\beta}'} \) in Section 2. The sequence \( S \) of braid moves is chosen so that \( \mu_{\text{braid}} \) and \( \pi_{\text{braid}} \) are as in Table 1(c–d).

We have the relabeling maps \( \lambda_{\beta}: [\tilde{m}] \to [m] \) and \( \lambda_{\beta'}: [\tilde{m}] \to [m] \) as in Section 5.2, where \( \beta, \beta' \) (resp., \( \tilde{\beta}, \tilde{\beta}' \)) are on \( m \) (resp., on \( \tilde{m} \)) letters. By construction, we can extend the action of \( \sigma \) from \( I \) to \( \tilde{J}_{\tilde{\beta}} \) and \( \tilde{J}_{\tilde{\beta}'} \). Specifically, for each letter \( i \) in \( \beta \), \( \sigma \) permutes the letters \( i_{c} \) in the corresponding consecutive subword \( \lambda_{\beta}^{-1}(c) \) of \( \tilde{\beta} \), and similarly for \( \beta' \).

Proposition 6.3. Let \( C \subset J_{\beta} \) be the set of indices \( c \) such that none of \( c' \in c \) is used in \( \mu_{\text{braid}} \), and let \( C := \lambda_{\beta}^{-1}(C) \). Let \( C' \subset J_{\beta'} \) and \( C' \subset J_{\beta'} \) be defined similarly. Then
\[ \ell' \circ (\Sigma_{\beta} \setminus C) \sim \Sigma_{\beta} \setminus C \quad \text{and} \quad \ell' \circ (\Sigma_{\beta'} \setminus C') \sim \Sigma_{\beta'} \setminus C'. \]

Proof. We focus on the first quasi-equivalence. By Lemma 6.2, it suffices to show that for each \( e \in J_{\beta} \setminus C \) and \( e' \in \lambda_{\beta}^{-1}(e) \), we have \( \ell'(x_{e'}) = x_{e} \). Let us fix such \( e, e' \). Choose also \( c \in [0, p] \), \( k \in I \), and \( k' \in k \). It is enough to show the statement
\[ \text{ord}_{\Delta_{c,k}} \Delta_{c,k} = \text{ord}_{\Delta_{c',k'}} \Delta_{c',k'}. \]
where \( V'_\rho \subset Y'_\rho \) is the Deodhar hypersurface corresponding to \( \hat{x}_\rho \in \mathbb{C}[\hat{R}_{\beta}] \cong \mathbb{C}[\hat{R}'_{\beta}] \) and \( \Delta'_{c,k'} \) was defined in Section 5.3.

We observe that the hollow crossings in \( \delta, \delta' \) (and thus in \( \tilde{\delta}, \tilde{\delta}' \)) have a very special form: one of \( \delta, \delta' \) has hollow crossings in positions \([r+1,p]\), while the other one has hollow crossings in positions \([r,p-1]\), for some \( r \); cf. Table 1(a–b). In this case, computing \( \text{ord}_{V_{\rho}} \Delta_{c,k} \) is straightforward. First, suppose that \( r \leq r \). Then all crossings in \([p]\) to the left of \( e \) are solid. It follows from Propositions 2.20 and 2.21 and Corollary 2.11 that for \( c \in [0,p] \) and \( k \in I \), we have \( \text{ord}_{V_{\rho}} \Delta_{c,k} = 1 \) if \( (c,k) \in \{ (e-1,i_e), (e-2,i_e) \} \) and \( \text{ord}_{V_{\rho}} \Delta_{c,k} = 0 \) otherwise. Applying the same argument to compute \( \text{ord}_{V'_{\rho}} \Delta'_{c,k'} \), we obtain (6.3). It remains to consider the case \( e = p \) when the hollow crossings are in positions \([r,p-1]\). The crossings \( r-1 \) and \( r-2 \) are solid, so Proposition 2.21 implies that \( \text{ord}_{V_{\rho}} \Delta_{c,k} = 0 \) for \( k = i_{r-1}, c < r-1 \) or \( k = i_{r-2}, c < r-2 \). Here \( \{ i_{r-1}, i_{r-2} \} = \{ i,j \} \). For \( e = p-1 \), we have \( \text{ord}_{V_{\rho}} \Delta_{c,k} = (\omega_k, \alpha_{i_{p-1}}^\vee) \) by Propositions 2.20 and 2.21. Thus, for \( c \in [r,p-1] \), Lemma 2.6 implies that \( \text{ord}_{V_{\rho}} \Delta_{c,k} = (\omega_k, s_{i_{c+1}} \ldots s_{i_{p-1}} \alpha_{i_{p-1}}^\vee) \). We have thus determined the values \( \text{ord}_{V_{\rho}} \Delta_{c,k} \) for all \( (c,k) \in [0,p] \times \{ i,j \} \) except for \( (c,k) = (r-1, i_{r-2}) \). By Corollary 2.11, we have \( \text{ord}_{V_{\rho}} \Delta_{r-1,i_{r-2}} = \text{ord}_{V_{\rho}} \Delta_{r,i_{r-2}} = 0 \). It is clear that we have \( \text{ord}_{V_{\rho}} \Delta_{c,k} = 0 \) for \( k \in I \setminus \{ i,j \} \). Computing \( \text{ord}_{V'_{\rho}} \Delta'_{c,k'} \) via a similar argument, we obtain (6.3).

For the remainder of the section, let \( C, C, C', C' \) be as in Proposition 6.3.

The sequence \( \mu_{\text{braid}} \) is ill-adapted to folding, so we find another mutation sequence \( \mu_{\text{lift}} \) relating \( \Sigma_{\beta} \) and a relabeling \( \pi_{\text{lift}} \) of \( \phi^* \Sigma_{\beta} \). Explicitly, \( \mu_{\text{lift}} \) is a sequence of orbit-mutations lifting the sequence \( \mu_{\text{fold}} \) from Table 1(a–b) and \( \pi_{\text{lift}} \) is given in Table 1(e–f) Part 1 of the next result generalizes [FG06, Theorem 3.5], which concerns the “all solid” case.

**Proposition 6.4.** Let \( \mu_{\text{lift}} \) and \( \pi_{\text{lift}} \) be as listed in Table 1(e–f). Then

1. (1) \( \pi_{\text{braid}} \circ \mu_{\text{braid}}(\Sigma_{\beta}) = \pi_{\text{lift}} \circ \mu_{\text{lift}}(\Sigma_{\beta}) \).
2. (2) \( \mu_{\text{lift}} \) is a sequence of quasi-admissible mutations of \( \Sigma_{\beta}^C \).

We delay the proof of Proposition 6.4 to the end of the section. Proposition 5.8 and part (2) of Proposition 6.4 together imply the following result.

**Corollary 6.5.** We have \( \iota^*(\mu_{\text{lift}} \Sigma_{\beta}^C) \sim \mu_{\text{fold}}(\iota^* \Sigma_{\beta}^C) \).

**Proof of (Q) for long braid moves.** We have a string of quasi-equivalences:

\[
\iota^*(\mu_{\text{lift}} \Sigma_{\beta}^C) \sim \mu_{\text{fold}}(\iota^* \Sigma_{\beta}^C) \sim \mu_{\text{fold}} \Sigma_{\beta}^C,
\]

where the first quasi-equivalence is Corollary 6.5 and the second follows from Proposition 6.3 and Lemma 3.8. On the other hand,

\[
\iota^*(\mu_{\text{lift}} \Sigma_{\beta}^C) = \iota^*(\pi_{\text{lift}} \phi^* \Sigma_{\beta}^C) = \pi_{\text{lift}} \phi^* \Sigma_{\beta}^C = \pi_{\text{lift}} \phi^* \Sigma_{\beta}^C 
\]

where the first equality holds by Proposition 6.4 and (6.2), the second holds by direct computation (cf. Table 1(e–f)), the third holds by Lemma 6.1 and the final quasi-equivalence follows from Proposition 6.3 and the fact that \( \phi^* \) preserves quasi-equivalence.

Summarizing, we have that \( \mu_{\text{fold}} \Sigma_{\beta}^C \) is quasi-equivalent to (a relabeling of) \( \phi^* \Sigma_{\beta}^C \). Notice that the cluster variables of \( \mu_{\text{fold}} \Sigma_{\beta}^C \), resp., \( \phi^* \Sigma_{\beta}^C \) are equal to those of \( \mu_{\text{fold}} \Sigma_{\beta} \), resp., \( \phi^* \Sigma_{\beta} \). By assumption \( (\hat{R}_{\beta}, \Sigma_{\beta}) \) is a cluster variety, so Proposition 3.3 implies the cluster variables of \( \mu_{\text{fold}} \Sigma_{\beta} \) are irreducible elements of \( \mathbb{C}[\hat{R}_{\beta}] \). On the other hand, by Corollary 2.24 the cluster variables in \( \phi^* \Sigma_{\beta} \) are irreducible elements of \( \mathbb{C}[\hat{R}_{\beta}] \). Thus, the cluster variables in \( \mu_{\text{fold}} \Sigma_{\beta} \) and \( \phi^* \Sigma_{\beta} \) can differ only by units and \( \mu_{\text{fold}} \Sigma_{\beta} \) is quasi-equivalent to (a relabeling of) \( \phi^* \Sigma_{\beta} \).

**Proof of Proposition 6.4.** Recall that \( \tilde{\beta} = \beta_1 \tilde{\delta} \beta_2 \) and \( \tilde{\beta}' = \beta_1 \tilde{\delta}' \beta_2 \), and that we index the crossings of \( \delta \) by \( 1, \ldots, p \). Let \( J := J_{\delta} \setminus C \) be the set of indices which are mutated in \( \mu_{\text{fold}} \).
Table 1. The mutation sequences $\mu_{\text{fold}}$, $\mu_{\text{braid}}$, $\mu_{\text{lift}}$, and the relabelings $\pi_{\text{fold}}$, $\pi_{\text{braid}}$, $\pi_{\text{lift}}$ used in Section 6. Hollow crossings are underlined. We denote $\mu(a_1, \ldots, a_r) := \mu a_1 \circ \cdots \circ \mu a_r$. For the case $B_2/C_2$, we denote $i = \{i', i''\}$, $j = \{j'\}$, $\tilde{\delta} = i'' i' j '' j', \tilde{\delta}' = j' i'' i' j''$; for $G_2$, we denote $1 = \{1, 3, 4\}$, $2 = \{2\}$, $\delta = 134213421342$, $\delta' = 213421342134$. In (e) and (f), the cases where $\mu_{\text{lift}}$ and $\mu_{\text{braid}}$ coincide are omitted; we define $\pi_{\text{lift}} := \pi_{\text{braid}}$ in those cases.
Table 2. The quivers $\hat{Q}_{res}$ from Proposition 6.4 listed in the same order as in Table 1(e–f). In the cases where $\mu_{braid} = \mu_{lift}$, $\hat{Q}_{res}$ has no arrows.

We show part $[1]$. By [GSV08] Theorem 4], a seed in $A(\hat{\Sigma}_{\beta})$ is uniquely determined by its cluster, so we need only check $[1]$ at the level of cluster variables. This is easy to check for the cluster variables \{x_e : c \in C\} which are not touched by either mutation sequence.

Let $\hat{\Sigma}_{\beta} = (\hat{x}, \hat{Q})$, and let $\hat{Q}_{res}$ be the induced subquiver of $\hat{Q}$ on $\hat{J} := \lambda_{\beta}^{-1}(J)$. Let $\hat{Q}_{fr}^{res}$ be the framing of $\hat{Q}_{res}$; the extended exchange matrix of $\hat{Q}_{res}^{fr}$ is thus of size $2|\hat{J}| \times |\hat{J}|$ and the bottom $|\hat{J}| \times |\hat{J}|$ submatrix is the identity. We denote by $\hat{\Sigma}_{res}$ the seed $(\hat{y}, \hat{Q}_{res}^{fr})$ for some cluster $\hat{y}$. By [FZ07] Theorem 3.7, to show $[1]$, it suffices to check that

\[
\pi_{braid} \circ \mu_{braid}(\hat{\Sigma}_{res}) = \pi_{lift} \circ \mu_{lift}(\hat{\Sigma}_{res}).
\]

The relevant cluster variables in $\mu_{braid}(\hat{\Sigma}_{\beta})$ and $\mu_{fold}(\hat{\Sigma})$ can then be obtained from those in (6.4) by specialization.

To check (6.4), recall the description of the orders of vanishing of the cluster variables in $\hat{J}$ from the proof of Proposition 6.3. This description only depends on which crossings in $\delta, \delta'$ are hollow, which in turn is determined by which crossings in $\hat{\beta}_2$ are hollow. This implies that to compute $\hat{Q}_{res}$, we may assume $\hat{\beta}_2$ is a type $A_3$ braid word (in the $B_2/C_2$ case) or a type $D_4$ braid word (in the $G_2$ case) consisting entirely of hollow crossings. Applying the algorithm from Section 7 to the simply-laced braids $\hat{\beta}, \hat{\beta}'$; cf. Remark 7.3, we get that $\hat{Q}_{res}$ is as displayed in Table 2. Equation (6.4) may then be verified by computer.

Part $[2]$ is also established by direct computation in $\hat{Q}_{res}$.

This completes the proof of Theorem 4.2 for long braid moves.

6.3. Finishing the proof. We now have shown Theorem 4.2 for all braid moves. Theorem 1.1 for multiply-laced $G$ follows by the argument in Section 4.9. Repeating the proof of Proposition 4.12, we have the following.

Proposition 6.6. Suppose $\beta, \beta'$ are related by a braid move $[B1]$–$[B4]$. The seeds $\Sigma_\beta, \Sigma_{\beta'}$ are mutation equivalent (up to relabeling cluster variables).

Continuing Remark 4.11, we obtain the following.

Corollary 6.7. The seeds $\Sigma_{\beta}$ are really full rank for all $\beta$.

Combining Corollary 6.7 with the proof of Proposition 3.6 and [LS16] (see also [GLSBS22] Section 10), we obtain the following. (While [LS16] work in the skew-symmetric setting, the curious Lefschetz theorem therein generalizes to the skew-symmetrizable case.)

Theorem 6.8. Even-dimensional double braid varieties $\hat{\mathcal{R}}_{\beta}$ satisfy the curious Lefschetz property and thus curious Poincaré symmetry. Odd-dimensional $\hat{\mathcal{R}}_{\beta}$ satisfy curious Poincaré symmetry.

7. Combinatorial algorithm

The exchange matrices for our seeds $\Sigma_\beta$ are defined via Deodhar geometry. Let $e \in J_\beta$ and $c \leq e$. We give an algorithm to compute the order of vanishing of $\Delta_c$ on $V_e$, which determines our cluster algebras.
The function $h^\pm_\beta$ of Section 2.4 is an $H$-valued character on $T_\beta$. We may thus write $h^\pm_\beta = \prod_{e \in J_\beta} \gamma^\pm_{\beta,e}(x_e)$, where $\gamma^\pm_{\beta,e}$ are cocharacters of $H$ satisfying $\gamma^\pm_{\beta,e} = u_e \cdot \gamma^+_\beta$.

**Lemma 7.1.**

1. Suppose $\beta'$ is obtained from $\beta$ by removing the first $c-1$ letters. Then $\gamma^\pm_{\beta,e} = \gamma^\pm_{\beta',1,e-c+1}$.

2. Suppose $\beta'$ is obtained from $\beta$ by doing non-mutation moves involving indices greater than $c$, and let $e'$ be the image of $e$ under the resulting identification of cluster seeds. Then we have $\gamma^\pm_{\beta,e} = \gamma^\pm_{\beta',e'}$.

3. Suppose $\beta'$ is obtained from $\beta$ by removing solid crossings greater than $e$. Then $\gamma^\pm_{\beta,e} = \gamma^\pm_{\beta',e}$.

**Proof.** Part (1) follows from Lemma 2.25 and part (2) is immediate from the definitions. We prove (3). Suppose $\beta$ has a solid crossing $e' > e$. Using (1) and (2), we may assume that $e \in J^\text{mut}_\beta$, that $e'$ is the largest solid crossing, and that $i_{e'} = i_{e'+1} \in I$. Let $\beta''$ be obtained from $\beta$ by removing the letter $i_{e'}$ from $\beta$. Let $W \subset \hat{R}_\beta$ be the open subset obtained by removing the Deodhar hypersurface $V_e$, if $e'$ is mutable; otherwise, let $W := \hat{R}_\beta$. The projection $\pi: W \to \hat{R}_\beta$, given by forgetting the flags $(X_\nu, Y_\nu)$ is a fiber bundle with fiber $C^\times$. We have $\pi^*(\Delta^\nu_c) = \Delta^\beta_c$ and $\pi$ maps $V^\beta_c \cap W$ surjectively onto $V^\beta_c$. (Here we use the superscript to refer to the braid variety on which $\Delta^\beta_c$ and $\Delta^\beta_c$ are defined.) Since both $V^\beta_c \subset \hat{R}_\beta$ and $V^\beta_c \subset \hat{R}_\beta$ are hypersurfaces, it follows that the order of vanishing of $\Delta^\beta_c$ on $V^\beta_c$ is equal to that of $\Delta^\beta_c$ on $V^\beta_c$. Repeating this argument, we obtain (3). \hfill \Box

Using Lemma 7.1, we may assume that $c = 1, e = m - 1$ and $\beta = (-b^{\text{rev}})akkk$, where $a, b$ are words in $I$, and $(-b^{\text{rev}})$ is obtained by reversing $b$ and applying the map $i \mapsto -i^\circ$ to each letter, and $k = i_e = i_m \in I$. Define

$$\gamma(a,k,b) := \gamma((-b^{\text{rev}})akkk,1,m-1),$$

and let $a$ and $b$ denote the Demazure product of $a$ and $b$, respectively.

**Proposition 7.2.** Suppose that Theorems I, II and IV hold for $G$. Then the cocharacter $\gamma(a,k,b)$ satisfies, and is recursively defined by the following properties.

I) We have $\gamma(a,k,b) = 0$ if $a = w_0$ or $b = w_0$.

II) The cocharacter $\gamma(a,k,b) = \gamma(a,k,b)$ only depends on the Demazure products $a,b$.

III) We have $\gamma(a,k,0) = a \gamma^\circ_k$ if $a \not= k > a$ and $\gamma(a,k,0) = 0$ if $a \not= k < a$.

IV) Suppose that $a, b$ are reduced. We let $a' = ia$ and $b' = b'j$, where the Demazure products satisfy $a'' = s_{a'}a$ and $b'' = s_{b'}b'$.

1. If $a' * s_{k} * b = a * s_{k} * b$, then $\gamma(a',k,b) = s_{i} \cdot \gamma(a',k,b)$.

2. If $a' * s_{k} * b = a * s_{k} * b'$, then $\gamma(a',k,b) = \gamma(a',k,b')$.

3. If $w := a * s_{k} * b = a * s_{k} * b = a * s_{k} * b'$, write $\alpha^\gamma = \alpha^\gamma_k$ and $\beta^\gamma = -w^{-1} \alpha^\gamma_j$.

   (3a) Suppose that $\alpha^\gamma \not= \beta^\gamma$. Then $\gamma(a',k,b) = \gamma(a',k,b') + x\alpha^\gamma + y\beta^\gamma$ for $x,y \in \mathbb{Z}$, and we have $\gamma(a,k,b) = \gamma(a,k,b') + y\beta^\gamma$.

   (3b) Suppose that $\alpha^\gamma = \alpha^\gamma_j = \beta^\gamma$. Then $\gamma(a,k,b) \gamma(a',k,b') \in \mathbb{Z} \alpha^\gamma$, and

   $$\langle \omega_i, \gamma(a,k,b) \rangle = (\omega_i, \gamma(a',k,b')) + \min(\langle \omega_i, \gamma(a',k,b) \rangle)
   + \sum_{i \neq j} a_{ij} \langle \omega_i, \gamma(a',k,b') \rangle.$$  

**Proof.** We first argue that the stated properties determine $\gamma(a,k,b)$. By [I] and [III] we know $\gamma(a,k,b)$ when $a = w_0$ or $b = id$. If $a \not= w_0$ and $b \not= id$, property [IV] allows us to express $\gamma(a,k,b)$ in terms of $\gamma(a',k,b), \gamma(a,k,b'), \gamma(a',k,b')$ where $a' > a$ and $b' > b$. Thus, all values of $\gamma(a,k,b)$ are determined. We now prove [I] and [IV].

Suppose that $a = w_0$ or $b = w_0$. Then a generic point $(X_\bullet, Y_\bullet)$ in $V_e$ satisfies $Y_0 \not= Y_0$. It follows that $\Delta_1$ does not vanish on $V_e$, establishing [I].

We show [II]. It is clear that if $as_k < a$ then $\gamma(a,k,b) = 0$. Suppose that $as_k > a$. We apply the moves $\beta^{-1}$ $(-b^{\text{rev}})ak(-k^*)$ $(-b^{\text{rev}})ak(-k^*)k$ $(-b^{\text{rev}})(-k^*)ak$. Since $as_k > a$,
these are non-mutation moves, and thus part (2) of Lemma 7.1 applies. We may now remove the solid crossings from $a$ using part (3) of Lemma 7.1 and then reverse the procedure to put $\beta$ back into its original form with solid crossings removed from $a$.

We prove [III] If $a_k < a$, we have already shown that $\gamma(a, k, \emptyset) = 0$. Assume $a_k > a$. When $a = \id$, the result follows from Proposition 2.20. For $a = \id$, we apply induction, [III] and the hollow case of Lemma 2.6.

We prove [IV] For (1) adding the letter $i$ in front of $(-b^{\rev})akk$ produces a new hollow crossing, and the claim follows from Lemmas 2.6 and 7.1. Similarly, for (2) the letter $-j^*$ is a hollow crossing in $(-b^{\rev})akk$. Case (3) holds if both $i$ and $-j^*$ are solid crossings in the word $i(-j^*)(-b^*)akk$. If swapping the order of $i$ and $-j^*$ is a non-mutation move then we are in Case (3a) and the claim follows from Lemma 2.6 and the linear independence of $\alpha^\vee$ and $\beta^\vee$. If swapping the order of $i$ and $-j^*$ is a mutation, then we are in Case (3b) and the claim follows from (4.8) and the assumption that Theorems 1.1 and 4.2 have been shown for $G$.

The algorithm has been implemented at [Gal23], where some examples can be found.

**Remark 7.3.** The logical dependencies in our proof are summarized as follows. In Section 4, we give a complete proof of Theorems 1.1 and 4.2 for the case when $G$ is simply-laced. Thus, Proposition 7.2 applies in this case. The proof for the case when $G$ is multiply-laced is given in Section 6, it depends on Proposition 7.2 but only invokes it for the simply-laced group $\hat{G}$.

The following result follows from our algorithm, but we have been unable to show it directly from Deodhar geometry.

**Corollary 7.4.** Let $i: \hat{R}_{\beta} \to \hat{R}_{\beta'}$ and $\lambda: [m] \to [m]$ be as in Sections 5.2 and 6. Then for each $e \in J_\beta$ and $e' \in \lambda^{-1}(e)$, we have $i^*(\hat{x}_{e'}) = x_e$.

**Proof.** With notation as in the proof of Proposition 6.3, it suffices to show that $\ord_{V_e, c, k} = \ord_{V_{e'}, c', k'}$ for $c \leq e, k \in I$, and $k' \in k$. This follows from applying Proposition 7.2 to $\hat{R}_{\beta}$ and $\hat{R}_{\beta'}$ separately. □

**References**


BRAID VARIETY CLUSTER STRUCTURES, II: GENERAL TYPE


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