HEDETNIEMI'S CONJECTURE

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1. Introduction

A **graph** G is a set $V(G)$ equipped with a symmetric relation $E(G)$ (note that G may have loops). A function $\phi : V(G) \to V(H)$ is a **homomorphism** if it preserves the edge relation, i.e. if $(v, w) \in E(G)$, then $(\phi(v), \phi(w)) \in E(H)$. Let $\mathrm{Hom}(G, H)$ denote the set of homomorphisms from G to H. We write $G \leq H$ if $\mathrm{Hom}(G, H)$ is nonempty (this is usually denoted $G \to H$ in finite graph theory).

Given a cardinal number n (which may be infinite), let K_n denote the **complete graph** on n vertices: $V(K_n) = n$, and $(k, l) \in E(K_n)$ iff $k \neq l$ (we follow the set-theoretic convention of writing n for the set of k with $0 \leq k < n$). Given a graph G, a **proper n-colouring** of G is a homomorphism from G to K_n, and the **chromatic number** of G, denoted $\chi(G)$, is the minimal n such that $G \leq K_n$ (we write $\chi(G) = \infty$ if no such n exists, which occurs exactly when G is not simple).

Given graphs G and H, the **categorical product** $G \times H$ is the graph with $V(G \times H) = V(G) \times V(H)$ such that $((v, w), (v', w')) \in E(G \times H)$ iff $(v, v') \in E(G)$ and $(w, w') \in E(H)$. There are projections $G \times H \to G$ and $G \times H \to H$, so $\chi(G \times H)$ is bounded above by both $\chi(G)$ and $\chi(H)$. Hedetniemi asked in his PhD thesis if this bound is optimal:

Conjecture 1 (Hedetniemi [Hed66]). *Let G and H be finite graphs. Then*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}. \qquad (*)$$

Let us make a few remarks.

(1) If either G or H is not simple, then $(*)$ holds trivially, since if G is not simple, then $H \leq G \times H$. So the conjecture is really about simple graphs.
(2) The equality $(*)$ is known to hold in the following cases:
   (a) $\chi(G \times H) \leq 2$ (easy).
   (b) $\chi(G \times H) = 3$, due to El-Zahar and Sauer [ES85].
   (c) When $\chi(G)$ is infinite and $\chi(H)$ is finite, due to Hajnal [Haj85] (for possibly infinite graphs).

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(d) For Borel chromatic numbers of analytic graphs, ($*$) holds when $\chi_B(G)$ and $\chi_B(H)$ are uncountable, as a corollary of the G_0-dichotomy [KST99, 6.11].

(3) The generalisation to infinite graphs does not hold in general. This was first explictly published by Hajnal [Haj85] (in the same issue of Combinatorica as El-Sahar and Zauer!).

In May 2019, Yaroslav Shitov refuted Hedetniemi's conjecture:

Theorem 1 (Shitov [Shi19])**.** *Hedetniemi's conjecture is false.*

We present his proof below.

2. Preliminaries

Given graphs G and H, the **exponential graph** H^G is the graph with $V(H^G) = V(H)^{V(G)}$, such that $(\alpha, \beta) \in E(H^G)$ iff for all $(v, w) \in E(G)$, we have $(\alpha(v), \beta(w)) \in E(H)$. It satisfies the following property:

$$\mathrm{Hom}(G \times H, K) \cong \mathrm{Hom}(G, K^H)$$

We can use this to simplify Hedetniemi's conjecture. The conjecture says that if G is a finite graph with $\chi(G) > n$, then any of the following equivalent statements holds:

- For every graph H, if $\chi(G \times H) \leq n$, then $\chi(H) \leq n$.
- For every graph H, if $G \times H \leq K_n$, then $H \leq K_n$.
- For every graph H, if $H \leq (K_n)^G$, then $H \leq K_n$.
- $(K_n)^G \leq K_n$.
- $\chi((K_n)^G) \leq n$.

So we have the following equivalent formulation of Hedetniemi's conjecture, first observed by El-Zahar and Sauer [ES85, Conjecture 2]:

Conjecture 2 (Hedetniemi v2)**.** *Let G be a finite graph with $\chi(G) > n$. Then $\chi((K_n)^G) \leq n$.*

We will be dealing with this version of the conjecture, and thus we will be interested in the graph $(K_n)^G$. This is the graph with vertex set $n^{V(G)}$, where α and β are adjacent iff for all $(v, w) \in E(G)$, we have $\alpha(v) \neq \beta(w)$. Let $\bar{k}$ denote the constant function taking the value k.

A **suited colouring** of $(K_n)^G$ is a proper n-colouring $\Phi : (K_n)^G \to K_n$ such that $\Phi(\bar{k}) = k$ for all $k \in n$. If $\chi((K_n)^G) \leq n$, then it is easily seen that there is a suited colouring of $(K_n)^G$, so we will work exclusively with suited colourings. The main convenience offered by suited colourings is the following fact, which we will use freely without further mention:

Proposition 1. *Let Φ be a suited colouring of $(K_n)^G$. Then for every $\alpha \in (K_n)^G$, we have $\Phi(\alpha) \in \mathrm{im}(\alpha)$.*

Proof. Since Φ is suited, we have $\Phi(\alpha) = \Phi(\overline{\Phi(\alpha)})$, so since Φ is a proper colouring, α and $\overline{\Phi(\alpha)}$ are not adjacent in $(K_n)^G$. In particular, their images must intersect, and thus $\Phi(\alpha) \in \mathrm{im}(\alpha)$. □

Given graphs G and H, the **strong product** $G \boxtimes H$ is the graph with vertex set $V(G) \times V(H)$ such that (v, w) and (v', w') are adjacent iff one of the following holds:

(1) $(v, v') \in E(G)$ and $(w, w') \in E(H)$.
(2) $(v, v') \in E(G)$ and $w = w'$.
(3) $v = v'$ and $(w, w') \in E(H)$.

We can now state the main theorem:

Theorem 2. *Let G be a finite graph with girth > 5 and radius > 2, and let $n > 6m$. If n is sufficiently large, then $\chi((K_n)^{G \boxtimes K_m}) > n$.*

To use this to refute Hedetniemi's conjecture, we will use the **fractional chromatic number**, which is defined as follows:

$$\chi_f(G) := \inf_m \frac{\chi(G \boxtimes K_m)}{m}$$

Proof of Theorem 1. Fix a finite graph G with girth > 5 and $\chi_f(G) > 7$ (for example, any graph with girth > 5 and independence number $< \frac{|G|}{7}$, see [Die17, 11.2.2]). G has radius > 2, since otherwise G would be a tree, which has $\chi_f(G) \leq 2$. Then for any m, we have

$$\chi(G \boxtimes K_m) \geq \chi_f(G) \cdot m > 7m,$$

Let $n = 7m$. By Theorem 2, if n is sufficiently large, then $\chi((K_n)^{G \boxtimes K_m}) > n$, refuting Conjecture 2. □

3. Condensed proof

We present a version of the proof requiring minimal overhead.

We will write $\mathbb{P}_k$ to denote the probability taken over $k \in n$ uniformly distributed, and similarly for $\mathbb{P}_\alpha$, where α ranges over n^G.

Proof of Theorem 2. For every $\alpha \in n^G$, write $\bar{\alpha} \in (K_n)^{G \boxtimes K_m}$ for the function $\bar{\alpha}(v, k) = \alpha(v)$.

Suppose that $\chi((K_n)^{G \boxtimes K_m}) \leq n$, and fix a suited colouring Ψ of $(K_n)^{G \boxtimes K_m}$.

First we find for each $v \in G$, some $\mu_v \in (K_n)^{G \boxtimes K_m}$ such that

(1) $|\mathrm{im}(\mu_v)| = 3m + 1$, and
(2) $\mu_v(w, i) = \Psi(\mu_v)$ iff $d(w, v) > 2$.

For $k \in \{3m, \ldots, n-1\}$, define $\mu_k \in (K_n)^{G \boxtimes K_m}$ via

$$\mu_k(w,i) = \begin{cases} i & w = v \\ i+m & d(w,v) = 1 \\ i+2m & d(w,v) = 2 \\ k & \text{otherwise} \end{cases}$$

Since G has radius > 2, the set $\{\mu_{3m}, \ldots, \mu_{n-1}\}$ has size $n - 3m$, and since G has girth > 5, it is a clique in $(K_n)^{G \boxtimes K_m}$. Thus since $n - 3m > 3m$, there is some μ_k such that $\Psi(\mu_k) \notin \{0, \ldots, 3m-1\}$. But since $\Psi(\mu_k) \in \operatorname{im}(\mu_k) = \{0, \ldots, 3m-1, k\}$, we must have $\Psi(\mu_k) = k$. Set $\mu_v = \mu_k$.

Next, we claim that for every $\alpha \in n^G$, if $\Psi(\bar{\alpha}) = \alpha(v) \notin \operatorname{im}(\mu_v)$, then there is some $v' \neq v$ such that $\alpha(v') \in \{\Psi(\mu_v), \alpha(v)\}$. To see this, define $\beta \in n^G$ as follows:

$$\beta(w) = \begin{cases} \Psi(\mu_v) & d(w,v) \le 1 \\ \alpha(v) & \text{otherwise} \end{cases}$$

Since $\alpha(v) \notin \operatorname{im}(\mu_v)$, we must have $\bar{\beta}$ and μ_v adjacent, and thus $\Psi(\bar{\beta}) \neq \Psi(\mu_v)$. Thus $\Psi(\bar{\beta}) = \alpha(v) = \Psi(\bar{\alpha})$. So $\bar{\beta}$ and $\bar{\alpha}$ are not adjacent, and thus there is some $v' \neq v$ such that $\alpha(v') \in \operatorname{im}(\beta) = \{\Psi(\mu_v), \alpha(v)\}$.

The claim, combined with the fact that $|\operatorname{im}(\mu_v)| = 3m + 1 < \frac{n}{2} + 1$, gives the following inequality:

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{n}\right)^{|G|} &< \prod_v \mathbb{P}_k[k \notin \operatorname{im}(\mu_v)] \\ &= \mathbb{P}_\alpha[\forall v\, [\alpha(v) \notin \operatorname{im}(\mu_v)]] \\ &\le \sum_v \mathbb{P}_\alpha[\Psi(\bar{\alpha}) = \alpha(v) \notin \operatorname{im}(\mu_v)] \\ &\le \sum_v \mathbb{P}_\alpha[\exists v' \neq v\, [\alpha(v') \in \{\Psi(\mu_v), \alpha(v)\}]] \\ &\le \sum_v \sum_{v' \neq v} \mathbb{P}_\alpha[\alpha(v') \in \{\Psi(\mu_v), \alpha(v)\}] \\ &\le \sum_v \sum_{v' \neq v} \frac{2}{n} \\ &= \frac{2|G|(|G|-1)}{n} \end{aligned}$$

This only holds for finitely many n, so we are done. □

Appendix A. Uncondensed proof

We will need the following important definition:

Definition 1 (Shitov)**.** Fix a suited colouring Φ of $(K_n)^G$. Then for a vertex $v \in G$, a colour $k \in n$ is v**-robust** if for every $\alpha \in (K_n)^G$ with $\Phi(\alpha) = k$, there is a vertex $w \in G$ with $d(w, v) \leq 1$ and $\alpha(w) = k$.

The proof strategy goes as follows. We will show that for any suited colouring, there is always a vertex with many robust colours. However, Hedetniemi's conjecture will provide us with too many suited colourings, in particular, one which does not have a vertex with many robust colours.

We first prove the existence of a vertex with many robust colours:

Proposition 2. *Let G be a finite graph and fix a suited colouring Φ of $(K_n)^G$. Then there is some $v \in G$ such that*

$$\mathbb{P}_k[k \text{ is not } v\text{-robust}] < \sqrt[|G|]{\frac{|G|^3}{n}}$$

Proof. For every v and k, define $\beta_{v,k} \in (K_n)^G$ as follows: if k is v-robust, pick $\beta_{v,k}$ arbitrarily; otherwise, pick $\beta_{v,k}$ witnessing the non-robustness, i.e. such that $\Phi(\beta_{v,k}) = k$ and for every v' with $d(v', v) \leq 1$, we have $\beta_{v,k}(v') \neq k$.

We claim that if $\Phi(\alpha) = \alpha(v)$ and $\alpha(v)$ is not v-robust, then $\exists v' \neq v$ such that $\alpha(v') \in \mathrm{im}(\beta_{v,\alpha(v)})$. If not, then α and β would be adjacent in $(K_n)^G$, contradicting the fact that $\Phi(\alpha) = \Phi(\beta)$.

This gives the following bound, from which the proposition follows immediately:

$$\begin{aligned}
&\prod_v \mathbb{P}_k[k \text{ is not } v\text{-robust}] \\
&= \mathbb{P}_\alpha[\forall v \, [\alpha(v) \text{ is not } v\text{-robust}]] \\
&\leq \sum_v \mathbb{P}_\alpha[\Phi(\alpha) = \alpha(v) \text{ and } \alpha(v) \text{ is not } v\text{-robust}] \\
&\leq \sum_v \sum_{v' \neq v} \mathbb{P}_\alpha\left[\alpha(v') \in \mathrm{im}\, \beta_{v,\alpha(v)}\right] \\
&\leq \sum_v \sum_{v' \neq v} \frac{|G|}{n} \\
&< \frac{|G|^3}{n}
\end{aligned}$$

$\square$

Given a graph G, the **reflexive closure** G°, is the graph obtained from G by adding every loop.

We will now see the consequences of having too many suited colourings:

Proposition 3. *Let G be a graph with girth > 5 and radius > 2, and let $n > 6m$. Suppose that $\chi((K_n)^{G\boxtimes K_m}) \leq n$. Then there is a suited colouring of $(K_n)^{G^\circ}$ such that for every $v \in G^\circ$,*

$$\mathbb{P}_k[k \text{ is } v\text{-robust}] < \frac{1}{2} + \frac{1}{n}$$

Proof. Note that there is a natural map $G \boxtimes K_m \to G^\circ$, and this induces a natural map $(K_n)^{G^\circ} \to (K_n)^{G\boxtimes K_m}$. For every $\alpha \in (K_n)^{G^\circ}$, let $\bar{\alpha} \in (K_n)^{G\boxtimes K_m}$ be the image of α under this map. Let Ψ be a suited colouring of $(K_n)^{G\boxtimes K_m}$, and let Φ be the suited colouring of $(K_n)^{G^\circ}$ obtained by composing Ψ with the map $(K_n)^{G^\circ} \to (K_n)^{G\boxtimes K_m}$.

Now fix $v \in G^\circ$. For $k \in \{3m, \ldots, n-1\}$, define $\mu_k \in (K_n)^{G\boxtimes K_m}$ via

$$\mu_k(w, i) = \begin{cases} i & w = v \\ i + m & d(w, v) = 1 \\ i + 2m & d(w, v) = 2 \\ k & \text{otherwise} \end{cases}$$

Since G has radius > 2, the set $\{\mu_{3m}, \ldots, \mu_{n-1}\}$ has size $n - 3m$, and since G has girth > 5, it is a clique in $(K_n)^{G\boxtimes K_m}$. Thus since $n - 3m > 3m$, there is some μ_k such that $\Psi(\mu_k) \notin \{0, \ldots, 3m-1\}$. But since $\Psi(\mu_k) \in \operatorname{im}(\mu_k) = \{0, \ldots, 3m-1, k\}$, we must have $\Psi(\mu_k) = k$.

It suffices to show that every v-robust colour is contained in $\operatorname{im}(\mu_k)$, since $|\operatorname{im}(\mu_k)| = 3m+1 < \frac{n}{2}+1$. To this end, let l be a v-robust colour. Define $\beta \in (K_n)^{G^\circ}$ as follows:

$$\beta(w) = \begin{cases} k & d(w, v) \leq 1 \\ l & \text{otherwise} \end{cases}$$

We must have $\Phi(\beta) = k$, since if $\Phi(\beta) = l$, then by robustness, we would have $\beta(w) = k$ for some w with $d(v, w) \leq 1$, and thus $l = k$. Thus $\Psi(\bar{\beta}) = \Phi(\beta) = k$. Since $\Psi(\mu_k) = k$ and Ψ is a proper colouring, μ_k and $\bar{\beta}$ are not adjacent, and thus we must have $l \in \operatorname{im}(\mu_k)$. □

Proof of Theorem 2. Suppose that $\chi((K_n)^{G\boxtimes K_m}) \leq n$. By Proposition 3, there is a suited colouring Φ of $(K_n)^{G^\circ}$ such that for every $v \in G^\circ$, we have

$$\mathbb{P}_k[k \text{ is } v\text{-robust}] < \frac{1}{2} + \frac{1}{n}.$$

But by Proposition 2, there is some $v \in G$ such that

$$\mathbb{P}_k[k \text{ is not } v\text{-robust}] < \sqrt[|G|]{\frac{|G|^3}{n}}.$$

There are only finitely many n such that both of these hold (since their sum is $\frac{1}{2} + o(1)$). ☐

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