BERNOULLI DISJOINTNESS (AFTER BERNSHTEYN)

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Fix an infinite (not necessarily countable) discrete group G. A **G-flow** is a nonempty compact Hausdorff space X equipped with a continuous action of G. A very important G-flow is the **Bernoulli shift** n^G, where n is finite.

A **subflow** of a G-flow X is a non-empty closed G-invariant subset of X. Given two G-flows X and Y, a **joining** of X and Y is a subflow of $X \times Y$ which projects onto X and Y. We say that X and Y are **disjoint**, denoted $X \perp Y$, if the only joining of X and Y is the trivial joining $X \times Y$. This is equivalent to saying that if Z is a G-flow which has X and Y as factors, then these factor maps factor through $X \times Y$.

A G-flow X is **minimal** if every orbit of X is dense. If a G-flow X is disjoint from n^G, then it is easy to show that X must be minimal. It was shown in [GTWZ] that this is the only restriction:

Theorem 1 (Glasner-Tsankov-Weiss-Zucker)**.** *$X \perp n^G$ for any minimal G-flow X.*

This property is called **Bernoulli disjointness** for obvious reasons.

1. Proof of Bernoulli disjointness

The original proof of Theorem 1 involved casework depending on various properties of G, and using many difficult results as a blackbox. Recently Anton Bernshteyn found a nicer proof of this result using the Lovász Local Lemma, which works uniformly for all groups G (see [Ber]). His proof is as follows:

Proof of Theorem 1. Let $Z \subset X \times n^G$ be a joining. To show that $Z = X \times n^G$, it suffices to show that Z is dense. So fix nonempty open sets $U \subset X$ and $V \subset n^G$. We need to show that $Z \cap (U \times V)$ is nonempty.

We claim that it suffices to find a subset $F \subset G$ satisfying the following two conditions:

(1) $\bigcap_{f\in F} f \cdot U$ meets every orbit (in X).
(2) $F \cdot V$ contains an orbit (in n^G).

To see this, suppose that the orbit $G \cdot y$ is contained in $F \cdot V$. Then there is some $x \in X$ with $(x, y) \in Z$. Now there is some $g \in G$ with $g \cdot x \in \bigcap_{f\in F} f \cdot U$. Since $g \cdot y \in F \cdot V$, there is some $f \in F$ with $g \cdot y \in f \cdot V$, so since $g \cdot x \in f \cdot U$ as well, we have $g \cdot (x, y) \in f \cdot (U \times V)$. Thus $f^{-1}g \cdot (x, y) \in U \times V$, and this is also in Z since Z is G-invariant.

We first find a family of subsets satisfying Condition 1. Fix any point $x_0 \in X$, and let $S = \{g \in G : x_0 \in g \cdot U\}$. Note that any finite subset F of S satisfies Condition 1, since the intersection is a nonempty open set (since it contains x_0) and thus meets every orbit by minimality of X. We claim that S is infinite. Indeed, let $T \subset G$ be a finite subset such that $X = T \cdot U$ (this exists by minimality and compactness, since minimality implies $X = G \cdot U$). Then for every $g \in G$, we have $X = gT \cdot U$, and thus there is some $t \in T$ with $x_0 \in gt \cdot U$,

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and thus $gt \in S$, i.e. $g \in t^{-1}S$. Thus $G = T^{-1}S$, so since T is finite, S must be infinite (in fact **left-syndetic**).

So S has arbitrarily large finite subsets, and thus it suffices to show that a sufficiently large subset of S satisfies Condition 2. We will show the following stronger fact, which is interesting in its own right:

Theorem 2 (Bernshteyn). *Let V be a non-empty open subset of n^G. Then for every sufficiently large finite subset $F \subset G$, the set $F \cdot V$ contains an orbit.*

Proof of Theorem 2. Let $F \subset G$ be a finite subset. Without loss of generality, we can shrink V so that $V = V_\phi$, where V_ϕ is the basic open neighbourhood defined by a finite partial function $\phi : G \to 2$, say with $\operatorname{dom} \phi = D$.

We claim that we can assume without loss of generality that F is **D-separated**, i.e. such that for any $f, f' \in F$, we have $fD \cap f'D = \varnothing$ (i.e. the D-balls in the Cayley graph are disjoint). To see that this, note that we can recursively construct a D-separated subset of F of size $\geq \frac{|F|}{|D|^2}$ as follows: pick any $f_0 \in F$, then pick any $f_1 \in F \setminus (D^{-1}Df_0)$, then pick any $f_2 \in F \setminus (D^{-1}D\{f_0, f_1\})$, and so on (each step removes at most $|D|^2$ elements from F).

Now for $F \cdot V_\phi$ to contain an orbit is equivalent to saying that the following intersection is nonempty:

$$\bigcap_{g \in G} gF \cdot V_\phi.$$

By compactness, it suffices to show that each finite intersection is nonempty.

Endow n^G with the product of the uniform probability measures. We recall the Lovász Local Lemma:

Theorem 3 (Lovász Local Lemma). *Let $\mathcal{A}$ be a set of events in a probability space, each with probability $\leq p$, such that for $A \in \mathcal{A}$, there is a subset $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{A} \setminus \mathcal{B}| \leq d$ such that A is independent from $\mathcal{B}$. If*

$$4p(d+1) < 1,$$

then for any $A_0, \ldots, A_k \in \mathcal{A}$, we have $\mathbb{P}[\bar{A}_0 \cdots \bar{A}_k] > 0$.

We verify the hypotheses of the Lovász Local Lemma for $\mathcal{A} = \{\neg(gF \cdot V_\phi)\}_{g \in G}$.

For a given $g \in G$, since F is D-separated, the sets $gf \cdot D$ are pairwise disjoint for distinct $f \in F$, and thus

$$\mathbb{P}[\neg(gF \cdot V_\phi)] = \prod_{f \in F} \mathbb{P}[\neg(V_{gf \cdot \phi})] = \left(1 - \frac{1}{n^{|D|}}\right)^{|F|}.$$

Now the event $gF \cdot V_\phi$ is independent with the set $\{hF \cdot V_\phi : gFD \text{ and } hFD \text{ are disjoint}\}$. If gFD and hFD are not disjoint, then

$$h \in gFDD^{-1}F^{-1},$$

and thus the event $gF \cdot V_\phi$ is independent with a set of cocardinality $\leq |D|^2|F|^2$. So for the Lovász Local Lemma to hold, we need the following inequality to hold:

$$4 \cdot \left(1 - \frac{1}{n^{|D|}}\right)^{|F|} \left(|D|^2|F|^2 + 1\right) < 1$$

which clearly holds for F sufficiently large. □

This concludes the proof of Bernoulli disjointness. □

Appendix A. Proof of the Lovász Local Lemma

We restate the Lovász Local Lemma.

Theorem 4 (Lovász Local Lemma)**.** *Let $\mathcal{A}$ be a set of events in a probability space, each with probability $\leq p$, such that for $A \in \mathcal{A}$, there is a subset $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{A} \setminus \mathcal{B}| \leq d$ such that A is independent from $\mathcal{B}$. If*

$$4p(d+1) < 1,$$

then for any $A_0, \ldots, A_k \in \mathcal{A}$, we have $\mathbb{P}[\bar{A}_0 \cdots \bar{A}_k] > 0$.

This is the original proof (see Lemma, p.616 in [EL]):

Proof. We prove the following stronger claim:

Proposition 1. *For any distinct $A_0, \ldots, A_k \in \mathcal{A}$, we have*
1. $\mathbb{P}[\bar{A}_1 \cdots \bar{A}_k] > 0$ *and*
2. $\mathbb{P}[A_0 | \bar{A}_1 \cdots \bar{A}_k] \leq 2p$.

Proof. We proceed by strong induction on k.

Note that (1) clearly holds when $k = 0$, and if $k > 0$, then $\mathbb{P}[A_1 | \bar{A}_2 \cdots \bar{A}_k] \leq 2p$, so

$$\mathbb{P}[\bar{A}_1 | \bar{A}_2 \cdots \bar{A}_k] \geq 1 - 2p > 2p \geq 0$$

where we use that $4p < 1$, and thus $\mathbb{P}[\bar{A}_1 \cdots \bar{A}_k] > 0$.

For (2), assume wlog that A_0 is independent from $\{A_{q+1}, \ldots, A_k\}$, where $q \leq d$. Then we have

$$\mathbb{P}[A_0 | \bar{A}_1 \cdots \bar{A}_k] = \frac{\mathbb{P}[A_0 \bar{A}_1 \cdots \bar{A}_q | \bar{A}_{q+1} \cdots \bar{A}_k]}{\mathbb{P}[\bar{A}_1 \cdots \bar{A}_q | \bar{A}_{q+1} \cdots \bar{A}_k]}$$

The numerator is $\leq p$ as follows:

$$\mathbb{P}[A_0 \bar{A}_1 \cdots \bar{A}_q | \bar{A}_{q+1} \cdots \bar{A}_k] \leq \mathbb{P}[A_0 | \bar{A}_{q+1} \cdots \bar{A}_k] \leq \mathbb{P}[A_0] \leq p$$

The denominator is $> \frac{1}{2}$ as follows:

$$\mathbb{P}[\bar{A}_1 \cdots \bar{A}_q | \bar{A}_{q+1} \cdots \bar{A}_k] \geq 1 - \sum_{i=1}^{q} \mathbb{P}[A_i | \bar{A}_{q+1} \cdots \bar{A}_k] \geq 1 - \sum_{i=1}^{q} 2p \geq 1 - 2pd > \frac{1}{2}$$

where the last inequality uses that $4pd < 1$. So we are done. □

□

References

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