HERIOT WATT UNIVERSITY

j.forlano@hw.ac.uk
macs.hw.ac.uk/∼jforlano/

Almost sure global well-posedness for the BBM equation with infinite L^2 initial data

Justin Forlano

MIGSAA
The Maxwell Institute Graduate School in Analysis and its Applications

1. (Deterministic) well-posedness of BBM

By factoring the time derivative, we write BBM as

$$\partial_t u + \varphi(\partial_x)u = -\tfrac{1}{2}\varphi(\partial_x)(u^2),$$

where $\varphi(\partial_x) := (1-\partial_x^2)^{-1}\partial_x$, which yields the integral form:

$$u(t) = \underbrace{e^{t\varphi(\partial_x)}u_0}_{\text{(linear evolution)}=:S(t)u_0} - \frac{1}{2}\int_0^t e^{(t-t')\varphi(\partial_x)}\varphi(\partial_x)(u^2(t'))dt'.$$

Well-posedness:

- Bona–Tzvetkov '07: BBM is GWP in $H^s(\mathbb{R})$ for any $s \geq 0$
- Roumégoux '10: extended Bona–Tzvetkov's GWP result to $\mathbb{T}$

Ill-posedness:

- Panthee '11, Bona–Dai '16: BBM is ill-posed below $L^2(\mathbb{T})$ in the sense that the solution map

$$\Phi : u_0 \in H^s(\mathbb{T}) \mapsto u \in C([0,T];H^s(\mathbb{T})) \quad \text{for } s<0,$$

experiences *norm-inflation at zero* ($u_0 = 0$, see Theorem 3)

▷ We extended this to norm inflation at *every* initial condition.

Theorem 3: Norm inflation at *general initial data*

Let $s<0$ and $u_0 \in H^s(\mathbb{T})$. Then, for any $\varepsilon > 0$, there exists a solution u_ε to BBM and $t_\varepsilon \in (0,\varepsilon)$ such that

$$\|u_\varepsilon(0) - u_0\|_{H^s(\mathbb{T})} < \varepsilon \quad \text{and} \quad \|u_\varepsilon(t_\varepsilon)\|_{H^s(\mathbb{T})} > \varepsilon^{-1}.$$

▷ Theorem 3 implies the solution map Φ is *discontinuous everywhere* in negative Sobolev spaces.

2. Why $\mathbf{u}_0^\omega$ as initial data?

The random Fourier series u_0^ω is a typical element in the support of Gaussian measures μ_α on periodic distributions on $\mathbb{T}$:

$$d\mu_\alpha = Z_\alpha^{-1}\exp\left(-\tfrac{1}{2}\|u\|_{H^\alpha(\mathbb{T})}^2\right)du.$$

▷ **Q:** What are the transport properties of μ_α under the BBM flow?

BBM conserves the energy:

$$E(u(t)) = \tfrac{1}{2}\|u(t)\|_{H^1}^2 = E(u(0)).$$

Hence, BBM has a naturally associated Gibbs measure:

$$d\mu_1 = Z_1^{-1}e^{-E(u)}du.$$

- de Suzzoni '14: μ_1 is *invariant* under the BBM flow
- Tzvetkov '15: the pushforward measure $\Phi(t)_\#\mu_\alpha$ is *quasi-invariant* (mutually absolutely continuous) with respect to $\mu_\alpha = \Phi(0)_\#\mu_\alpha$ for integers $\alpha \geq 2$

$\Rightarrow$ solutions with data in supp μ_α enjoy additional $W^{\alpha-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T})$-integrability (beyond deterministic methods)

- de Suzzoni–Tzvetkov '14: Links with wave-turbulence theory

Main results

We consider the Benjamin-Bona-Mahony equation (BBM):

$$\begin{cases}\partial_t u - \partial_{xxt}u + \partial_x u + \frac{1}{2}\partial_x(u^2) = 0 \\ u|_{t=0} = u_0.\end{cases} \qquad (x,t) \in \mathbb{T}\times\mathbb{R}_+,$$

- For $u_0 \in H^s(\mathbb{T})$, $s \geq 0$, BBM is globally well-posed (GWP).
- For $u_0 \in H^s(\mathbb{T})$, $s<0$, BBM is ill-posed.

Does (a form of) well-posedness persist below $L^2(\mathbb{T})$?

Yes, if we consider randomised initial data:

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1+|n|^{2\alpha}}}e^{inx}, \qquad \alpha \in \mathbb{R},$$

where $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables such that $g_{-n} = \overline{g_n}$ and g_0 is real.

Regularity: $\quad u_0^\omega \in H^{\alpha-\frac{1}{2}-\varepsilon}(\mathbb{T}) \setminus H^{\alpha-\frac{1}{2}}(\mathbb{T})$, for any $\varepsilon>0$, a.s.

▷ **Difficulty:** we consider $\alpha \leq \frac{1}{2}$ so that $u_0^\omega \notin L^2(\mathbb{T})$ almost surely

Theorem 1: Almost sure well-posedness

(i) Let $\alpha > \frac{1}{4}$. Then, BBM is **almost surely locally well-posed** with respect to the random initial data u_0^ω.

(ii) Let $\alpha = \frac{1}{2}$. Then, BBM is **almost surely globally well-posed** with respect to the random initial data u_0^ω.

▷ **Q:** Do our random solutions depend *continuously* on the random initial data?

Theorem 2: Approximation property of random solutions

Let $\alpha \in \left(\frac{1}{4},\frac{1}{2}\right]$, $s < \alpha - \frac{1}{2}$ and fix 'good' ω such that u^ω solves BBM with $u^\omega|_{t=0} = u_0^\omega$. Then,

(i) there are smooth (random) solutions $\{u_k^\omega\}_{k\in\mathbb{N}}$ to BBM such that

$$\lim_{k\to\infty}\|u_k^\omega(0) - u_0^\omega\|_{H^s} = 0 \quad \text{and} \quad \lim_{k\to\infty}\|u_k^\omega - u^\omega\|_{C([0,k^{-1}];H^s)} = \infty$$

(ii) the solutions $\{u_k^\omega\}_{k\in\mathbb{N}}$ from **mollified** data $u_{0,k}^\omega = \rho_k * u_0^\omega$ satisfy

$$\lim_{k\to\infty}\|u_{0,k}^\omega - u_0^\omega\|_{H^s} = 0 \quad \text{and} \quad \lim_{k\to\infty}\|u_k^\omega - u^\omega\|_{C([0,T];H^s)} = 0$$

Furthermore, the limits are **independent** of the choice of mollifier ρ

▷ **Ans: No**, since **(i)** implies there is a 'bad' way to approximate. However, **(ii)** implies we have a 'good' approximation property **when** we *regularise by mollification*: the smooth approximating sequence converges *independently* of the choice of mollifier.

- Similar stability statements typical in (singular) stochastic PDEs, e.g. see Hairer '13, Gubinelli, Imkeller and Perkowski '15.

3. Ideas of Theorem 1 (i): Local

We construct solutions which have the form:

$$u = z + v,$$

where $z(t) := S(t)u_0^\omega$ is the random linear solution and we solve a fixed point argument for the *'residual part'* v.

Key: v is almost surely smoother than z,

$v \in C([0,T];H^\sigma(\mathbb{T}))$ for $\sigma < 2\alpha$, because:

(i) randomisation leads to improved integrability:

$$z \in C([0,T];W^{\alpha-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T})) \text{ a.s.}$$

(ii) *nonlinear smoothing under randomisation* of second Picard iterate:

$$\mathcal{N}(z) := \int_0^t e^{(t-t')\varphi(\partial_x)}\varphi(\partial_x)\big(z(t')^2\big)dt \in C([0,T];H^\sigma(\mathbb{T})) \text{ a.s.}$$

for any $\sigma < 2\alpha$, **provided** $\alpha > \frac{1}{4}$.

- The expansion above follows McKean '95, Bourgain '96 and Da Prato–Debussche '03.
- 'Sharpness' of $\alpha > \frac{1}{4}$: if $\alpha \leq \frac{1}{4}$, $\mathcal{N}(z) \notin C([0,T];\mathcal{D}'(\mathbb{T}))$ a.s.

4. Ideas of Theorem 1 (ii): Global

▷ **Goal:** Show *for any* $T>0$, we have a bound

$$\|v(t)\|_{H^\sigma} \leq C(T), \quad \text{for } t \leq T \text{ and } \sigma < 2\alpha. \qquad (*)$$

▷ **Problem:** $E(v(t)) = +\infty$ a.s.!

Smooth out v: Use I-method of Colliander, Keel, Staffilani, Takaoka, Tao '02.

▷ **New goal:** Control growth of $E(I_N v) := \frac{1}{2}\|I_N v\|_{H^1}^2$, $N \in \mathbb{N}$, where $I_N = \text{Id}$ on 'low frequencies' and is smoothing of order $1-\sigma$ on 'high frequencies'.

Using the equation $I_N v$ solves:

$$\frac{d}{dt}E(I_N v) \text{ “}\sim\text{” } \underbrace{\int_{\mathbb{T}}(\partial_x I_N v)\big[(I_N v)^2 - I_N(v^2)\big]}_{(1)=\text{(Commutator)}} + \underbrace{\int_{\mathbb{T}}(\partial_x I_N v)I_N v\, I_N z}_{(2)}$$

(1): I_N does not commute with products so (1) does not vanish:

$$(1) \lesssim N^{-\theta_1}E^{\frac{3}{2}}(I_N v) \quad \text{for some } \theta_1 > 0.$$

Hence, ODE for $E(I_N v)$ *blows-up in finite time* $T^*(N)$.

(2): $z \notin L^2(\mathbb{T})$, so $I_N z \in L^2(\mathbb{T})$ at a cost of a 'bad' growth in N:

$$(2) \lesssim \|I_N z\|_{L^2}^2 E(I_N v) \sim \phi_\alpha(N)^{\frac{1}{2}}E(I_N v),$$

where $\phi_\alpha(N) \sim \log N$ if $\alpha = \frac{1}{2}$ and polynomial if $\alpha < \frac{1}{2}$.

- Gronwall argument: $T^*(N) \sim \phi_\alpha(N)^{-\frac{1}{2}}\log N$.
- Given $T>0$, we **must** choose $\alpha = \frac{1}{2}$ and then $N = N(T)$ so that $T^*(N) > T$, which yields $(*)$.