Dynamics near the ground state energy for the Cubic Nonlinear Klein-Gordon equation in 3D

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Justin Forlano

Supervisors: Tadahiro Oh, Oana Pocovnicu

We consider the cubic nonlinear Klein-Gordon equation (NLKG):

$$\begin{cases} \partial_t^2 u - \Delta u + u = u^3, & (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ (u(0), \partial_t u(0)) \in \mathcal{H} := H^1_{\text{rad}}(\mathbb{R}^3) \times L^2_{\text{rad}}(\mathbb{R}^3) \end{cases}$$

Conserved Energy:

$$E(u, \partial_t u)(t) := \int_{\mathbb{R}^3} \left(\frac{1}{2}u^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{1}{4}u^4 \right) dx.$$

Dynamics of NLKG understood below the ground state, Q, the unique, smooth, positive, radial solution in H^1 of $-\Delta Q + Q = Q^3$.

Dichotomy below the ground state (Payne-Sattinger Theory) [1, 2]

Let $K_0(\varphi) := \int \varphi^2 + |\nabla \varphi|^2 - \varphi^4 \, dx$, and define the regions

$$\mathcal{PS}_+ = \{\vec{u} \in \mathcal{H} \; : \; E(\vec{u}) < J(Q), \;\; K_0(u) \geq 0\},$$
$$\mathcal{PS}_- = \{\vec{u} \in \mathcal{H} \; : \; E(\vec{u}) < J(Q), \;\; K_0(u) < 0\}.$$

Then $\mathcal{PS}_\pm$ are *invariant* under the (NLKG) flow and solutions in $\mathcal{PS}_+$ exists *globally in time* and *scatter*, while those in $\mathcal{PS}_-$ *blow-up in finite time* ($t \to \pm\infty$).

Above the ground state

We now enter the perturbative regime

$$E(Q,0) \leq E(u, \partial_t u) < E(Q,0) + \epsilon^2,$$

where $\epsilon \ll 1$, which was studied by Nakanishi-Schlag [3, 4, 5]. Their theory combines Dispersive PDE, Dynamical systems and Spectral analysis techniques.

Linearise (NLKG) about $(Q,0)$ to obtain the system ($u =: Q + v$):

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix}}_{=:A} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 \\ N(v) \end{pmatrix},$$

where $L_+ := -\Delta + 1 - 3Q^2$. Analysing the spectra of A reveals we can write

$$u(t,x) = Q + \lambda(t)\rho(x) + \gamma(t,x),$$

where λ contains *stable* and *unstable* modes and γ is 'radiation.'

Projecting away unstable modes of λ: Construction of center-stable (invariant) manifold W^{cs}.

Dynamics off the center-stable manifold

Q: What happens *away* from $(\pm Q, 0)$?

Idea: Mimic Payne-Sattinger theory $\Longrightarrow$ Control sign $K_0(u)$ by studying unstable mode of λ.

The facts (about a 2ϵ-ball around $(Q,0)$ in d_Q):

1. sign K_0 can *only change* if you re-enter the 2ϵ-ball (Variational)
2. Solutions *not trapped* by 2ϵ-ball are *ejected* (Ejection Lemma)
3. Upon exit from 2ϵ-ball, solution *cannot* re-enter (One-Pass)

$\Longrightarrow$ "Either Trapped or Ejected."

Main technical tool: The *non-linear distance function*

$$d_Q(\vec{u}(t)) \simeq \|\vec{u}(t) - (Q,0)\|_{\mathcal{H}}.$$

When $d_Q(\vec{u}) \leq \delta_E \ll 1$, we have

$$d_Q^2(\vec{u}(t)) = E(\vec{u}) - J(Q) + k^2\lambda(t)^2 < \epsilon^2 + k^2\lambda(t)^2.$$

$$\Longrightarrow \lambda \text{ dominance: } d_Q(\vec{u}) \simeq |\lambda|$$

Ejection-Lemma [4]

There exists an abs. constant $0 < \delta_X \leq \delta_E$ with the following property. Let $u(t)$ be an NLKG solution satisfying

$$d_Q(\vec{u}(0)) \leq \delta_X, \quad E(\vec{u}) < J(Q) + \epsilon^2, \quad \text{and} \quad \frac{d}{dt}\Big|_{t=0} d_Q(\vec{u}(t)) \geq 0.$$

Then $d_Q(\vec{u}(t))$ monotonically increases until hitting δ_X while

$$d_Q(\vec{u}(t)) \simeq d_Q(\vec{u}(0))e^{kt}, \quad \mathfrak{s}K_0(u(t)) \gtrsim d_Q(\vec{u}(t)) - C_* d_Q(\vec{u}(0)),$$

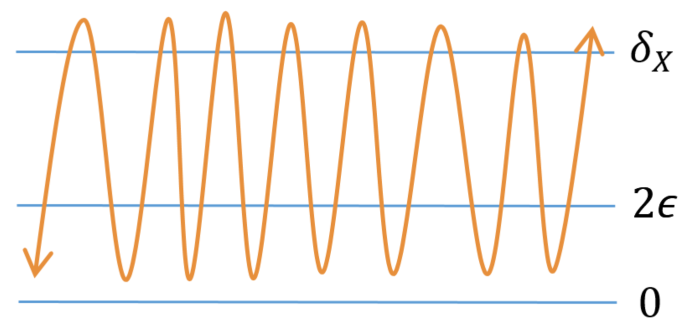
where $\mathfrak{s} \in \{\pm 1\}$ is a fixed sign and C_* an abs. constant.

The ejection lemma has two important immediate corollaries:

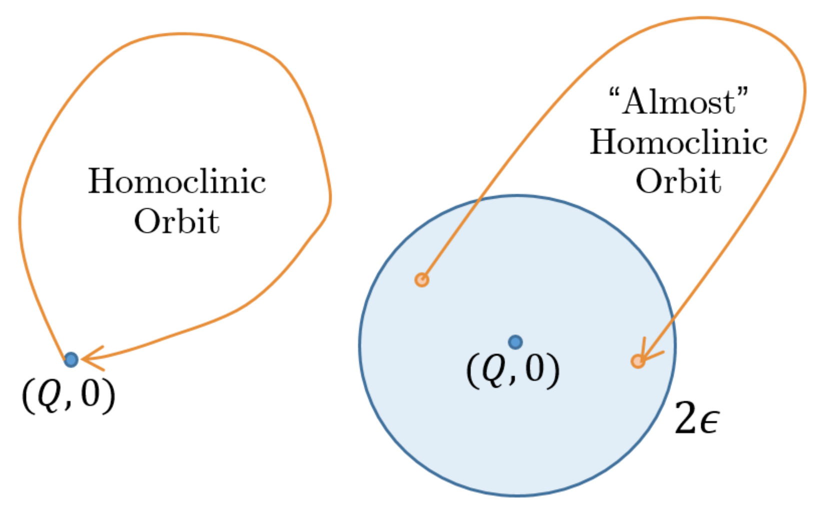
Corollary 1: There is no solution which *circulates* ($2\epsilon < d_Q(\vec{u}(t)) < \delta_X$ for all $t \geq 0$)

Corollary 2: Suppose $d_Q(\vec{u}(0)) \ll \delta_X$ and $\vec{u}(t)$ is *not trapped* by the 2ϵ-ball about $(Q,0)$. Then $\vec{u}$ is *ejected* to δ_X.

(1)+(2)=Insufficient: No chance for sign K_0 to stabilize!



Idea: Limit number of times solution can return to 2ϵ-ball. Equivalent to excluding *almost homoclinic orbits*.

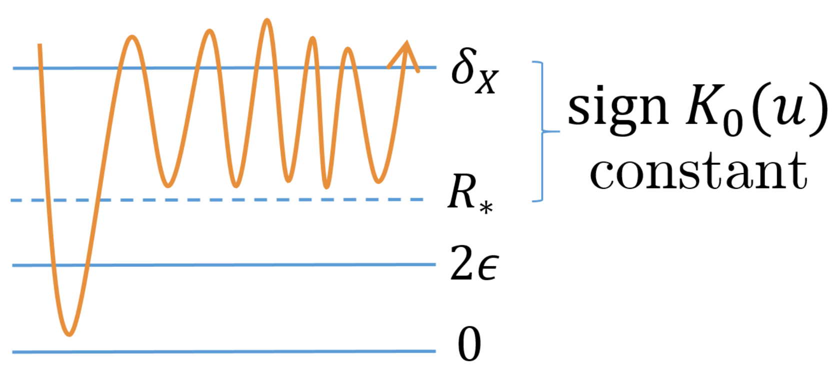


One-Pass Theorem [4]

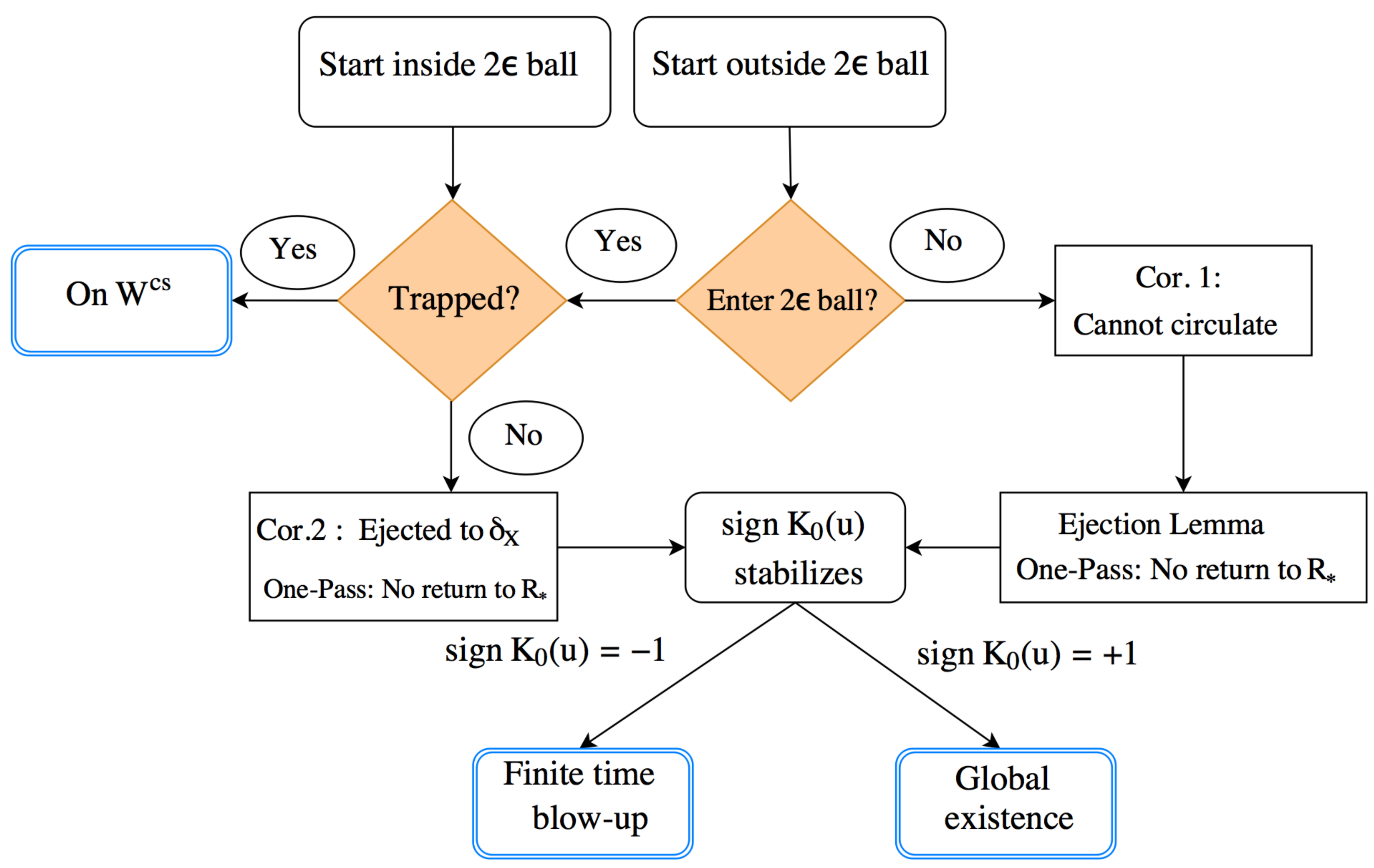
There exists an abs. constant $2\epsilon \ll R_* \ll \delta_X$ such that if an NLKG solution u satisfies for some $R \in (2\epsilon, R_*]$ and $t_1 < t_2$,

$$E(\vec{u}) < J(Q) + \epsilon^2, \quad d_Q(\vec{u}(t_1)) < R = d_Q(\vec{u}(t_2)),$$

then for all $t > t_2$, $d_Q(\vec{u}(t)) \geq R$.



This leads to the following classification of the behaviour:



Nine-set Theorem [4]

The set of solutions to (NLKG) are characterised by three possibilities, each of which can occur either forward or backward in time: *scattering to zero, finite time blow-up* or *trapping by the ground states*. Thus the solution set splits into nine non-empty sets.

[1] L.E. Payne, D.H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel Journal of Mathematics, 22 (1975), no. 3, 273–303.

[2] S. Ibrahim, N. Masmoudi, and K. Nakanishi, *Scattering thershold for the focusing nonlinear Klein-Gordon equation*, Analysis and PDE 4 (2011), no. 3, 405—460.

[3] K. Nakanishi, W. Schlag, *Invariant Manifolds and Dispersive Hamiltonian Evolution Equations*, Zurich Lectures in Advanced Mathematics, EMS Publishing House (2011).

[4] K. Nakanishi, W. Schlag, *Global dynamics above the ground state energy for the focusing nonlinear Klein-Gordon equation*, J. Diff. Eq 250, (2011), no. 5, 2299–2333.

[5] K. Nakanishi, W. Schlag, *Global dynamics above the ground state for the nonlinear Klein-Gordon equation without a radial assumption*, Archive for Rational Mechanics and Analysis 203 (2012), no. 3, pp. 809–851.