# Almost sure global well-posedness for the BBM equation with infinite $L^2$ initial data

Justin Forlano

HERIOT WATT UNIVERSITY

MIGSAA The Maxwell Institute Graduate School in Analysis and its Applications

## (Deterministic) well-posedness theory

We consider the Benjamin-Bona-Mahony equation (BBM) on the torus $\mathbb{T}$:

$$\begin{cases}\partial_t u - \partial_{xxt}u + \partial_x u + \frac{1}{2}\partial_x(u^2) = 0 \\ u|_{t=0} = u_0.\end{cases} \qquad (x,t)\in\mathbb{T}\times\mathbb{R}_+,$$

By factoring the time derivative, we write BBM as

$$\partial_t u + \varphi(\partial_x)u = -\tfrac{1}{2}\varphi(\partial_x)(u^2),$$

where $\varphi(\partial_x) := (1-\partial_x^2)^{-1}\partial_x$, which yields the integral form:

$$u(t) = \underbrace{e^{t\varphi(\partial_x)}u_0}_{\text{(linear evolution)}=:S(t)u_0} - \frac{1}{2}\int_0^t e^{(t-t')\varphi(\partial_x)}\varphi(\partial_x)(u^2(t'))dt'.$$

**Well-posedness:**

- Bona-Tzvetkov '07, Roumégoux '10: BBM is globally well-posed (GWP) in $H^s(\mathbb{T})$, for any $s\geq 0$

**Ill-posedness:**

- Panthee '11, Bona-Dai '16: BBM is ill-posed below $L^2(\mathbb{T})$, since the solution map

$$\Phi : u_0 \in H^s(\mathbb{T}) \mapsto u \in C([0,T]; H^s(\mathbb{T})) \;\text{ for } s<0,$$

experiences *norm-inflation at zero* ($u_0 = 0$, see Theorem 1)

▷ We extended this to norm inflation at *every* initial condition.

### Theorem 1: Norm inflation at *general initial data*

Let $s<0$ and $u_0\in H^s(\mathbb{T})$. Then, for any $\varepsilon>0$, there exists a solution $u_\varepsilon$ to BBM and $t_\varepsilon\in(0,\varepsilon)$ such that

$$\|u_\varepsilon(0)-u_0\|_{H^s(\mathbb{T})} < \varepsilon \quad\text{and}\quad \|u_\varepsilon(t_\varepsilon)\|_{H^s(\mathbb{T})} > \varepsilon^{-1}.$$

▷ Theorem 1 implies $\Phi$ is *discontinuous everywhere* in negative Sobolev spaces.

## Almost sure well-posedness theory

### Q: Does (a form of) well-posedness persist *below* $L^2(\mathbb{T})$?

Yes, if we consider randomised initial data:

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1+|n|^{2\alpha}}}e^{inx}, \qquad \alpha\in\mathbb{R},$$

where $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables such that $g_{-n} = \overline{g_n}$ and $g_0$ is real.

Regularity: $\quad u_0^\omega \in H^{\alpha-\frac{1}{2}-\varepsilon}(\mathbb{T})\setminus H^{\alpha-\frac{1}{2}}(\mathbb{T})$, for any $\varepsilon>0$, a.s.

▷ Difficulty: we consider $\alpha\leq\frac{1}{2}$ so that $u_0^\omega\notin L^2(\mathbb{T})$ almost surely

**On the nonlinearity below $L^2$:**

We interpret the nonlinearity $\varphi(\partial_x)(u^2)$ via the frequency side as:

$$N(u) = \sum_{n\neq 0} i\varphi(n)e^{inx}\sum_{n=n_1+n_2}\widehat{u}(n_1)\widehat{u}(n_2) \text{ “ = ” } \varphi(\partial_x)\Big(u^2 - \int_{\mathbb{T}} u^2 dx\Big)$$

- When $u\in L^2(\mathbb{T})$, $N(u) = \varphi(\partial_x)(u^2)$
- Quadratic derivative nonlinearity $\implies$ no renormalisation (e.g. stochastic Burgers equation)

**Why $u_0^\omega$ as initial data?**

The random Fourier series $u_0^\omega$ is a typical element in the support of Gaussian measures $\mu_\alpha$ on periodic distributions on $\mathbb{T}$:

$$d\mu_\alpha = Z_\alpha^{-1}\exp\Big(-\tfrac{1}{2}\|u\|^2_{H^\alpha(\mathbb{T})}\Big)\,du.$$

▷ Q: What are the transport properties of $\mu_\alpha$ under the BBM flow?

BBM conserves the energy:

$$E(u(t)) = \tfrac{1}{2}\|u(t)\|^2_{H^1} = E(u(0)).$$

Hence, BBM has a naturally associated Gibbs measure: $d\mu_1 = Z_1^{-1}\exp(-E(u))\,du$.

- de Suzzoni '14: $\mu_1$ is *invariant* under the BBM flow
- Tzvetkov '15: the pushforward measure $\Phi(t)_\#\mu_\alpha$ is *quasi-invariant* (mutually absolutely continuous) with respect to $\mu_\alpha = \Phi(0)_\#\mu_\alpha$ for integers $\alpha\geq 2$

$\implies$ solutions with data in supp $\mu_\alpha$ enjoy additional $W^{\alpha-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T})$-integrability (beyond deterministic methods)

### Theorem 2: Almost sure local existence

Let $\alpha>\frac{1}{4}$. Then, BBM is almost surely locally well-posed with respect to the random initial data $u_0^\omega$.

**Ideas of Theorem 2:**

We construct solutions which have the form:

$$u = z + v,$$

where $z(t) := S(t)u_0^\omega$ is the random linear solution and we solve a fixed point argument for the `residual part' $v$.

Key: $v$ is almost surely smoother than $z$,

$$v\in C([0,T];H^\sigma(\mathbb{T})) \;\text{ for } \sigma<2\alpha, \;\text{ because:}$$

(i) randomisation leads to improved integrability: $z\in C([0,T];W^{\alpha-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}))$ a.s.

(ii) *nonlinear smoothing under randomisation* of second Picard iterate:

$$\mathcal{N}(z) := \int_0^t e^{(t-t')\varphi(\partial_x)}N(z(t'))dt' \in C([0,T];H^\sigma(\mathbb{T})) \;\text{ a.s.}$$

for any $2\alpha$, provided $\alpha>\frac{1}{4}$.

- This expansion follows McKean '95, Bourgain '96 and Da Prato-Debussche '03.
- "Sharpness" of $\alpha>\frac{1}{4}$: if $\alpha\leq\frac{1}{4}$, then $\mathcal{N}(z)\notin C([0,T];\mathcal{D}'(\mathbb{T}))$ a.s.

$\Downarrow$

No improvement from *higher-order* expansions of $u$

### Theorem 3: Almost sure global existence

Let $\alpha=\frac{1}{2}$. Then, BBM is almost surely globally well-posed with respect to the random initial data $u_0^\omega$.

**Ideas of Theorem 3:**

▷ Goal: Show *for any* $T>0$, we have a bound

$$\|v(t)\|_{H^\sigma}\leq C(T), \quad \text{for } t\leq T \;\text{ and }\; \sigma<2\alpha. \qquad (*)$$

▷ Problem: $E(v(t)) = +\infty$ a.s.!

Smooth out $v$: Use "$I$-method" of Colliander, Keel, Staffilani, Takaoka, Tao '02.

▷ New goal: Control growth of $E(I_N v) := \frac{1}{2}\|I_N v\|^2_{H^1}$, $N\in\mathbb{N}$, where $I_N = \text{Id}$ on 'low frequencies' and is smoothing of order $1-\sigma$ on 'high frequencies'.

Using the equation $I_N v$ solves:

$$\frac{d}{dt}E(I_N v) \text{“} \sim \text{”} \underbrace{\int_{\mathbb{T}}(\partial_x I_N v)\big[(I_N v)^2 - I_N(v^2)\big]}_{(1)=\text{(Commutator)}} + \underbrace{\int_{\mathbb{T}}(\partial_x I_N v)I_N v\, I_N z}_{(2)}$$

(1): $I_N$ does not commute with products so (1) does not vanish:

$$(1)\lesssim N^{-\theta_1}E^{\frac{3}{2}}(I_N v) \;\text{ for some }\; \theta_1>0.$$

Hence, the ODE for $E(I_N v)$ *blows-up in finite time* $T^*(N)$.

(2): $z\notin L^2(\mathbb{T})$, so $I_N z\in L^2(\mathbb{T})$ at a cost of a 'bad' growth in $N$:

$$(2)\lesssim \|I_N z\|^2_{L^2}E(I_N v)\sim \phi_\alpha(N)^{\frac{1}{2}}E(I_N v),$$

where $\phi_\alpha(N)\sim\log N$ if $\alpha=\frac{1}{2}$ and polynomial if $\alpha<\frac{1}{2}$.

- Gronwall argument: $T^*(N)\sim\phi_\alpha(N)^{-\frac{1}{2}}\log N$.
- Given $T>0$, we **must** choose $\alpha=\frac{1}{2}$ and then $N=N(T)$ so that $T^*(N)>T$, which yields $(*)$.
- $I$-method for singular stochastic nonlinear wave equations: Gubinelli-Koch-Oh-Tolomeo '18 and Tolomeo'18.

## Stability property of random solutions

▷ Q: Do our random solutions depend *continuously* on the random initial data?

### Theorem 4: Approximation property of random solutions

Let $\alpha\in\big(\frac{1}{4},\frac{1}{2}\big]$, $s<\alpha-\frac{1}{2}$ and fix `good' $\omega$ such that $u^\omega$ solves BBM with $u^\omega|_{t=0} = u_0^\omega$. Then,

(i) there are smooth (random) solutions $\{u_k^\omega\}_{k\in\mathbb{N}}$ to BBM such that

$$\lim_{k\to\infty}\|u_k^\omega(0)-u_0^\omega\|_{H^s} = 0 \quad\text{and}\quad \lim_{k\to\infty}\|u_k^\omega - u^\omega\|_{C([0,k^{-1}];H^s)} = \infty$$

(ii) the solutions $\{u_k^\omega\}_{k\in\mathbb{N}}$ from mollified data $u_{0,k}^\omega = \rho_k * u_0^\omega$ satisfy

$$\lim_{k\to\infty}\|u_{0,k}^\omega - u_0^\omega\|_{H^s} = 0 \quad\text{and}\quad \lim_{k\to\infty}\|u_k^\omega - u^\omega\|_{C([0,T];H^s)} = 0$$

Furthermore, the limits are independent of the choice of mollifier $\rho$

▷ Ans: No, since (i) implies there is a `bad' way to approximate. However, (ii) implies we have a `good' approximation property **when** we *regularise by mollification*: the smooth approximating sequence converges *independently* of the choice of mollifier.

- Similar stability statements typical in (singular) stochastic PDEs, e.g. see Hairer '13, Gubinelli-Imkeller-Perkowski '15.

## Extension: Non-Gaussian randomisation

We consider now the random initial data

$$u_0^{\omega,h}(x) = \sum_{n\in\mathbb{Z}}\frac{h_n(\omega)}{\sqrt{1+|n|^{2\alpha}}}e^{inx}, \qquad \alpha\in\mathbb{R},$$

where $\{h_n(\omega)\}_{n\in\mathbb{Z}}$ is a sequence of independent, mean-zero complex-valued *sub-Gaussian* random variables such that $h_{-n} = \overline{h_n}$ and $h_0$ is real.

Under an additional *non-degeneracy assumption* (as in Burq-Tzvetkov '08), the randomisation does *not* regularise: $u_0^{\omega,h}\in H^{\alpha-\frac{1}{2}-\varepsilon}(\mathbb{T})\setminus H^{\alpha-\frac{1}{2}}(\mathbb{T})$, for any $\varepsilon>0$, a.s.

### Theorem 5: Non-Gaussian existence theory

With respect to the random initial data $u_0^{\omega,h}$, BBM is (i) almost surely locally well-posed, when $\alpha>\frac{1}{4}$, and (ii) almost surely globally well-posed, when $\alpha=\frac{1}{2}$.

Main difficulty: Lack of the Gaussian specific tool, *The Wiener chaos estimate*:

"If $S_2(\omega) = \sum_{|n_1|,|n_2|\leq N}a(n_1,n_2)g_{n_1}(\omega)g_{n_2}(\omega)$, then for any $p\geq 2$,

$$\|S_2(\omega)\|_{L^p(\omega)}\lesssim_p \|S_2(\omega)\|_{L^2(\Omega)}."$$

- Need to estimate high-moments of $\mathcal{N}(z^h)$ `by-hand':

Case-by-case analysis on $\mathbb{E}[h_{n_1}(\omega)h_{n_2}(\omega)\overline{h_{n_3}(\omega)h_{n_4}(\omega)}h_{n_5}(\omega)h_{n_6}(\omega)\overline{h_{n_7}(\omega)h_{n_8}(\omega)}]$