NON-TORSION BRAUER GROUPS IN POSITIVE CHARACTERISTIC

LOUIS ESSER

1. Introduction

One way of extending the notion of the classical Brauer group of a field to any scheme X is by defining the Brauer-Grothendieck group $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$. Just as for fields, this group is torsion for any regular integral noetherian scheme [5, Corollaire 1.8]. However, this no longer holds for singular schemes. In fact, if R is the local ring of the vertex of a cone over a smooth curve of degree $d \geq 4$ in $\mathbb{P}^2_{\mathbb{C}}$, then $\mathrm{Br}(R)$ contains the additive group of $\mathbb{C}$. There are then affine Zariski open neighborhoods of the vertex with non-torsion Brauer group [3, Example 8.2.2]. However, such examples work only in characteristic zero. This leaves open the following question, suggested by Colliot-Thélène and Skorobogatov:

Question 1.1. If X is an integral normal quasi-projective variety over a field k of positive characteristic, is $\mathrm{Br}(X)$ a torsion group?

To analyze this question, we use a result concerning the Brauer group of a normal variety X with only isolated singularities $p_1, \ldots, p_n$. Suppose X is defined over an algebraically closed field k of arbitrary characteristic. Let K be the function field of X, $R_i = \mathcal{O}_{X,p_i}$ be the local ring at each singularity, and R_i^{sh} its strict henselization. Then, $\mathrm{Br}(X)$ is given by the exact sequence (see [3, Section 8.2], elaborating on [5, §1, Remarque 11 (b)])

$$0 \to \mathrm{Pic}(X) \to \mathrm{Cl}(X) \to \bigoplus_{i=1}^{n} \mathrm{Cl}(R_i^{\text{sh}}) \to \mathrm{Br}(X) \to \mathrm{Br}(K). \tag{1}$$

This sequence indicates that one way for $\mathrm{Br}(X)$ to be large is for a singularity to have a large henselian local class group with divisors that do not extend globally to X. This idea is illustrated by a counterexample given by Burt Totaro, which shows that $\mathrm{Br}(X)$ is non-torsion for X a hypersurface of degree $d \geq 3$ in $\mathbb{P}^4_k$ with a single node [3, Proposition 8.2.3]. Here k is any algebraically closed field with characteristic not 2.

The below is a summary of the example due to Totaro, which was included in previous versions of the paper.

suppose that $X \subset \mathbb{P}^4$ is a hypersurface of degree $d \geq 3$ with a single node p. Then $Y = \mathrm{Bl}_p(X)$ is a smooth, ample divisor in $\mathrm{Bl}_p\mathbb{P}^4$. By the Grothendieck-Lefschetz theorem for Picard groups, the restriction $\mathrm{Pic}(\mathrm{Bl}_p\mathbb{P}^4) \to \mathrm{Pic}(Y)$ is an isomorphism.

If E is the exceptional divisor, then the sequence $0 \to \mathbb{Z} \cdot [E] \to \mathrm{Pic}(Y) \to \mathrm{Cl}(X) \to 0$ yields $\mathrm{Cl}(X) \cong \mathbb{Z} \cdot \mathcal{O}_X(1)$, so that the restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(R^{\text{sh}})$ is zero. However, since a threefold node is étale-locally the cone over a smooth quadric surface, one can show that the henselian local class group contains a copy of $\mathbb{Z}$. Thus, $\mathrm{Br}(X)$ is not torsion.

We will show that counterexamples to Question 1.1 also exist in dimension 2, provided that one works over a large algebraically closed field k.

Acknowledgements. I thank Burt Totaro for suggesting this problem to me and for his advice. This work was partially supported by National Science Foundation grant DMS-1701237.

2. A Surface Counterexample

The following construction is taken from [6], Example 1.27. Take a smooth cubic curve D in the projective plane and a quartic curve Q that meets it tranversally. Let $Y = \mathrm{Bl}_{q_1,\ldots,q_{12}}\mathbb{P}^2$, where $q_1, \ldots, q_{12}$ are the points of intersection. On Y, the strict transform C of D satisfies $C^2 = -3$. Unlike rational curves, not all negative self-intersection higher genus curves may be contracted to yield a projective surface. Rather, the contraction might only exist as an algebraic space. However, in this case, C is contractible.

Proposition 2.1. *There exists a normal projective surface X and a contraction morphism $Y \to X$ whose exceptional locus is C.*

Proof. The Picard group of Y is the free abelian group on H, the pullback of a general line in $\mathbb{P}^2$, and the exceptional divisors $E_1, \ldots, E_{12}$. Then, we claim that the line bundle $\mathcal{O}_Y(4H - \sum_i E_i) = \mathcal{O}_Y(H + C)$ defines a basepoint-free linear system on Y. Indeed, no point outside of the E_i can be a base point, and the transforms of both D + a line and Q belong to the linear system. These don't intersect on the exceptional locus by the transversality assumption. Therefore, the system defines a morphism $Y \to X'$ to a projective surface X'.

This contracts only C: if C' is another (irreducible) curve with $C' \cdot (H + C) = 0$, clearly C' is not supported on the E_i, or the intersection would be positive. Therefore, $H \cdot C' > 0$, meaning $C \cdot C' < 0$. Therefore, $C' = C$, as desired. Finally, passing to the normalization X of X', we may assume X is a normal projective surface. □

The resulting singularity p of X has minimal resolution with exceptional set exactly C, a smooth elliptic curve. Singularities satisfying this condition are *simple elliptic singularities*. Over $\mathbb{C}$, such singularities are completely classified. A simple elliptic singularity with $C^2 = -3$ is known as an $\tilde{E}_6$ singularity, and is complex analytically isomorphic to a cone over C [7]. Here, we present a computation of the henselian local class group $\mathrm{Cl}(R^{\mathrm{sh}})$ of the singularity that works in any characteristic.

Consider the pullback of this desingularization to a "henselian neighborhood":

$$\begin{array}{ccc} Y^h & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R^{\mathrm{sh}}) & \longrightarrow & X. \end{array}$$

The scheme Y^h is regular and $Y^h \setminus C \cong \mathrm{Spec}(R^{\mathrm{sh}}) \setminus \{\mathfrak{m}\}$, so we have an exact sequence $0 \to \mathbb{Z} \cdot [C] \to \mathrm{Pic}(Y^h) \to \mathrm{Cl}(R^{\mathrm{sh}}) \to 0$. It suffices, therefore, to compute $\mathrm{Pic}(Y^h)$. To do so, we'll first consider infinitesimal neighborhoods of C in Y.

Detailed explanation of why Y^h is a regular scheme: essentially, this is because henselization behaves well with regularity and tensor products. We first base change to R; the Zariski local rings of Y are regular, so this gives $Y_R \to \mathrm{Spec}(R)$ with Y_R regular. Then, to cover Y^h, one can consider affine opens in Y_R. For any such $\mathrm{Spec}(S) \subset R$, we have a ring map $R \to S$ and the corresponding open in Y^h is $R^{\mathrm{sh}} \otimes_R S$. The local rings there can be identified with the henselizations of the local rings of S, which are regular (see, e.g. [8, 0BSK]).

The sequence of infinitesimal neighborhoods $C = C_1 \subset C_2 \subset \cdots$ is defined by powers of the ideal sheaf $\mathcal{I}_C$. Notably, these C_n are the same regardless of whether we consider them inside Y or inside the henselian neighborhood Y^h. The normal bundle to C in Y gives obstructions to extending line bundles to successive neighborhoods, but we'll show that all line bundles extend uniquely. The group $\varprojlim_n \mathrm{Pic}(C_n)$ in the proposition below is also the Picard group of the formal neighborhood of C in Y.

Proposition 2.2. *The restriction map* $\varprojlim_n \mathrm{Pic}(C_n) \to \mathrm{Pic}(C)$ *is an isomorphism.*

Proof. It's enough to show that the maps $\mathrm{Pic}(C_{n+1}) \to \mathrm{Pic}(C_n)$ are all isomorphisms for $n \geq 1$. Each extension $C_n \subset C_{n+1}$ is a first-order thickening, since C_n is defined in C_{n+1} by the square-zero ideal sheaf $\mathcal{I}_C^n/\mathcal{I}_C^{n+1}$. Associated to such a thickening is a long exact sequence in cohomology [8, 0C6Q]

$$\cdots \to H^1(C, \mathcal{I}_C^n/\mathcal{I}_C^{n+1}) \to \mathrm{Pic}(C_{n+1}) \to \mathrm{Pic}(C_n) \to H^2(C, \mathcal{I}_C^n/\mathcal{I}_C^{n+1}) \to \cdots .$$

We may take the outer cohomology groups to be over C since the underlying topological spaces are the same. As sheaves of abelian groups on C, we have $\mathcal{I}_C^n/\mathcal{I}_C^{n+1} \cong \mathcal{O}_C(-nC)$, a multiple of the conormal bundle. But $C^2 = -3$ in Y so this last bundle has degree $3n > 0$. Therefore, its higher cohomology vanishes and $\mathrm{Pic}(C_{n+1}) \to \mathrm{Pic}(C_n)$ is an isomorphism for all n. □

Detailed explanation as to why outer cohomology groups are over C, and not a thickening: we consider every sheaf in the sequence

$$0 \to \mathcal{I}_C^n/\mathcal{I}_C^{n+1} \to \mathcal{O}_{C_{n+1}}^* \to \mathcal{O}_{C_n}^* \to 0$$

to be a sheaf of an abelian groups, and forget the module structure (this doesn't change cohomology). All the C_n have the same underlying topological space so these sheaves are all on that topological space. All that changes is the ringed space structure.

Now, we need only compare $\varprojlim_n \mathrm{Pic}(C_n)$ to $\mathrm{Pic}(Y_h)$. Using the Artin approximation theorem [1, Theorem 3.5], the map $\mathrm{Pic}(Y^h) \to \varprojlim_n \mathrm{Pic}(C_n)$ is injective with dense image. However, the topology of the latter group is discrete in this setting because each group of the inverse limit is $\mathrm{Pic}(C)$. Therefore, the map is surjective also and $\mathrm{Pic}(Y^h) \cong \mathrm{Pic}(C)$.

Detailed explanation: The formulation of Artin approximation in [1] is as follows: let A be a local henselian ring with maximal ideal $\mathfrak{m}$ (A here is the henselization of a finite type algebra over the field k) and let $\hat{A}$ be its completion. Then for any functor $F : \{A-\text{algebras}\} \to \{\text{sets}\}$ that is locally of finite presentation, any $\hat{\xi} \in F(\hat{A})$, and any integer n, there exists $\xi \in F(A)$ such that

$$\hat{\xi} \equiv \xi \mod \mathfrak{m}^n.$$

In the proof of theorem 3.5, it is noted that $H^1(X \times_{\mathrm{Spec}(A)} -, \mathbb{G}_m)$ is a functor locally of finite presentation for $X \to A$ proper. Hence in our setting, we can approximate any line bundle on the formal neighborhood up to C_n. Since all the Picard groups are the same, this demonstrates surjectivity.

Theorem 2.3. *Let k be an algebraically closed field that is not the algebraic closure of a finite field. Then,* $\mathrm{Br}(X)$ *is non-torsion.*

Proof. From the above, we have the identification $\mathrm{Cl}(R^{\mathrm{sh}}) \cong \mathrm{Pic}(Y^h)/\mathbb{Z} \cdot \mathcal{O}_{Y^h}(C)$ $\cong \mathrm{Pic}(C)/\mathbb{Z} \cdot \mathcal{O}_C(C)$. Since $\deg_C \mathcal{O}_C(C) = 3$, the class group is then an extension of $\mathbb{Z}/3$ by $\mathrm{Pic}^0(C) \cong C(k)$. If $k = \overline{\mathbb{F}}_p$, then $C(k)$ is torsion, because every point is defined over $\mathbb{F}_{p^m}$ for some m. In contrast, if k satisfies the hypothesis of the theorem, $C(k)$ has infinite rank [4, Theorem 10.1].

However, the global class group of X is quite small: $\mathrm{Cl}(Y) = \mathrm{Pic}(Y) \cong \mathbb{Z}^{13}$ since Y is the blow up of $\mathbb{P}^2$ in 12 points and $\mathrm{Cl}(X) \cong \mathrm{Cl}(Y)/\mathbb{Z} \cdot [C]$. Therefore, the cokernel of the restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(R^{\mathrm{sh}})$ in (1) contains non-torsion elements, so $\mathrm{Br}(X)$ does too. ∎

To complement the above result, we also prove

Theorem 2.4. *Suppose that X is an integral normal surface over the algebraic closure $k = \overline{\mathbb{F}}_p$ of a finite field. Then* $\mathrm{Br}(X)$ *is torsion.*

Proof. The strategy is similar to the previous theorem. Here, the crucial fact is that all possible "building blocks" of the henselian local class group - abelian varieties over k, the additive group of k, and the multiplicative group of k - are all torsion.

In the exact sequence (1), $\mathrm{Br}(K)$ is always torsion, so if we can prove $\oplus_{i=1}^n \mathrm{Cl}(R_i^{\mathrm{sh}})$ is as well, the result follows. Therefore, we focus on the desingularization $\pi : Y^h \to \mathrm{Spec}(R^{\mathrm{sh}})$ of the henselian local ring at just one point p. We may choose Y^h such that the exceptional set $E = \pi^{-1}(p)$ is a union of irreducible curves E_j which are smooth and meet pairwise transversely, with no three containing a common point.

The following argument is due to Artin [2, p. 491]. Suppose G is the free abelian group of divisors supported on E, and consider the map $\mathrm{Pic}(Y^h) \to \mathrm{Hom}(G, \mathbb{Z})$ given by $L \mapsto (D \mapsto D \cdot L)$. This restricts to a map $G \to \mathrm{Hom}(G, \mathbb{Z})$ that is injective because the intersection matrix of the curves E_j is negative definite. In particular, the first map in the excision sequence of class groups $0 \to G \to \mathrm{Pic}(Y^h) \to \mathrm{Cl}(R^{\mathrm{sh}}) \to 0$ is injective. Since G and $\mathrm{Hom}(G, \mathbb{Z})$ are free abelian groups of equal rank, $G \to \mathrm{Hom}(G, \mathbb{Z})$ also has finite cokernel. This allows us to find a cycle $Z = \sum_j r_j E_j$ with all $r_j > 0$ and $\mathcal{O}_Z(-Z)$ ample. We'll examine infinitesimal neighborhoods of the closed subscheme Z.

As before, for every $n \geq 1$, there is an exact sequence

$$\cdots \to H^1(Z, \mathcal{O}_Z(-nZ)) \to \mathrm{Pic}(Z_{n+1}) \to \mathrm{Pic}(Z_n) \to H^2(Z, \mathcal{O}_Z(-nZ)) \to \cdots.$$

Because $\dim(Z) = 1$, the last group is always zero. Since $\mathcal{O}_Z(-Z)$ is ample, the first group is zero for $n \gg 0$. Therefore, the inverse limit $\varprojlim_n \mathrm{Pic}(Z_n)$ is constructed as a finite series of extensions of $\mathrm{Pic}(Z)$ by finite-dimensional k-vector spaces. While Z is generally non-reduced and not equal to E, we may still apply Artin approximation. This is because for any large n, the scheme E_n is nested between two infinitesimal neighborhoods of Z, where all restrictions of Picard groups are surjective (use a similar exact sequence to the above, e.g. [8, 09NY]). It follows that $\varprojlim_n \mathrm{Pic}(Z_n) \cong \varprojlim \mathrm{Pic}(E_n)$, so $\mathrm{Pic}(Y^h) \cong \varprojlim_n \mathrm{Pic}(Z_n)$.

Next, let $\bar{Z}$ be the disjoint union of the schemes $r_j E_j$. Then $f : \bar{Z} \to Z$ is a finite map that is an isomorphism away from the finite set of intersection points and such that $\mathcal{O}_Z \subset f_* \mathcal{O}_{\bar{Z}}$. It follows (see [8, 0C1H]) that $\mathrm{Pic}(Z)$ is a finite sequence of extensions of $\mathrm{Pic}(\bar{Z})$ by quotients of $(k, +)$ or $(k, *)$. Lastly, $\mathrm{Pic}(\bar{Z}) = \oplus_j \mathrm{Pic}(r_j E_j)$, each of which is built from finite-dimensional k-vector spaces, and $\mathrm{Pic}(E_j) \cong \mathbb{Z} \oplus \mathrm{Pic}^0(E_j)$. Since the $\mathrm{Pic}^0(E_j)$ are groups of k-points of abelian varieties over k, they are all torsion.

Over large fields of characteristic p, the key to non-torsion Brauer groups is some non-simple-connectedness of the exceptional locus, either via a cycle of curves (which gives a k^* because you "link up" two points on the same connected component) or via $\mathrm{Pic}(C)$ for C of positive genus.

Taken together, all of this implies that G and $\mathrm{Pic}(Y^h)$ are equal rank. Because $G \to \mathrm{Pic}(Y^h)$ is injective, the quotient $\mathrm{Cl}(R^{\mathrm{sh}})$ is a torsion group, as desired. ☐

REFERENCES

[1] M. Artin, *Algebraic approximation of structures over complete local rings*. Publ. Math., Inst. Hautes Étud. Sci. 36 (1969), 23–58. 3

[2] M. Artin, *Some numerical criteria for contractability of curves on algebraic surfaces*. Am. J. Math., 84 (1962), 485-496. 4

[3] J.-L. Colliot-Thélène and A. Skorobogatov, *The Brauer-Grothendieck group*, volume 71 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer International Publishing, 2021. 1

[4] G. Frey and M. Jarden, *Approximation theory and the rank of abelian varieties over large algebraic fields*. Proc. London Math. Soc. 28 (1974), 112-128. 4

[5] A. Grothendieck, *Le groupe de Brauer, II, Théorie cohomologique*. Dix exposés sur la cohomologie des schémas, North-Holland (1968), 67-87. 1

[6] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*. Cambridge University Press, 1998. 2

[7] K. Saito, *Einfach-elliptische Singularitäten*. Invent. Math. 23 (1974), 289–325. 2

[8] The Stacks Project Authors, *The Stacks Project*. `https://stacks.columbia.math.edu`, 2021. 3, 5

UCLA MATHEMATICS DEPARTMENT, BOX 951555, LOS ANGELES, CA 90095-1555

Email address: `esserl@math.ucla.edu`