NON-TORSION BRAUER GROUPS IN POSITIVE CHARACTERISTIC

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1. INTRODUCTION

One way of extending the notion of the classical Brauer group of a field to any scheme $X$ is by defining the Brauer-Grothendieck group $\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$. Just as for fields, this group is torsion for any regular integral noetherian scheme [5, Corollaire 1.8]. However, this no longer holds for singular schemes. In fact, if $R$ is the local ring of the vertex of a cone over a smooth curve of degree $d \geq 4$ in $\mathbb{P}^2$, then $\text{Br}(R)$ contains the additive group of $\mathbb{C}$. There are then affine Zariski open neighborhoods of the vertex with non-torsion Brauer group [3, Example 8.2.2]. However, such examples work only in characteristic zero. This leaves open the following question, suggested by Colliot-Thélène and Skorobogatov:

**Question 1.1.** If $X$ is an integral normal quasi-projective variety over a field $k$ of positive characteristic, is $\text{Br}(X)$ a torsion group?

To analyze this question, we use a result concerning the Brauer group of a normal variety $X$ with only isolated singularities $p_1, \ldots, p_n$. Suppose $X$ is defined over an algebraically closed field $k$ of arbitrary characteristic. Let $K$ be the function field of $X$, $R_i = \mathcal{O}_{X, p_i}$ be the local ring at each singularity, and $R_i^{\text{sh}}$ its strict henselization. Then, $\text{Br}(X)$ is given by the exact sequence (see [3, Section 8.2], elaborating on [5, §1, Remarque 11 (b)])

\[
0 \to \text{Pic}(X) \to \text{Cl}(X) \to \bigoplus_{i=1}^n \text{Cl}(R_i^{\text{sh}}) \to \text{Br}(X) \to \text{Br}(K).
\]

This sequence indicates that one way for $\text{Br}(X)$ to be large is for a singularity to have a large henselian local class group with divisors that do not extend globally to $X$. This idea is illustrated by a counterexample given by Burt Totaro, which shows that the answer to Question 1.1 is “no” for threefolds: suppose that $X \subset \mathbb{P}^4$ is a hypersurface of degree $d \geq 3$ with a single node $p$. Then $Y = \text{Bl}_p(X)$ is a smooth, ample divisor in $\text{Bl}_p\mathbb{P}^4$. By the Grothendieck-Lefschetz theorem for Picard groups, the restriction $\text{Pic}(\text{Bl}_p\mathbb{P}^4) \to \text{Pic}(Y)$ is an isomorphism.

If $E$ is the exceptional divisor, then the sequence $0 \to \mathbb{Z} \cdot [E] \to \text{Pic}(Y) \to \text{Cl}(X) \to 0$ yields $\text{Cl}(X) \cong \mathbb{Z} \cdot \mathcal{O}_X(1)$, so that the restriction map $\text{Cl}(X) \to \text{Cl}(R^{\text{sh}})$ is zero. However, since a threefold node is étale-locally the cone over a smooth quadric surface, one can show that the henselian local class group contains a copy of $\mathbb{Z}$. Thus, $\text{Br}(X)$ is not torsion.

We will show that counterexamples also exist in dimension 2, provided that one works over a large algebraically closed field $k$. 

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2. A Surface Counterexample

The following construction is taken from [6], Example 1.27. Take a smooth cubic curve $D$ in the projective plane and a quartic curve $Q$ that meets it tranversally. Let $Y = \text{Bl}_{q_1, ..., q_{12}} \mathbb{P}^2$, where $q_1, ..., q_{12}$ are the points of intersection. On $Y$, the strict transform $C$ of $D$ satisfies $C^2 = -3$. Unlike rational curves, not all negative self-intersection higher genus curves may be contracted to yield a projective surface. Rather, the contraction might only exist as an algebraic space. However, in this case, $C$ is contractible.

**Proposition 2.1.** There exists a normal projective surface $X$ and a contraction morphism $Y \to X$ whose exceptional locus is $C$.

**Proof.** The Picard group of $Y$ is the free abelian group on $H$, the pullback of a general line in $\mathbb{P}^2$, and the exceptional divisors $E_1, ..., E_{12}$. Then, we claim that the line bundle $\mathcal{O}_Y(4H - \sum_i E_i) = \mathcal{O}_Y(H + C)$ defines a basepoint-free linear system on $Y$. Indeed, no point outside of the $E_i$ can be a base point, and the transforms of both $D + a$ line and $Q$ belong to the linear system. These don’t intersect on the exceptional locus by the transversality assumption. Therefore, the system defines a morphism $Y \to X'$ to a projective surface $X'$.

This contracts only $C$: if $C'$ is another (irreducible) curve with $C' \cdot (H + C) = 0$, clearly $C'$ is not supported on the $E_i$, or the intersection would be positive. Therefore, $H \cdot C' > 0$, meaning $C \cdot C' < 0$. Therefore, $C' = C$, as desired. Finally, passing to the normalization $X$ of $X'$, we may assume $X$ is a normal projective surface. $\square$

The resulting singularity $p$ of $X$ has minimal resolution with exceptional set exactly $C$, a smooth elliptic curve. Singularities satisfying this condition are simple elliptic singularities. Over $\mathbb{C}$, such singularities are completely classified. A simple elliptic singularity with $C^2 = -3$ is known as an $\tilde{E}_6$ singularity, and is complex analytically isomorphic to a cone over $C$ [7]. Here, we present a computation of the henselian local class group $\text{Cl}(R^{sh})$ of the singularity that works in any characteristic.

Consider the pullback of this desingularization to a “henselian neighborhood”:

$$
\begin{array}{ccc}
Y^h & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}(R^{sh}) & \longrightarrow & X.
\end{array}
$$

The scheme $Y^h$ is regular and $Y^h \setminus C \cong \text{Spec}(R^{sh}) \setminus \{m\}$, so we have an exact sequence $0 \to \mathbb{Z} \cdot [C] \to \text{Pic}(Y^h) \to \text{Cl}(R^{sh}) \to 0$. It suffices, therefore, to compute $\text{Pic}(Y^h)$. To do so, we’ll first consider infinitesimal neighborhoods of $C$ in $Y$. 
Detailed explanation of why \( Y^h \) is a regular scheme: essentially, this is because henselization behaves well with regularity and tensor products. We first base change to \( R \); the Zariski local rings of \( Y \) are regular, so this gives \( Y_R \to \text{Spec}(R) \) with \( Y_R \) regular. Then, to cover \( Y^h \), one can consider affine opens in \( Y_R \). For any such \( \text{Spec}(S) \subset R \), we have a ring map \( R \to S \) and the corresponding open in \( Y^h \) is \( R^h \otimes_R S \). The local rings there can be identified with the henselizations of the local rings of \( S \), which are regular (see, e.g. [8, 0BSK]).

The sequence of infinitesimal neighborhoods \( C = C_1 \subset C_2 \subset \cdots \) is defined by powers of the ideal sheaf \( \mathcal{I}_C \). Notably, these \( C_n \) are the same regardless of whether we consider them inside \( Y \) or inside the henselian neighborhood \( Y^h \). The normal bundle to \( C \) in \( Y \) gives obstructions to extending line bundles to successive neighborhoods, but we’ll show that all line bundles extend uniquely. The group \( \lim_{\leftarrow n} \text{Pic}(C_n) \) in the proposition below is also the Picard group of the formal neighborhood of \( C \) in \( Y \).

**Proposition 2.2.** The restriction map \( \lim_{\leftarrow n} \text{Pic}(C_n) \to \text{Pic}(C) \) is an isomorphism.

**Proof.** It’s enough to show that the maps \( \text{Pic}(C_{n+1}) \to \text{Pic}(C_n) \) are all isomorphisms for \( n \geq 1 \). Each extension \( C_n \subset C_{n+1} \) is a first-order thickening, since \( C_n \) is defined in \( C_{n+1} \) by the square-zero ideal sheaf \( \mathcal{I}_C/\mathcal{I}_C^{n+1} \). Associated to such a thickening is a long exact sequence in cohomology [8, 0C6Q]

\[
\cdots \to H^1(C, \mathcal{I}_C^n/\mathcal{I}_C^{n+1}) \to \text{Pic}(C_{n+1}) \to \text{Pic}(C_n) \to H^2(C, \mathcal{I}_C^n/\mathcal{I}_C^{n+1}) \to \cdots .
\]

We may take the outer cohomology groups to be over \( C \) since the underlying topological spaces are the same. As sheaves of abelian groups on \( C \), we have \( \mathcal{I}_C^n/\mathcal{I}_C^{n+1} \cong \mathcal{O}_C(-nC) \), a multiple of the conormal bundle. But \( C^2 = -3 \) in \( Y \) so this last bundle has degree \( 3n > 0 \). Therefore, its higher cohomology vanishes and \( \text{Pic}(C_{n+1}) \to \text{Pic}(C_n) \) is an isomorphism for all \( n \).

Detailed explanation as to why outer cohomology groups are over \( C \), and not a thickening: we consider every sheaf in the sequence

\[
0 \to \mathcal{I}_C^n/\mathcal{I}_C^{n+1} \to \mathcal{O}_{C_{n+1}}^* \to \mathcal{O}_{C_n}^* \to 0
\]

to be a sheaf of an abelian groups, and forget the module structure (this doesn’t change cohomology). All the \( C_n \) have the same underlying topological space so these sheaves are all on that topological space. All that changes is the ringed space structure.

Now, we need only compare \( \lim_{\leftarrow n} \text{Pic}(C_n) \) to \( \text{Pic}(Y^h) \). Using the Artin approximation theorem [1, Theorem 3.5], the map \( \text{Pic}(Y^h) \to \lim_{\leftarrow n} \text{Pic}(C_n) \) is injective with dense image. However, the topology of the latter group is discrete in this setting because each group of the inverse limit is \( \text{Pic}(C) \). Therefore, the map is surjective also and \( \text{Pic}(Y^h) \cong \text{Pic}(C) \).

Detailed explanation: The formulation of Artin approximation in [1] is as follows: let \( A \) be a local henselian ring with maximal ideal \( \mathfrak{m} \) (\( A \) here is the henselization of a finite type algebra over the field \( k \)) and let \( \hat{A} \) be its completion. Then for any functor \( F : \{ A - \text{algebras} \} \to \{ \text{sets} \} \) that is locally of finite presentation, any \( \xi \in F(\hat{A}) \), and any integer \( n \), there exists \( \xi \in F(A) \) such that
\[ \hat{\xi} \equiv \xi \mod m^n. \]

In the proof of theorem 3.5, it is noted that \( H^1(X \times_{\text{Spec}(A)} - , \mathbb{G}_m) \) is a functor locally of finite presentation for \( X \to A \) proper. Hence in our setting, we can approximate any line bundle on the formal neighborhood up to \( C_n \). Since all the Picard groups are the same, this demonstrates surjectivity.

**Theorem 2.3.** Let \( k \) be an algebraically closed field that is not the algebraic closure of a finite field. Then, \( \text{Br}(X) \) is non-torsion.

**Proof.** From the above, we have the identification \( \text{Cl}(R^\text{sh}) \cong \text{Pic}(Y^h)/\mathbb{Z} \cdot \mathcal{O}_{Y^h}(C) \cong \text{Pic}(C)/\mathbb{Z} \cdot \mathcal{O}_C(C) \). Since \( \deg_C \mathcal{O}_C(C) = 3 \), the class group is then an extension of \( \mathbb{Z}/3 \) by \( \text{Pic}^0(C) \cong C(k) \). If \( k = \mathbb{F}_p \), then \( C(k) \) is torsion, because every point is defined over \( \mathbb{F}_p \) for some \( m \). In contrast, if \( k \) satisfies the hypothesis of the theorem, \( C(k) \) has infinite rank [4, Theorem 10.1].

However, the global class group of \( X \) is quite small: \( \text{Cl}(Y) = \text{Pic}(Y) \cong \mathbb{Z}^{13} \) since \( Y \) is the blow up of \( \mathbb{P}^2 \) in 12 points and \( \text{Cl}(X) \cong \text{Cl}(Y)/\mathbb{Z} \cdot [C] \). Therefore, the cokernel of the restriction map \( \text{Cl}(X) \to \text{Cl}(R^\text{sh}) \) in (1) contains non-torsion elements, so \( \text{Br}(X) \) does too. \( \square \)

To complement the above result, we also prove

**Theorem 2.4.** Suppose that \( X \) is an integral normal surface over the algebraic closure \( k = \overline{\mathbb{F}}_p \) of a finite field. Then \( \text{Br}(X) \) is torsion.

**Proof.** The strategy is similar to the previous theorem. Here, the crucial fact is that all possible “building blocks” of the henselian local class group - abelian varieties over \( k \), the additive group of \( k \), and the multiplicative group of \( k \) - are all torsion.

In the exact sequence (1), \( \text{Br}(K) \) is always torsion, so if we can prove \( \bigoplus_{i=1}^n \text{Cl}(R^\text{sh}_i) \) is as well, the result follows. Therefore, we focus on the desingularization \( \pi : Y^h \to \text{Spec}(R^\text{sh}) \) of the henselian local ring at just one point \( p \). We may choose \( Y^h \) such that the exceptional set \( E = \pi^{-1}(p) \) is a union of irreducible curves \( E_j \) which are smooth and meet pairwise transversely, with no three containing a common point.

The following argument is due to Artin [2, p. 491]. Suppose \( G \) is the free abelian group of divisors supported on \( E \), and consider the map \( \text{Pic}(Y^h) \to \text{Hom}(G, \mathbb{Z}) \) given by \( L \mapsto (D \mapsto D \cdot L) \). This restricts to a map \( G \to \text{Hom}(G, \mathbb{Z}) \) that is injective because the intersection matrix of the curves \( E_j \) is negative definite. In particular, the first map in the excision sequence of class groups \( 0 \to G \to \text{Pic}(Y^h) \to \text{Cl}(R^\text{sh}) \to 0 \) is injective. Since \( G \) and \( \text{Hom}(G, \mathbb{Z}) \) are free abelian groups of equal rank, \( G \to \text{Hom}(G, \mathbb{Z}) \) also has finite cokernel. This allows us to find a cycle \( Z = \sum_j r_j E_j \) with all \( r_j > 0 \) and \( \mathcal{O}_Z(-Z) \) ample. We’ll examine infinitesimal neighborhoods of the closed subscheme \( Z \).

As before, for every \( n \geq 1 \), there is an exact sequence

\[ \cdots \to H^1(Z, \mathcal{O}_Z(-nZ)) \to \text{Pic}(Z_{n+1}) \to \text{Pic}(Z_n) \to H^2(Z, \mathcal{O}_Z(-nZ)) \to \cdots. \]
Because \( \dim(Z) = 1 \), the last group is always zero. Since \( \mathcal{O}_Z(-Z) \) is ample, the first group is zero for \( n \gg 0 \). Therefore, the inverse limit \( \varprojlim_n \text{Pic}(Z_n) \) is constructed as a finite series of extensions of \( \text{Pic}(Z) \) by finite-dimensional \( k \)-vector spaces. While \( Z \) is generally non-reduced and not equal to \( E \), we may still apply Artin approximation. This is because for any large \( n \), the scheme \( E_n \) is nested between two infinitesimal neighborhoods of \( Z \), where all restrictions of Picard groups are surjective (use a similar exact sequence to the above, e.g. [8, 09NY]). It follows that \( \varprojlim_n \text{Pic}(Z_n) \cong \varprojlim_n \text{Pic}(E_n) \), so \( \text{Pic}(Y^h) \cong \varprojlim_n \text{Pic}(Z_n) \).

Next, let \( \bar{Z} \) be the disjoint union of the schemes \( r_j E_j \). Then \( f : \bar{Z} \to Z \) is a finite map that is an isomorphism away from the finite set of intersection points and such that \( \mathcal{O}_Z \subset f^* \mathcal{O}_Z \). It follows (see [8, 0C1H]) that \( \text{Pic}(Z) \) is a finite sequence of extensions of \( \text{Pic}(\bar{Z}) \) by quotients of \( (k,+) \) or \( (k,*) \). Lastly, \( \text{Pic}(\bar{Z}) = \bigoplus_j \text{Pic}(r_j E_j) \), each of which is built from finite-dimensional \( k \)-vector spaces, and \( \text{Pic}(E_j) \cong \mathbb{Z} \oplus \text{Pic}^0(E_j) \). Since the \( \text{Pic}^0(E_j) \) are groups of \( k \)-points of abelian varieties over \( k \), they are all torsion.

Over large fields of characteristic \( p \), the key to non-torsion Brauer groups is some non-simple-connectedness of the exceptional locus, either via a cycle of curves (which gives a \( k^* \) because you “link up” two points on the same connected component) or via \( \text{Pic}(C) \) for \( C \) of positive genus.

Taken together, all of this implies that \( G \) and \( \text{Pic}(Y^h) \) are equal rank. Because \( G \to \text{Pic}(Y^h) \) is injective, the quotient \( \text{Cl}(R^h) \) is a torsion group, as desired. \( \square \)

References