NON-TORSION BRAUER GROUPS IN POSITIVE CHARACTERISTIC

LOUIS ESSER

1. INTRODUCTION

One way of extending the notion of the classical Brauer group of a field to any scheme $X$ is by defining the Brauer-Grothendieck group $\text{Br}(X) = H^2(X_{\text{ét}}, \mathbb{G}_m)$. Just as for fields, this group is torsion for any regular integral noetherian scheme [5, Corollaire 1.8]. However, this no longer holds for singular schemes. In fact, there exist affine integral normal complex surfaces $X$ with one singular point such that $\text{Br}(X)$ contains the additive group of $\mathbb{C}$ [3, Example 8.2.2]. Specifically, the affine cone over a curve of degree $d \geq 4$ in $\mathbb{P}^2_\mathbb{C}$ has this property. However, these examples leave open the following question, suggested by Colliot-Thélène and Skorobogatov:

**Question 1.1.** If $X$ is an integral normal quasi-projective variety over a field $k$ of positive characteristic, is $\text{Br}(X)$ a torsion group?

To analyze this question, we use a result concerning the Brauer group of a normal variety $X$ with only isolated singularities $p_1, \ldots, p_n$. Suppose $X$ is defined over an algebraically closed field $k$ of arbitrary characteristic. Let $K$ be the function field of $X$, $R_i = \mathcal{O}_{X,p_i}$ be the local ring at each singularity, and $R_i^{\text{sh}}$ its strict henselization. Then, $\text{Br}(X)$ is given by the exact sequence (see [3, Section 8.2], elaborating on [5, §1, Remarque 11 (b)])

$$
0 \rightarrow \text{Pic}(X) \rightarrow \text{Cl}(X) \rightarrow \bigoplus_{i=1}^n \text{Cl}(R_i^{\text{sh}}) \rightarrow \text{Br}(X) \rightarrow \text{Br}(K).
$$

This sequence indicates that one way for $\text{Br}(X)$ to be large is when a singularity has a large henselian local class group with divisors that do not extend globally to $X$. This idea is illustrated by a counterexample given by Burt Totaro, which shows that the answer to Question 1.1 is “no” for threefolds: suppose that $X \subset \mathbb{P}^4$ is a hypersurface of degree $d \geq 3$ with a single node $p$. Then $Y = \text{Bl}_p(X)$ is a smooth, ample divisor in $\text{Bl}_p \mathbb{P}^4$. By the Grothendieck-Lefschetz theorem for Picard groups, the restriction $\text{Pic}(\text{Bl}_p \mathbb{P}^4) \rightarrow \text{Pic}(Y)$ is an isomorphism.

If $E$ is the exceptional divisor, then the sequence $0 \rightarrow \mathbb{Z} \cdot [E] \rightarrow \text{Pic}(Y) \rightarrow \text{Cl}(X) \rightarrow 0$ yields $\text{Cl}(X) \cong \mathbb{Z} \cdot \mathcal{O}_X(1)$, so that the restriction map $\text{Cl}(X) \rightarrow \text{Cl}(R^{\text{sh}})$ is zero. However, since a threefold node is étale-locally the cone over a smooth quadric surface, one can show that the henselian local class group contains a copy of $\mathbb{Z}$. Thus, $\text{Br}(X)$ is not torsion.

We will show that counterexamples also exist in dimension 2, provided that one works over a large algebraically closed field $k$. 
Acknowledgements. I thank Burt Totaro for suggesting this problem to me and for his advice.

2. A Surface Counterexample

The following construction is taken from [6], Example 1.27. Take a smooth cubic curve \( D \) in the projective plane and a quartic curve \( Q \) that meets it transversally. Let \( Y = \text{Bl}_{q_1, \ldots, q_{12}} \mathbb{P}^2 \), where \( q_1, \ldots, q_{12} \) are the points of intersection. On \( Y \), the strict transform \( C \) of \( D \) satisfies \( C^2 = -3 \). Unlike rational curves, not all negative self-intersection higher genus curves may be contracted to yield a projective surface. Rather, the contraction might only exist as an algebraic space. However, in this case, \( C \) is contractible.

**Proposition 2.1.** There exists a normal projective surface \( X \) and a contraction morphism \( Y \to X \) whose exceptional locus is \( C \).

**Proof.** The Picard group of \( Y \) is the free abelian group on \( H \), the pullback of a general line in \( \mathbb{P}^2 \), and the exceptional divisors \( E_1, \ldots, E_{12} \). Then, we claim that the line bundle \( \mathcal{O}_Y(4H - \sum_i E_i) = \mathcal{O}_Y(H + C) \) defines a basepoint-free linear system on \( Y \). Indeed, no point outside of the \( E_i \) can be a base point, and the transforms of both \( D + \text{a line} \) and \( Q \) belong to the linear system. These don’t intersect on the exceptional locus by the transversality assumption. Therefore, the system defines a morphism \( Y \to X' \) to a projective surface \( X' \).

This contracts only \( C \): if \( C' \) is another (irreducible) curve with \( C' \cdot (H + C) = 0 \), clearly \( C' \) is not supported on the \( E_i \), or the intersection would be positive. Therefore, \( H \cdot C' > 0 \), meaning \( C \cdot C' < 0 \). Therefore, \( C' = C \), as desired. Finally, passing to the normalization \( X \) of \( X' \), we may assume \( X \) is a normal projective surface. \( \square \)

The resulting singularity \( p \) of \( X \) has minimal resolution with exceptional set exactly \( C \), a smooth elliptic curve. Singularities satisfying this condition are simple elliptic singularities. Over \( \mathbb{C} \), such singularities are completely classified. A simple elliptic singularity with \( C^2 = -3 \) is known as an \( \tilde{E}_6 \) singularity, and is complex analytically isomorphic to a cone over \( C \) [7]. Here, we present a computation of the henselian local class group \( \text{Cl}(R^{\text{sh}}) \) of the singularity that works in any characteristic.

Consider the pullback of this desingularization to a "henselian neighborhood":

\[
\begin{array}{ccc}
Y^h & \to & Y \\
\downarrow & & \downarrow \\
\text{Spec}(R^{\text{sh}}) & \to & X.
\end{array}
\]

The scheme \( Y^h \) is regular and \( Y^h \setminus C \cong \text{Spec}(R^{\text{sh}}) \setminus \{m\} \), so we have an exact sequence \( 0 \to \mathbb{Z} \cdot [C] \to \text{Pic}(Y^h) \to \text{Cl}(R^{\text{sh}}) \to 0 \). It suffices, therefore, to compute \( \text{Pic}(Y^h) \). To do so, we’ll first consider infinitesimal neighborhoods of \( C \) in \( Y \).

The sequence of infinitesimal neighborhoods \( C = C_1 \subset C_2 \subset \cdots \) is defined by powers of the ideal sheaf \( \mathcal{I}_C \). Notably, these \( C_n \) are the same regardless of whether we consider
them inside $Y$ or inside the henselian neighborhood $Y^h$. The normal bundle to $C$ in $Y$ gives obstructions to extending line bundles to successive neighborhoods, but we’ll show that all line bundles extend uniquely. The group $\lim_{\leftarrow n} \text{Pic}(C_n)$ in the proposition below is also the Picard group of the formal neighborhood of $C$ in $Y$.

**Proposition 2.2.** The restriction map $\lim_{\leftarrow n} \text{Pic}(C_n) \to \text{Pic}(C)$ is an isomorphism.

*Proof.* It’s enough to show that the maps $\text{Pic}(C_{n+1}) \to \text{Pic}(C_n)$ are all isomorphisms for $n \geq 1$. Each extension $C_n \subset C_{n+1}$ is a first-order thickening, since $C_n$ is defined in $C_{n+1}$ by the square-zero ideal sheaf $T^n_C/T^{n+1}_C$. Associated to such a thickening is a long exact sequence in cohomology [8, 0C6Q]

$$
\cdots \to H^1(C, T^n_C/T^{n+1}_C) \to \text{Pic}(C_{n+1}) \to \text{Pic}(C_n) \to H^2(C, T^n_C/T^{n+1}_C) \to \cdots.
$$

We may take the outer cohomology groups to be over $C$ since the underlying topological spaces are the same. As sheaves of abelian groups on $C$, we have $T^n_C/T^{n+1}_C \cong \mathcal{O}_C(-nC)$, a multiple of the conormal bundle. But $C^2 = -3$ in $Y$ so this last bundle has degree $3n > 0$. Therefore, its higher cohomology vanishes and $\text{Pic}(C_{n+1}) \to \text{Pic}(C_n)$ is an isomorphism for all $n$. \hfill \Box

Now, we need only compare $\lim_{\leftarrow n} \text{Pic}(C_n)$ to $\text{Pic}(Y^h)$. Using the Artin approximation theorem [1, Theorem 3.5], the map $\text{Pic}(Y^h) \to \lim_{\leftarrow n} \text{Pic}(C_n)$ is injective with dense image. However, the topology of the latter group is discrete in this setting because each group of the inverse limit is $\text{Pic}(C)$. Therefore, the map is surjective also and $\text{Pic}(Y^h) \cong \text{Pic}(C)$.

**Theorem 2.3.** Let $k$ be an algebraically closed field that is not the algebraic closure of a finite field. Then, $\text{Br}(X)$ is non-torsion.

*Proof.* From the above, we have the identification $\text{Cl}(R^\text{sh}) \cong \text{Pic}(Y^h)/\mathbb{Z} \cdot \mathcal{O}_Y(Y^h)(C)$ $\cong \text{Pic}(C)/\mathbb{Z} \cdot \mathcal{O}_C(C)$. Since $\deg_C \mathcal{O}_C(C) = 3$, the class group is then an extension of $\mathbb{Z}/3$ by $\text{Pic}^0(C) \cong C(k)$. If $k = \mathbb{F}_p$, then $C(k)$ is torsion, because every point is defined over $\mathbb{F}_p^m$ for some $m$. In contrast, if $k$ satisfies the hypothesis of the theorem, $C(k)$ has infinite rank [4, Theorem 10.1].

However, the global class group of $X$ is quite small: $\text{Cl}(Y) = \text{Pic}(Y) \cong \mathbb{Z}^{13}$ since $Y$ is the blow up of $\mathbb{P}^2$ in 12 points and $\text{Cl}(X) \cong \text{Cl}(Y)/\mathbb{Z} \cdot [C]$. Therefore, the cokernel of the restriction map $\text{Cl}(X) \to \text{Cl}(R^\text{sh})$ in (1) contains non-torsion elements, so $\text{Br}(X)$ does too. \hfill \Box

To complement the above result, we also prove

**Theorem 2.4.** Suppose that $X$ is an integral normal projective surface over the algebraic closure $k = \mathbb{F}_p$ of a finite field. Then $\text{Br}(X)$ is torsion.

*Proof.* The strategy is similar to the previous theorem. Here, the crucial fact is that all possible “building blocks” of the henselian local class group - abelian varieties over $k$, the additive group of $k$, and the multiplicative group of $k$ - are all torsion.
In the exact sequence (1), Br(K) is always torsion, so if we can prove \( \bigoplus_{i=1}^{n} \text{Cl}(R_i^{sh}) \) is as well, the result follows. Therefore, we focus on the desingularization \( \pi: Y^h \to \text{Spec}(R^{sh}) \) of the henselian local ring at just one point \( p \). We may choose \( Y^h \) such that the exceptional set \( E = \pi^{-1}(p) \) is a union of irreducible curves \( E_j \) which are smooth and meet pairwise transversely, with no three containing a common point.

The following argument is due to Artin [2, p. 491]. Suppose \( G \) is the free abelian group of divisors supported on \( E \), and consider the map \( \text{Pic}(Y^h) \to \text{Hom}(G, \mathbb{Z}) \) given by \( L \mapsto (D \mapsto D \cdot L) \). This restricts to a map \( G \to \text{Hom}(G, \mathbb{Z}) \) that is injective because the intersection matrix of the curves \( E_j \) is negative definite. In particular, the first map in the excision sequence of class groups \( 0 \to G \to \text{Pic}(Y^h) \to \text{Cl}(R^{sh}) \to 0 \) is injective. Since \( G \) and \( \text{Hom}(G, \mathbb{Z}) \) are free abelian groups of equal rank, \( G \to \text{Hom}(G, \mathbb{Z}) \) also has finite cokernel. This allows us to find a cycle \( Z = \sum_{j} r_j E_j \) with all \( r_j > 0 \) and \( O_Z(-Z) \) ample. We’ll examine infinitesimal neighborhoods of the closed subscheme \( Z \).

As before, for every \( n \geq 1 \), there is an exact sequence

\[
\cdots \to H^1(Z, O_Z(-nZ)) \to \text{Pic}(Z_{n+1}) \to \text{Pic}(Z_n) \to H^2(Z, O_Z(-nZ)) \to \cdots.
\]

Because \( \dim(Z) = 1 \), the last group is always zero. Since \( O_Z(-Z) \) is ample, the first group is zero for \( n \gg 0 \). Therefore, the inverse limit \( \varprojlim_n \text{Pic}(Z_n) \) is constructed as a finite series of extensions of \( \text{Pic}(Z) \) by finite-dimensional \( k \)-vector spaces. While \( Z \) is generally non-reduced and not equal to \( E \), we may still apply Artin approximation. This is because for any large \( n \), the scheme \( E_n \) is nested between two infinitesimal neighborhoods of \( Z \), where all restrictions of Picard groups are surjective (use a similar exact sequence to the above, e.g. [8, 09NY]). It follows that \( \varprojlim_n \text{Pic}(Z_n) \cong \varprojlim \text{Pic}(E_n) \), so \( \text{Pic}(Y^h) \cong \varprojlim_n \text{Pic}(Z_n) \).

Next, let \( \tilde{Z} \) be the disjoint union of the schemes \( r_j E_j \). Then \( f: \tilde{Z} \to Z \) is a finite map that is an isomorphism away from the finite set of intersection points and such that \( O_Z \subset f_* O_{\tilde{Z}} \). It follows (see [8, 0C1H]) that \( \text{Pic}(Z) \) is a finite sequence of extensions of \( \text{Pic}(\tilde{Z}) \) by quotients of \( (k, +) \) or \( (k, *) \). Lastly, \( \text{Pic}(\tilde{Z}) = \bigoplus \text{Pic}(r_j E_j) \), each of which is built from finite-dimensional \( k \)-vector spaces, and \( \text{Pic}(E_j) \cong \mathbb{Z} \oplus \text{Pic}^0(E_j) \). Since the \( \text{Pic}^0(E_j) \) are groups of \( k \)-points of abelian varieties over \( k \), they are all torsion.

Taken together, all of this implies that \( G \) and \( \text{Pic}(Y^h) \) are equal rank. Because \( G \to \text{Pic}(Y^h) \) is injective, the quotient \( \text{Cl}(R^{sh}) \) is a torsion group, as desired. \( \square \)

**References**


UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555

Email address: esserl@math.ucla.edu