ABSTRACT. Unlike the classical Brauer group of a field, the Brauer-Grothendieck group of a singular scheme need not be torsion. We show that there exist integral normal projective surfaces over a large field of positive characteristic with non-torsion Brauer group. In contrast, we demonstrate that such examples cannot exist over the algebraic closure of a finite field.

1. INTRODUCTION

One way of extending the notion of the classical Brauer group of a field to any scheme $X$ is by defining the Brauer-Grothendieck group $\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$. Just as for fields, this group is torsion for any regular integral noetherian scheme [5, Corollaire 1.8]. However, this no longer holds for singular schemes. Chapter 8 of [3] lists several counterexamples. For instance, if $R$ is the local ring of the vertex of a cone over a smooth curve of degree $d \geq 4$ in $\mathbb{P}^2_\mathbb{C}$, then $\text{Br}(R)$ contains the additive group of $\mathbb{C}$. There are then affine Zariski open neighborhoods of the vertex with non-torsion Brauer group [3, Example 8.2.2]. However, the additive group of $k$ is torsion when $k$ has positive characteristic, so analogous constructions do not work there. There are also reducible varieties in arbitrary characteristic with non-torsion Brauer group [3, Section 8.1]. This leaves open the following question, communicated by Colliot-Thélène and Skorobogatov in an unpublished draft of [3]:

Question 1.1. If $X$ is an integral normal quasi-projective variety over a field $k$ of positive characteristic, is $\text{Br}(X)$ a torsion group?

To analyze this question, we use a result concerning the Brauer group of a normal variety $X$ with only isolated singularities $p_1, \ldots, p_n$. Suppose $X$ is defined over an algebraically closed field $k$ of arbitrary characteristic. Let $K$ be the function field of $X$, $R_i = \mathcal{O}_{X,p_i}$ be the local ring at each singularity, and $R^h_i$ its henselization. Then, $\text{Br}(X)$ is given by the exact sequence (see [3, Section 8.2], elaborating on [5, §1, Remarque 11 (b)])

$$0 \to \text{Pic}(X) \to \text{Cl}(X) \to \bigoplus_{i=1}^n \text{Cl}(R^h_i) \to \text{Br}(X) \to \text{Br}(K).$$

This sequence indicates that one way for $\text{Br}(X)$ to be large is for a singularity to have a large henselian local class group with divisors that do not extend globally to $X$. This
idea is illustrated by a counterexample given by Burt Totaro, which shows that $\text{Br}(X)$ is non-torsion for $X$ a hypersurface of degree $d \geq 3$ in $\mathbb{P}^4_k$ with a single node [3, Proposition 8.2.3]. Here $k$ is any algebraically closed field with characteristic not 2.

We will show that counterexamples to Question 1.1 exist in dimension 2 if and only if $k$ is not the algebraic closure of a finite field.

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2. A Surface Counterexample

The following construction is taken from [7], Example 1.27. Take a smooth cubic curve $D$ in the projective plane and a quartic curve $Q$ that meets it transversally. Let $Y = \text{Bl}_{q_1,\ldots,q_{12}} \mathbb{P}^2$, where $q_1,\ldots,q_{12}$ are the points of intersection. On $Y$, the proper transform $C$ of $D$ satisfies $C^2 = -3$. Unlike rational curves, not all negative self-intersection higher genus curves may be contracted to yield a projective surface. Rather, the contraction might only exist as an algebraic space. However, in this case, $C$ is contractible.

Proposition 2.1. There exists a normal projective surface $X$ and a proper birational morphism $Y \to X$ whose exceptional locus is exactly $C$.

Proof. The Picard group of $Y$ is the free abelian group on $H$, the pullback of a general line in $\mathbb{P}^2$, and the exceptional divisors $E_1,\ldots,E_{12}$. Then, we claim that the line bundle $L := \mathcal{O}_Y(4H - \sum_i E_i) = \mathcal{O}_Y(H + C)$ defines a basepoint-free linear system on $Y$. Indeed, no point outside of the $E_i$ can be a base point, and the proper transforms of both $D + a$ line and $Q$ belong to the linear system. These don’t intersect on the exceptional locus by the transversality assumption. Therefore, the system defines a morphism from $Y$ to projective space with image $X'$. This morphism is birational because we have an injective map $H^0(Y, \mathcal{O}_Y(H)) \hookrightarrow H^0(Y, L)$ and $\mathcal{O}_Y(H)$ is the pullback of a very ample line bundle on $\mathbb{P}^2$.

The exceptional locus of the morphism $Y' \to X'$ is precisely the union of the irreducible curves in $Y$ on which $L$ has degree zero. If $C'$ an irreducible curve with $C' \cdot (H + C) = 0$, clearly $C'$ is not supported on the $E_i$, or the intersection would be positive. Therefore, $H \cdot C' > 0$, meaning $C \cdot C' < 0$. This means $C' = C$, so the exceptional locus is the curve $C$, which is mapped to a point. Thus, $X'$ is a surface birational to $Y$ and $Y \to X'$ is an isomorphism away from $C$. Finally, passing to the normalization $X$ of $X'$, we may assume $X$ is a normal projective surface; the normalization will also be an isomorphism away from $C$, and the image of $C$ will again be a point in $X$. □

The resulting singularity $p$ of $X$ has minimal resolution with exceptional set exactly $C$, a smooth elliptic curve. Singularities satisfying this condition are simple elliptic singularities. Over $\mathbb{C}$, such singularities are completely classified. A simple elliptic singularity with $C^2 = -3$ is known as an $\tilde{E}_6$ singularity, and is complex analytically isomorphic to a cone over $C$ [8]. Here, we present a computation of the henselian local class group $\text{Cl}(R^h)$ of the singularity that works in any characteristic.
Consider the pullback of the desingularization $Y \to X$ to a “henselian neighborhood”:

$$
\begin{array}{ccc}
Y^h & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}(R^h) & \longrightarrow & X.
\end{array}
$$

The scheme $Y^h$ is regular and $Y^h \setminus C \cong \text{Spec}(R^h) \setminus \{m\}$, so we have an exact sequence $0 \to \mathbb{Z} \cdot [C] \to \text{Pic}(Y^h) \to \text{Cl}(R^h) \to 0$. Here, the first map is injective since $\mathcal{O}_{Y^h}(C)$ has nonzero degree on $C$. It suffices, therefore, to compute $\text{Pic}(Y^h)$. To do so, we’ll first consider infinitesimal neighborhoods of $C$ in $Y$.

The sequence of infinitesimal neighborhoods $C = C_1 \subset C_2 \subset \cdots$ is defined by powers of the ideal sheaf $\mathcal{I}_C$. Notably, these $C_n$ are the same regardless of whether we consider them inside $Y$ or inside the henselian neighborhood $Y^h$. The normal bundle to $C$ in $Y$ gives obstructions to extending line bundles to successive neighborhoods, but we’ll show that all line bundles extend uniquely. The group $\varprojlim_n \text{Pic}(C_n)$ in the proposition below is also the Picard group of the formal neighborhood of $C$ in $Y$.

**Proposition 2.2.** The restriction map $\varprojlim_n \text{Pic}(C_n) \to \text{Pic}(C)$ is an isomorphism.

**Proof.** It’s enough to show that the maps $\text{Pic}(C_{n+1}) \to \text{Pic}(C_n)$ are all isomorphisms for $n \geq 1$. Each extension $C_n \subset C_{n+1}$ is a first-order thickening, since $C_n$ is defined in $C_{n+1}$ by the square-zero ideal sheaf $\mathcal{I}_C^n/\mathcal{I}_C^{n+1}$. Associated to such a thickening is a long exact sequence in cohomology [9, 0C6Q]

$$
\cdots \to H^1(C, \mathcal{I}_C^n/\mathcal{I}_C^{n+1}) \to \text{Pic}(C_{n+1}) \to \text{Pic}(C_n) \to H^2(C, \mathcal{I}_C^n/\mathcal{I}_C^{n+1}) \to \cdots.
$$

We may take the outer cohomology groups to be over $C$ since the underlying topological spaces are the same. As sheaves of abelian groups on $C$, we have $\mathcal{I}_C^n/\mathcal{I}_C^{n+1} \cong \mathcal{O}_C(-nC)$, a multiple of the conormal bundle. But $C^2 = -3$ in $Y$ so this last bundle has degree $3n > 0$. Since $C$ is genus 1, the higher cohomology of $\mathcal{O}_C(-nC)$ vanishes and $\text{Pic}(C_{n+1}) \to \text{Pic}(C_n)$ is an isomorphism for all $n$. \(\square\)

Now, we need only compare $\varprojlim_n \text{Pic}(C_n)$ to $\text{Pic}(Y^h)$. Using the Artin approximation theorem [1, Theorem 3.5], the map $\text{Pic}(Y^h) \to \varprojlim_n \text{Pic}(C_n)$ is injective with dense image. However, the topology of the latter group is discrete in this setting because each group of the inverse limit is $\text{Pic}(C)$. Therefore, the map is surjective also and $\text{Pic}(Y^h) \cong \text{Pic}(C)$.

**Theorem 2.3.** Let $k$ be an algebraically closed field that is not the algebraic closure of a finite field and $X$ be the surface defined in the proof of Proposition 2.1. Then, $\text{Br}(X)$ is non-torsion.

**Proof.** From the above, we have the identification $\text{Cl}(R^h) \cong \text{Pic}(Y^h)/\mathbb{Z} \cdot \mathcal{O}_{Y^h}(C) \cong \text{Pic}(C)/\mathbb{Z} \cdot \mathcal{O}_C(C)$. Since $\deg_C \mathcal{O}_C(C) = 3$, the class group is then an extension of $\mathbb{Z}/3$ by $\text{Pic}^0(C) \cong C(k)$, where $C(k)$ is the group of $k$-rational points of the elliptic curve $C$ with a chosen identity point. Since $k \neq \mathbb{F}_p$, $C(k)$ has infinite rank [4, Theorem 10.1]. Note that
in contrast, $C(k)$ is torsion for an elliptic curve $C$ over $\overline{\mathbb{F}}_p$, because every point is defined over $\mathbb{F}_{p^m}$ for some $m$.

However, the global class group of $X$ is quite small: $\text{Cl}(Y) = \text{Pic}(Y) \cong \mathbb{Z}^{13}$ since $Y$ is the blow up of $\mathbb{P}^2$ in 12 points and $\text{Cl}(X) \cong \text{Cl}(Y)/\mathbb{Z} \cdot [C]$. Therefore, the cokernel of the restriction map $\text{Cl}(X) \to \text{Cl}(R^h)$ in (1) contains non-torsion elements, so $\text{Br}(X)$ does too.

To complement the above result, we also prove:

**Theorem 2.4.** Suppose that $X$ is an integral normal surface over the algebraic closure $k = \overline{\mathbb{F}}_p$ of a finite field. Then $\text{Br}(X)$ is torsion.

*Proof.* The strategy is similar to the previous theorem. Here, the crucial fact is that all possible “building blocks” of the henselian local class group - abelian varieties over $k$, the additive group of $k$, and the multiplicative group of $k$ - are torsion.

Since the singularities of a normal surface are isolated, we may apply the exact sequence (1). The group $\text{Br}(K)$ is always torsion, so if we can prove $\oplus_{i=1}^n \text{Cl}(R^h_i)$ is as well, the result follows. Let $Y \to X$ be a desingularization. We focus on the base change $\pi: Y^h \to \text{Spec}(R^h)$ to the henselian local ring at just one singular point $p$. Let $E = \pi^{-1}(p)$ be the scheme-theoretic fiber. We may choose $Y$ such that $E_{\text{red}}$ is a union of irreducible curves $F_j$ which are smooth and meet pairwise transversely, with no three containing a common point.

The following argument is due to Artin [2]. Suppose $G$ is the free abelian group of divisors supported on $E$, and consider the map $\alpha: \text{Pic}(Y^h) \to \text{Hom}(G, \mathbb{Z})$ given by $L \mapsto (D \mapsto D \cdot L)$. This restricts to a map $\alpha|_G: G \to \text{Hom}(G, \mathbb{Z})$ that is injective because the intersection matrix of the curves $F_j$ is negative definite. Since $G$ and $\text{Hom}(G, \mathbb{Z})$ are free abelian groups of equal rank, $G \to \text{Hom}(G, \mathbb{Z})$ also has finite cokernel. This allows us to find an effective Cartier divisor $Z = \sum_j r_j F_j$ with all $r_j > 0$ such that $\alpha(Z) = \alpha(-H)$ for an ample line bundle $H$ on $Y^h$ [2, p. 491]. If we restrict this Cartier divisor to the scheme associated to $Z$, the resulting line bundle $\mathcal{O}_Z(-Z)$ has positive degree on every irreducible component of $Z$, so it is ample. We’ll examine infinitesimal neighborhoods of the closed subscheme $Z$ in $Y^h$.

As before, for every $n \geq 1$, there is an exact sequence

$$\cdots \to H^1(Z, \mathcal{O}_Z(-nZ)) \to \text{Pic}(Z_{n+1}) \to \text{Pic}(Z_n) \to H^2(Z, \mathcal{O}_Z(-nZ)) \to \cdots.$$ 

Because $\dim(Z) = 1$, the last group is always zero. Since $\mathcal{O}_Z(-Z)$ is ample, the first group is zero for $n \gg 0$ by Serre vanishing, which holds on any projective scheme [6, Theorem II.5.2]. Therefore, the inverse limit $\varprojlim_n \text{Pic}(Z_n)$ is constructed as a finite series of extensions of $\text{Pic}(Z)$ by finite-dimensional $k$-vector spaces. Applying Artin approximation, we have that $\text{Pic}(Y^h) \to \varprojlim_n \text{Pic}(E_n)$ is injective with dense image. However, for large $n$, the scheme $E_n$ is nested between two infinitesimal neighborhoods of $Z$, where all restrictions of Picard groups are surjective (use a similar exact sequence to the above,
e.g. [9, 09NY]). It follows that \( \lim_{\longleftarrow n} \text{Pic}(E_n) \cong \lim_{\longleftarrow n} \text{Pic}(Z_n) \) and that both have the discrete topology, so \( \text{Pic}(Y^h) \cong \lim_{\longleftarrow n} \text{Pic}(Z_n) \).

Next, let \( Z \) be the disjoint union of the schemes \( r_jF_j \), where \( r_jF_j \) is the subscheme of \( Y^h \) cut out by the ideal sheaf of \( F_j \) to the power \( r_j \). Then \( f : Z \to Z \) is a finite map that is an isomorphism away from the finite set of intersection points and such that \( O_Z \subset f_*O_Z \). It follows (see [9, 0C1M, 0C1N]) that \( \text{Pic}(Z) \) is a finite sequence of extensions of \( \text{Pic}(Z) \) by quotients of \((k, +)\) or \((k, \ast)\). Lastly, \( \text{Pic}(Z) = \oplus_j \text{Pic}(r_jF_j) \), where each summand is built from finite-dimensional \( k \)-vector spaces and \( \text{Pic}(F_j) \cong \mathbb{Z} \oplus \text{Pic}^0(F_j) \). Since the \( \text{Pic}^0(F_j) \) are groups of \( k \)-points of abelian varieties over \( k \), they are all torsion.

Taken together, all of this implies that \( G \) and \( \text{Pic}(Y^h) \) have equal rank. Since \( G \to \text{Hom}(G, \mathbb{Z}) \) is injective, the first map in the excision sequence of class groups \( 0 \to G \to \text{Pic}(Y^h) \to \text{Cl}(R^h) \to 0 \) is injective also. Therefore, the quotient \( \text{Cl}(R^h) \) is a torsion group, as desired. \( \square \)

**References**


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