

The Space of 2×2 Complex Matrices

Abstract

The purpose of this article is to visualize the space of 2×2 matrices over the complex numbers both geometrically and topologically. We focus on the interaction of several main structures including the unitary group, the variety of singular matrices, and the normal matrices (those that are unitarily diagonalizable).

1 Setting

The 2×2 complex matrices correspond as usual to points in \mathbb{C}^4 , or in Euclidean space \mathbb{R}^8 . From here there are two possible ways to proceed. The first, following [1], would be to discard the zero matrix at the origin and scale the remaining points down to the sphere $S^7(\sqrt{2})$ of radius $\sqrt{2}$ in \mathbb{R}^8 . This ensures that the sum of the squares of the entries is 2, so that the unitary group lies on the sphere. The second approach, which we follow here, allows scaling by nonzero complex scalars instead of only real scalars, reducing $\mathbb{C}^4 \setminus \{0\}$ down to $\mathbb{C}\mathbb{P}^3$, which has six (real) dimensions.

We lose some of the concreteness of the spherical picture by descending into projective space. Nevertheless, the $\mathbb{C}\mathbb{P}^3$ of 2×2 matrices inherits useful structure from S^7 , since it can be regarded not only as a quotient of \mathbb{C}^4 but also as the quotient space of the fibration $S^1 \hookrightarrow S^7 \twoheadrightarrow \mathbb{C}\mathbb{P}^3$. In terms of matrices, the action of $S^1 = U(1)$ on S^7 is multiplication by a norm 1 complex scalar $e^{i\phi}$. The induced metric from the unit sphere on the quotient $\mathbb{C}\mathbb{P}^3$ is called the Fubini-Study metric, and is the canonical metric on complex projective space. By adopting it, we are free to identify distances, tangent spaces, and geodesics on $\mathbb{C}\mathbb{P}^3$ with those on S^7 , provided that those under consideration are always “horizontal,” that is, orthogonal to the fibers of the $U(1)$ action.

2 Features

2.1 Initial View

The first two important structures in $\mathbb{C}\mathbb{P}^3$ are the projective unitary group $PU(2)$ and the rank 1 singular matrices Y^1 . $PU(2)$ is a three-dimensional real submanifold appearing as a real projective 3-space in $\mathbb{C}\mathbb{P}^3$. One way to see this is to note that $SU(2) \cong S^3$ double-covers $PU(2)$. This is because $\det(e^{i\phi}U) = e^{2i\phi} \det(U)$ so every projective unitary point has two special unitary representatives.

Unlike $PU(2)$, the variety Y^1 of rank 1 matrices is a complex submanifold of $\mathbb{C}\mathbb{P}^3$. In particular, it is a $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \cong S^2 \times S^2$. Explicitly, the map $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^3$ is given by the “outer product” of complex vectors

$$(w, z) \mapsto wz^* = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \end{pmatrix} = \begin{pmatrix} w_1\bar{z}_1 & w_1\bar{z}_2 \\ w_2\bar{z}_1 & w_2\bar{z}_2 \end{pmatrix}$$

The resulting matrix always has rank 1 as long as w and z are nonzero, and forgetting about scalar multiples of the matrix corresponds precisely to forgetting scalar multiples of w and z . Therefore, the map is well-defined in projective space. Note that this map is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ over the complex numbers and cuts out a smooth quadric surface. The defining equation of this surface is just the requirement that the determinant of the 2×2 matrix is zero, i.e.

$$z_1z_4 - z_2z_3 = 0.$$

2.2 Tangents and Geodesics at the Identity

In this section, we consider geodesics in $\mathbb{C}\mathbb{P}^3$ passing through the identity I , and give interpretations of these curves in terms of matrix subspaces of $\mathbb{C}\mathbb{P}^3$. The Hermitian metric h on $\mathbb{C}\mathbb{P}^n$, the *Fubini-Study metric*, is inherited from the ‘round’ metric on S^{2n+1} pulled back from the standard metric on \mathbb{R}^{2n} . Explicitly, if $\{z_i\}_{i=1,\dots,n}$ are affine coordinates on a standard chart U of $\mathbb{C}\mathbb{P}^3$, we have a frame $\{\partial_i, \bar{\partial}_i\}_{i=1,\dots,n}$ on the complexified tangent bundle $T_{\mathbb{C}}U$. Then in this frame the components of h are given by

$$h_{ij} := h(\partial_i, \bar{\partial}_j) = \frac{(1 + \|z\|^2)\delta_{ij} - \bar{z}_i z_j}{(1 + \|z\|^2)^2}$$

With $\|\cdot\|$ the standard Hermitian metric on \mathbb{C}^n . Note that this metric h is Kahler, its imaginary part is a non-degenerate closed real $(1, 1)$ form ω , which is given in the same chart by

$$\omega_{ij} = i\partial\bar{\partial}\log(1 + \delta_{ij}z_i\bar{z}_j)$$

Rather than work directly with the explicit form of h , it is easier to remember that it descends from the metric on S^7 (we are now in the case $n = 3$). Since $\mathbb{C}\mathbb{P}^3$ is the quotient of S^7 by the isometric and properly discontinuous action of S^1 , any geodesic γ on $\mathbb{C}\mathbb{P}^3$ lifts to a geodesic $\tilde{\gamma}$ on S^7 such that $\pi(\tilde{\gamma}) = \gamma$. Indeed, π is a local isometry when $\mathbb{C}\mathbb{P}^3$ is equipped with the quotient metric. Since the verification that $\tilde{\gamma}$ is a geodesic amounts to checking that the covariant derivative of $\tilde{\gamma}$ at any $p \in S^7$ along $\tilde{\gamma}'(p)$ vanishes, a condition that is always checked locally for each p , it follows that γ is a geodesic if and only if $\tilde{\gamma}$ is. Therefore the geodesics in $\mathbb{C}\mathbb{P}^3$ passing through I are the images of great circles passing through its lift to S^7 .

Remark Although the lift of γ is a geodesic, the lift of a closed geodesic γ is not necessarily a great circle. Of course, since $\mathbb{C}\mathbb{P}^3$ has a nontrivial fundamental group $\mathbb{Z}/2$, we will see that closed geodesics on $\mathbb{C}\mathbb{P}^3$ will admit lifts to semicircles in S^7 .

To get a handle on the latter, we can determine the tangent space to S^7 at I . We can decompose $T_I\mathbb{R}^8 = T_I S^7 \oplus N_I S^7$, where $N_I S^7$ is the normal space at I . That space is of course spanned by I itself. Under the identification $\mathbb{R}^8 \cong \mathbb{C}^{4 \times 4}$, we have $T_I\mathbb{R}^8 \cong H \oplus \overline{H}$, where H and \overline{H} denote hermitian and skew hermitian matrices respectively. This is clear, since any matrix A can be rewritten $A = (A + A^*) + (A - A^*)$. Any skew hermitian matrix A can be written $A = iB$ for B hermitian, so multiplication by i interchanges H and \overline{H} . In other words, $T_I\mathbb{R}^8 = H \oplus iH$. For convenience, we can make a choice of basis for H .

$$H = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\rangle$$

Let H' be the subspace generated by the last three matrices, which we denote $\sigma_1, \sigma_2, \sigma_3$. Said in a less explicit way, H' is the complement in H of $\mathbb{R}I$. Then our remarks above demonstrate that $T_I S^7 = H' \oplus iH$.

S^1 acts effectively on S^7 , and the orbit of any given point is a great circle. In the particular, the orbit of I is the geodesic generated by iI . One way of seeing this is as follows. $\mathbb{C}\mathbb{P}^3$ can be obtained by successive quotients of \mathbb{C}^4 , first by multiplication by positive real scalars and secondly by multiplication by unit norm complex numbers $e^{i\theta} \in S^1$. The quotient by the first action yields S^7 , and the second action then has orbits $e^{i\theta} A$ for any matrix $A \in S^7$. In particular, the orbit of I is

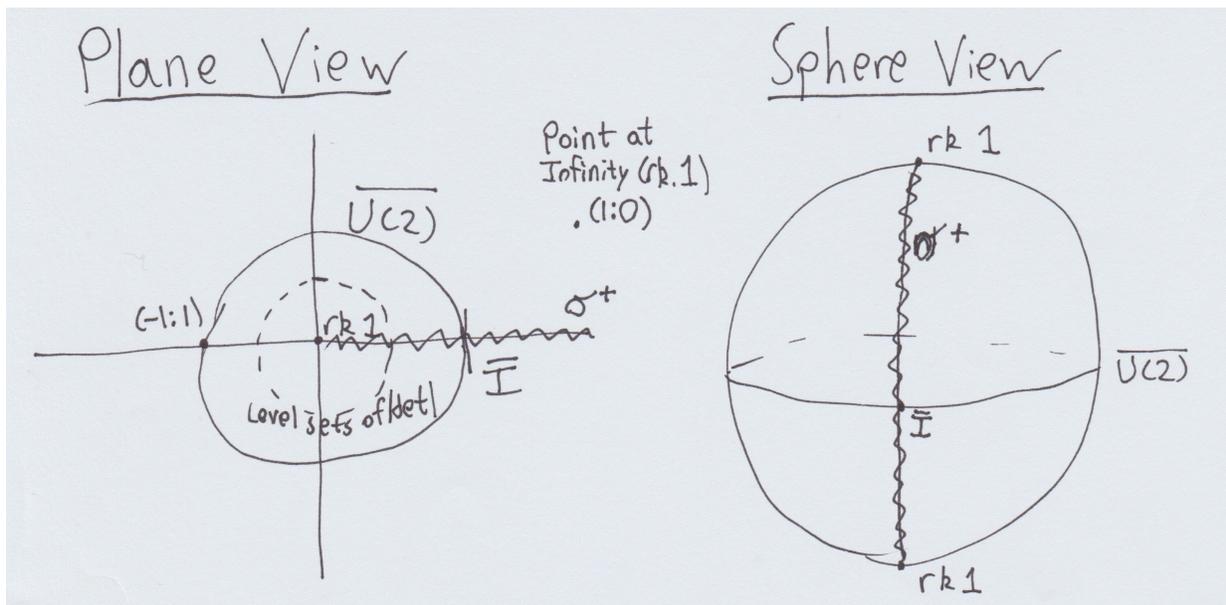
$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

Which has tangent vector iI at I . We then have that $T_I\mathbb{C}\mathbb{P}^3 = T_I S^7 / (i\mathbb{R}I)$, or in other words, $T_I\mathbb{C}\mathbb{P}^3 = H' + iH'$. Now, the tangent space to $PU(2)$ at I is the quotient $T_I U(2) / iI$. Therefore $T_I PU(2)$ is just iH' , the set of skew hermitian matrices modulo the identity. Hence we actually have a decomposition $T_I\mathbb{C}\mathbb{P}^3 = T_I PU(2) \oplus H'$. Therefore, out of the 6 tangent directions to I in $\mathbb{C}\mathbb{P}^3$, 3 lead directly through the projective unitary group $PU(2)$. The remaining directions can then of course be interpreted as the normal space to $PU(2)$.

We can find geodesics passing through I with tangent vectors $i\sigma_k$, and extend these to closed geodesics on $\mathbb{C}\mathbb{P}^3$ based at I . The purpose of doing this is to see if these curves remain in $PU(2)$. Indeed, $\gamma(\theta) = e^{i\sigma_k\theta}$ is a unitary matrix for all $\theta \in [0, \pi]$, and the class of $e^{\pi i\sigma_k}$ in $\mathbb{C}\mathbb{P}^3$ is I . Since the FS metric on $\mathbb{C}\mathbb{P}^3$ descends from S^7 , and we know that the curve γ has length π in the round metric, it follows that the ‘diameter’ of $\mathbb{C}\mathbb{P}^3$ and hence $PU(2)$ is π . Notice that γ lifts to a semicircle on S^7 , even though it is closed in $\mathbb{C}\mathbb{P}^3$. We see that γ represents the nontrivial homotopy class in $\pi_1(PU(2))$, where $PU(2) \cong \mathbb{R}\mathbb{P}^3$.

Alternatively, we can follow the directions normal to $PU(2)$. If we explicitly write down geodesics, we see that such curves will inevitably arrive at the rank 1 matrices after a distance of $\pi/4$.

2.3 Diagonal Matrices



The diagonal 2×2 complex matrices form a \mathbb{CP}^1 (or a 2-sphere) inside \mathbb{CP}^3 . The origin and point at infinity of this projective line are the two rank 1 diagonal points and the equator. Located halfway in between (again at a distance of $\frac{\pi}{4}$) is the circle of projective diagonal unitary matrices. The projective identity I is located on this equator.

Also significant is the perpendicular arc passing through I and extending to the two rank 1 points. This contains (the images of) all diagonal matrices with positive, real entries on the diagonal. Though this property is not preserved under complex scalar multiplication, all representatives of points on this line share the geometric property that their eigenvectors in the complex plane are parallel (in the real sense) and point in the same direction. Refer to this arc as L .

2.4 The Tubular Neighborhood of $PU(2)$

After removing the singular matrices Y^1 , the remaining space forms a tubular neighborhood over the real projective space $PU(2)$. The construction of this neighborhood occurs in two steps, beginning with the arc of L considered above. Consider the conjugation action of the unitary matrices $U(2)$ on L . The resulting space consists precisely of the (representatives of) the positive-definite Hermitian matrices, since by the spectral theorem, all Hermitian matrices are unitarily diagonalizable. Further, those that are positive-definite must have positive, real eigenvalues. We will argue that these form a round 3-cell D^3 in \mathbb{CP}^3 , with the identity I at the center.

Consider any non-identity point A in \mathbb{CP}^3 . We are interested in the diffeomorphism type of the orbit of A under unitary conjugation. Since conjugating by $e^{i\phi}U$ gives the same result as conjugating by U , we are welcome to restrict our attention to conjugation by matrices only in $SU(2)$ to obtain the same orbit. Since our point A is not the identity in projective space,

it has distinct eigenvalues so that the only elements of $SU(2)$ that fix A are the diagonal matrices, of the form

$$\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

The isotropy subgroup is therefore the circle of diagonal matrices in $SU(2)$. The quotient $SU(2)/U(1)$ by the action of the above diagonal matrices is one definition of the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$. Therefore, the orbits are 2-spheres. Moreover, they are round 2-spheres in $\mathbb{C}\mathbb{P}^3$, since conjugation by a unitary matrix is distance preserving. We can therefore picture the orbits of points on L as concentric spheres about I , forming in total the open 3-cell D^3 of radius $\frac{\pi}{4}$.

Next, we translate the fiber D^3 via left multiplication by an arbitrary unitary matrix U . This is another distance preserving transformation, so we obtain as a result a round 3-cell about each point of $PU(2)$. The fact that this forms a tubular neighborhood of non-intersecting fibers is essentially a statement of the uniqueness of polar decomposition. Polar decomposition states that any matrix A may be written as a product

$$A = UP,$$

where U is unitary and P is positive-definite or positive-semidefinite Hermitian. P is always unique, and in the case where A is full rank, U is also unique. As a tubular neighborhood, $\mathbb{C}\mathbb{P}^3 \setminus Y^1$ is diffeomorphic to the normal bundle of $PU(2)$ inside $\mathbb{C}\mathbb{P}^3$. In fact, the normal bundle is trivial over $PU(2)$. We saw already that

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\}$$

is a collection of traceless Hermitian tangent vectors at the identity that generate the normal space to $U(2)$ on S^7 . On the sphere, it is clear that left multiplication by a unitary matrix U takes these vectors to an orthonormal frame for the normal space to $U(2)$ at the point U . These global sections trivialize the bundle. Points on S^7 differing by only an overall phase factor $e^{i\phi}$ have their frames multiplied by the same phase factor, so each of these sections is well-defined on the quotient $\mathbb{C}\mathbb{P}^3$. The normal space to $U(2)$ in S^7 is the same as that of $PU(2)$ in $\mathbb{C}\mathbb{P}^3$ so the global sections remain orthonormal. Therefore, the normal bundle, and hence the tubular neighborhood, are trivial over $PU(2)$.

The tubular neighborhood just constructed represents geometrically the process of finding the “nearest unitary neighbor” to a given full-rank matrix A . To find the nearest neighborhood, simply move inside the tubular neighborhood fiber to which A belongs toward the center of the 3-cell. At the center of the 3-cell is the unitary matrix U that is closest to the given A .

2.5 Line Bundles Over Rank 1 Matrices

There are two naturally defined holomorphic line bundles over the space $Y^1 \cong \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ of rank 1 matrices. We define the **kernel bundle** \mathcal{K} over Y^1 to be the line bundle with fiber

over $M \in Y^1$ being the 1 dimensional complex vector space $\ker(M)$. Similarly, we define the **image bundle** \mathcal{J} over Y^1 to be the line bundle with fiber over $M \in Y^1$ given by $\text{im}(M)$. These definitions make sense, since the kernel and image of a matrix M are unaffected by multiplication by complex scalars, and therefore the line bundles \mathcal{K} and \mathcal{J} are well defined in $Y^1 \subset \mathbb{CP}^3$. To be more precise, we have given holomorphic maps $f, g : Y^1 \rightarrow \mathbb{CP}^1$ sending (the equivalence class of) a matrix M to $\ker(M) \subset \mathbb{C}^2$ and $\text{im}(M) \subset \mathbb{C}^2$ respectively. Then the line bundles \mathcal{K} and \mathcal{M} are defined as the pullback of the tautological bundle $\mathcal{O}(-1)$ over \mathbb{CP}^1 along the maps f and g respectively. Let's recall the complex isomorphism $\phi : \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow Y^1$. This map is (the restriction of) the Segre embedding, sending the subspaces generated by $v, w \in \mathbb{C}^2$ to the equivalence class of the matrix vw^* . Composition of the maps $f, g : Y^1 \rightarrow \mathbb{CP}^1$ with ϕ yields maps $\mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$

$$(f \circ \phi)(\bar{v}, \bar{w}) = \langle w \rangle$$

$$(g \circ \phi)(\bar{v}, \bar{w}) = \langle v \rangle$$

So we see that $g \circ \phi = \pi_1$ and $f \circ \phi = \pi_2$. Therefore we conclude that $\phi^*\mathcal{K} \cong \pi_2^*\mathcal{O}(-1)$ and $\phi^*\mathcal{J} \cong \pi_1^*\mathcal{O}(-1)$. In terms of $\text{Pic}(\mathbb{CP}^1 \times \mathbb{CP}^1) \cong \mathbb{Z}^2$, these are the line bundles given by $(-1, 0)$ and $(0, -1)$. Therefore we've identified \mathcal{K} and \mathcal{J} within $\text{Pic}(Y^1)$ via the isomorphism $\text{Pic}(Y^1) \rightarrow \text{Pic}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ induced by ϕ .

3 The Normal Matrices

3.1 Introduction

We now consider a third structure on \mathbb{CP}^3 , the space of normal matrices. A matrix A is said to be normal iff it commutes with its own Hermitian adjoint:

$$AA^* = A^*A.$$

Hermitian and unitary matrices, satisfying $A = A^*$ and $U^* = U^{-1}$, respectively, are special cases of normal matrices. By the spectral theorem, a matrix satisfies this condition iff it is unitarily diagonalizable, that is, there exists a unitary U such that $A = UDU^{-1} = UDU^*$. Geometrically, this means that eigenspaces of A corresponding to distinct eigenvalues are necessarily orthogonal, since the eigenvectors are just a collection of standard basis vectors under some rotation. Normal operators hold some significance in quantum physics as the broadest class of operators that could be considered “observables,” since they have a complete basis of orthonormal eigenstates. Usually operators corresponding to observables are required to be Hermitian, but situations where complex measurements are allowed, normal operators offer an appropriate generalization.

The normal matrices \mathcal{N}^4 inside \mathbb{CP}^3 form a four-dimensional “variety” in this space. However, this is not a complex projective variety because the conditions $AA^* - A^*A = 0$ involve conjugates, and yield only real polynomial conditions on the matrix entries, considered as 8-tuples in \mathbb{R}^8 .

All points in $\mathbb{C}\mathbb{P}^3$ besides the identity represent matrices with distinct eigenvalues. This set \mathcal{N}_d^4 forms a smooth 4-manifold in $\mathbb{C}\mathbb{P}^3$. This may be seen by the following double cover (analogous to a result in [2]):

$$D_d \times SU(2)/U(1) \cong D_d \times S^2 \twoheadrightarrow \mathcal{N}_d^4,$$

where D_d is the collection of diagonal matrices with distinct eigenvalues (all but the identity) and the S^2 is the orbit of each matrix under unitary conjugation, as discussed earlier. The map sends $(D, U) \mapsto UDU^{-1}$. Since normal matrices are unitarily diagonalizable, all normal matrices (except, of course, the identity) are contained in the image. Moreover, there are exactly two ways to obtain each normal matrix, given by switching the entries of the diagonal matrix D and permuting U the same way. D_d is the sphere with one point removed in $\mathbb{C}\mathbb{P}^3$, so its product with S^2 is 4-dimensional, as is the quotient \mathcal{N}^4 .

In contrast, I is a singularity of the variety \mathcal{N}^4 because the operation of conjugation degenerates there. One way to see this is to note that all six tangent vectors listed in our basis for $T_I\mathbb{C}\mathbb{P}^3$ are also tangent to \mathcal{N}^4 , since they are either “unitary directions” or “Hermitian directions,” both of which are normal. Hence \mathcal{N}^4 has too many tangents at the identity to be smooth there.

3.2 A Parametrization of the Normal Matrices

In this section, we exhibit a parametrization of the normal matrices that makes the form of the singularity at the identity more explicit. To start, we identify the form of a normal matrix in general (ignoring for now the projection into $\mathbb{C}\mathbb{P}^3$). Note first that the sum of two normal matrices A and B is normal as long as A and B commute. In particular, it is always possible to add any multiple of the identity to a normal matrix and have it remain normal. Therefore, for any normal M ,

$$M = \frac{1}{2}\text{tr}(M)I + B,$$

where B is now a traceless normal matrix. By expanding out the expression $AA^* - A^*A = 0$, one can show that the off-diagonal elements of a 2×2 normal matrix must be of the same magnitude, so a traceless normal matrix B is of the form

$$B = \begin{pmatrix} \rho e^{i\phi} & r e^{i\theta} \\ r e^{i\psi} & -\rho e^{i\phi} \end{pmatrix},$$

with the following additional condition: either $\rho = 0$ or $r = 0$ or $\theta + \psi = 2\phi \pmod{2\pi}$. In any of these cases, B is a phase multiple of a traceless Hermitian matrix, since for example by factoring out $e^{i\phi}$ the phase condition becomes $\theta = -\psi \pmod{2\pi}$.

In projective space, we can eliminate the trace of M and consider only traceless Hermitian matrices that are normalized: $M = (\cos t)I + (\sin t)e^{i\phi}A$, where t measures the distance from the identity, and

$$A = \begin{pmatrix} \rho & r e^{i\theta} \\ r e^{-i\theta} & -\rho \end{pmatrix},$$

where in addition, $r^2 + \rho^2 = 1$ (or some such normalization condition). In this form, it is clear we have four degrees of freedom, say t, ϕ, ρ , and θ . The singularity occurs when $\sin t = 0$, for this yields the identity regardless of the choice of A . Also, the normal directions from the identity are those that are phase multiples of traceless Hermitian matrices. If $\phi = 0$, we get the Hermitian matrices themselves, and if $\phi = \frac{\pi}{2}$, we obtain the unitary matrices. The intersection of these two components at I is “bridged” by varying ϕ between these two values.

3.3 Normal and Nilpotent Matrices of Rank 1

One particular subset of \mathcal{N}^4 of interest is the set of normal matrices of rank 1. Geometrically, these are all those linear transformations whose image and kernel spaces are mutually orthogonal. On $\mathbb{C}\mathbb{P}^3$, these have a particularly simple form, as they are all unitary conjugates of the rank 1 diagonal matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

since we need only rotate the kernel and image to find every such matrix. Equivalently, they come from outer products of the form vv^* for a complex vector v . Hence normal rank 1 matrices form an S^2 inside rank 1. Further, this S^2 is the boundary of the 3-cell D^3 of our tubular neighborhood about the identity I because the diagonal matrix above is the endpoint of the segment L . Hence every normal rank 1 point is at the boundary of this entirely normal 3-cell.

At the other extreme of rank 1 matrices lay the nilpotent transformations, i.e. those whose kernel and image coincide. Analogously, these are the unitary conjugates of the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

or outer products of the form $v^\perp v^*$. If these are considered to be vectors in $\mathbb{C}\mathbb{P}^1$, then v^\perp is uniquely defined. As in the first case, these form an S^2 inside of rank 1.

A natural question arises: do these spheres correspond to non-trivial members of $H_2(Y^1) = H_2(S^2 \times S^2) = \mathbb{Z} \oplus \mathbb{Z}$? Using the kernel and image line bundles constructed above, it is clear that the pullbacks of these bundles along the inclusion of each S^2 are nontrivial. This is because the image and kernel correspond to the first and second members of the outer product uv^* , respectively, and each of these runs all around $\mathbb{C}\mathbb{P}^1$ as we pass around the normal rank 1 or nilpotent matrices. If either sphere were homotopically trivial, the pullback bundle over it must be isomorphic to a pullback bundle over a point, contradicting the above observation.

However, these two spheres are homologous to one another via the deformation

$$U \begin{pmatrix} \cos \theta & \sin \theta \\ 0 & 0 \end{pmatrix} U^{-1},$$

of the points corresponding to conjugation by U . The above matrix remains in rank 1 throughout the deformation.

3.4 Normal Matrices and Rank 1 Boundaries in the Tubular Neighborhood

To visualize the normal matrices in more detail, it is useful to see how they interact with the tubular neighborhood of full-rank matrices about the unitary matrices. The fiber D^3 about the identity I consists entirely of normal matrices, since every point in D^3 is positive-definite Hermitian. However, this is not true for any other fiber. Algebraically, we have the following characterization of when a matrix is normal in terms of its polar decomposition:

Lemma 1. *Let U be a unitary matrix and A a Hermitian matrix. Then the product UA is normal iff the matrices U and A commute.*

Proof. UA is normal iff $UA(UA)^* = (UA)^*UA$ holds. This equality may be equivalently written $UAA^*U^* = A^*U^*UA$. Utilizing the given properties of U and A , this simplifies to $UA^2U^{-1} = A^2$, which is precisely the statement that U and A^2 commute. This is the same as saying that U and A^2 preserve one another's eigenspaces. A has the same eigenspaces as A^2 since it is positive-definite Hermitian, so U and A^2 commute iff U and A do. ■

Products of this type are exactly what appears in the construction of our tubular neighborhood allowing an explicit characterization of normal matrices inside each fiber.

Proposition 2. *The normal matrices inside UD^3 (the translate of D^3 via left multiplication by U) for a non-identity point U are precisely those $N = UV D_1 V^{-1}$ (with $V \in U(2)$, $D_1 \in L$) such that V unitarily diagonalizes U .*

Proof. If N is of the required form, then $U = VD'V^{-1}$ for some diagonal unitary D' . Setting $A = VD_1V^{-1}$, we get that $UA = VD'V^{-1}VD_1V^{-1} = VD'D_1^2V^{-1} = VD_1D'V^{-1} = AU$, so U and A commute. By the lemma above, $N = UA$ is normal. Conversely, let $N = UA = UV D_1 V^{-1}$ be a normal member of the fiber. The lemma shows that U and A commute so that $UV D_1 V^{-1} = VD_1V^{-1}U$, which is equivalent to the statement $(V^{-1}U^{-1}V)D_1(V^{-1}U^{-1}V)^{-1} = D_1$. $D_1 \in L$, so either D_1 is the identity (in which case any V may be chosen to make U have the required form) or D_1 has distinct eigenvalues. Conjugating a diagonal matrix with distinct eigenvalues and getting the same matrix may only occur if $V^{-1}U^{-1}V$ is a diagonal matrix D' , so $U = VD'V^{-1}$, providing the required diagonalization. The V that diagonalizes U is unique as a coset in $SU(2)/U(1)$ so $UVLV^{-1}$ is the entire normal set. ■

Geometrically, this characterization means that in any 3-cell fiber of a unitary matrix besides the identity, there is a single diameter through the 3-cell's center containing normal matrices. This diameter is a rotated and translated copy of the diameter L of diagonal matrices whose ratio of eigenvalues is real that sits inside D^3 . The endpoints of the diameter L are the rank 1 outer products $e_1e_1^*$ and $e_2e_2^*$ of standard basis vectors, so if $U = VD'V^{-1}$ is the unitary matrix in question, we can identify the endpoints of the normal line $UVLV^{-1}$ in UD^3 :

$$UV(e_1e_1^*)V^{-1} = VD'V^{-1}Ve_1(e_1^*V^*) = (VD'e_1)(Ve_1)^*.$$

Since $D'e_1$ is just a scalar multiple of e_1 (which changes nothing in projective space), the endpoints of the normal line are

$$(Ve_1)(Ve_1)^*$$

and

$$(Ve_2)(Ve_2)^*.$$

It is clear from how these are written that they are also points of the form vv^* that lie at the boundary of D^3 ! In fact, this had to be so, as the 2-sphere boundary of D^3 already accounts for all normal rank 1 matrices. What we have shown is that the boundary of the fiber about U intersects the boundary of the fiber at I in exactly two antipodal points, determined only by a matrix V diagonalizing U . Forgetting about normality for a moment, we can easily adapt this reasoning to the intersection of any two boundaries in the tubular neighborhood. This yields the following result:

Theorem 3. *For distinct unitary points U and V in projective space, the 2-sphere boundaries of the 3-cells UD^3 and VD^3 intersect exactly in two antipodal points.*

References

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