Rigidity for Lorentzian metrics having the same length of null-geodesics

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Zoom Inverse Problems Seminar, May 26, 2022
Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$. Consider a Lorentzian metric in the cylinder $\mathbb{R} \times \Omega$

$$
\sum_{j,k=0}^{n} g_{jk}(x) dx_j dx_k, \quad (1)
$$

where $x_0 \in \mathbb{R}, x \in \mathbb{R}^n, x_0$ is the time variable, $x = (x_1, \ldots, x_n) \in \Omega$. We assume that metric tensor $[g_{jk}(x)]_{j,k=0}^{n}$ is independent of $x_0$, and the signature of the matrix $[g_{jk}]_{j,k=0}^{n}$ is $(+, -, \ldots, -)$.

Let $[g^{jk}(x)]_{j,k=0}^{n} = ([g_{jk}]_{j,k=0}^{n})^{-1}$ be the inverse metric tensor. Denote by $H(x, \xi_0, \xi)$ the Hamiltonian, i.e.

$$
H(x, \xi_0, \xi) = \frac{1}{2} \sum_{j,k=0}^{n} g^{jk}(x) \xi_j \xi_k. \quad (2)
$$
The Hamiltonian system corresponding to the Hamiltonian $H(x, \xi_0, \xi)$ is

$$\frac{dx}{dt} = \frac{\partial H}{\partial \xi}, \quad \frac{d\xi}{dt} = -\frac{\partial H}{\partial x},$$

(3)

$$\frac{dx_0}{dt} = \frac{\partial H}{\partial \xi_0}, \quad \frac{d\xi_0}{dt} = -\frac{\partial H}{\partial x_0} = 0,$$

(4)

since $H$ is independent of $x_0$. Here $x = (x_1, ..., x_n)$, $\xi = (\xi_1, ..., \xi_n)$ and since $\frac{d\xi_0}{dt} = 0$, $\xi_0$ is independent of $t$, $\xi_0(t) = \eta_0$. 
The initial conditions for (3), (4) are

\[ x(0) = y, \quad \xi(0) = \eta, \quad \xi_0(t) = \xi_0(0) = \eta_0, \quad x_0(0) = 0. \quad (5) \]

We have

\[
\frac{dH(x(t), \xi_0(t), \xi(t))}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial \xi} \frac{d\xi}{dt} + \frac{\partial H}{\partial \xi_0} \frac{d\xi_0}{dt}. \quad (6)
\]

Substituting (3), (4) into (6) we get

\[
\frac{dH(x(t), \xi_0(t), \xi(t))}{dt} = \frac{\partial H}{\partial x} \frac{\partial H}{\partial \xi} - \frac{\partial H}{\partial \xi} \frac{\partial H}{\partial x} = 0,
\]

since \( \frac{d\xi_0}{dt} = 0 \). Thus \( H(x(t), \xi_0(t), \xi(t)) = H(y, \eta_0, \eta), \forall t. \)
The solution \((x_0(t), x(t), \xi_0(t), \xi(t))\) of (3), (4) such that \(H(y, \eta_0, \eta) = 0\), is called null-bicharacteristic.

The restriction of \((x_0(t), x(t), \xi_0(t), \xi(t))\) to the \(x\)-space is called null-geodesics.

We also call the restriction of \((x_0(t), x(t), \xi_0(t), \xi(t))\) to \((x_0, x)\) space the time-space null-geodesics.

The null-bicharacteristic or null-geodesic are called forward if \(\frac{dx_0}{dt} > 0\). It follows from (4) that the null-bicharacteristic is forward if \(\frac{\partial H}{\partial \xi_0} > 0\).

Note that when \(\hat{x} = (x_0, x)\) and if \(s(\hat{x})\) is a diffeomorphism of \(\mathbb{R} \times \mathbb{R}^n\), \(s(\hat{x}) = \hat{x}\) for \(|\hat{x}|\) large, then the inverse metric tensor has the form

\[
\left(\frac{\partial s}{\partial \hat{x}}\right)^T g(\hat{x}) \frac{\partial s(\hat{x})}{\partial \hat{x}}
\]

in new coordinates.
We can use the change of coordinates to simplify the inverse metric tensor. In particular, we can make the change of variables such that
\[
\hat{g}^{00} = 1, \quad \hat{g}^{0j} = \hat{g}^{j0} = 0, \quad 1 \leq j \leq n. \tag{8}
\]
Then the Hamiltonian system, after simplifying notations, will have the form
\[
\frac{dx_0}{dt} = 1, \quad \frac{dx}{dt} = \hat{g} \xi, \quad \frac{d\xi}{dt} = -\frac{1}{2} \frac{\partial \hat{g}}{\partial x} \xi \cdot \xi, \tag{9}
\]
where \( \hat{g} = [\hat{g}^{jk}]_{j,k=1}^n \).
Denote by $L(q, T, y, \eta)$ the integral over the time-space geodesics $x_0 = x_0(t, y, \eta), x = x(t, y, \eta)$ starting at $(0, y, \eta)$ at $t = 0$ and ending when $t = T$ at the point $x_0(T), x_T = x(T, y, \eta)$. We have that the length of this time-space geodesic is

$$L(q, T, y, \eta) = \int_0^T \sqrt{\left(\frac{dx_0}{dt}\right)^2 + |x_t|^2} dt, \quad |x_t|^2 = \sum_{k=1}^n \left|\frac{dx_k}{dt}\right|^2. \quad (10)$$

Let $y_0 \in \partial \Omega$ and $\gamma_{y_0}(\varepsilon)$ be an $\varepsilon$-neighborhood of $y_0$ in $\partial \Omega$. Denote by $\Gamma_{y_0}$ the union of all time-space null-geodesics in $\mathbb{R} \times \Omega$ starting on $\gamma_{y_0}(\varepsilon)$. Consider all null-bicharacteristics $x = x(t, y, \eta), \xi = \xi(t, y, \eta)$ where $t \geq 0, y \in \gamma_{y_0}(\varepsilon), \eta$ is fixed. Let $T = T(y, \eta)$ be such that $x_T = x(T, y, \eta) \in \partial \Omega$. We shall call such $T(y, \eta)$ maximal.

Consider the time-space forward null-geodesic $x_0 = x_0(t, y, \eta), x = x(t, y, \eta)$ that enters the cylinder $\mathbb{R} \times \Omega$ at point $(0, y)$ at the time $t = 0$, stays in $\mathbb{R} \times \Omega$ for $0 < t < T$ and reaches again $\mathbb{R} \times \partial \Omega$ at point $x_0(T), x_T = x(T, y, \eta)$ at the time $t = T$. 
The main result of the talk is the following theorem:

**Theorem**

Let $\sum_{j,k=0}^{n} q_{pjk}(x) dx_j dx_k$ be two metrics, $p = 1, 2$, and let $(x_0(q_p), x(q_p)))$ be the time-space null-geodesics with the same initial conditions $x_0 = 0, x = y$ and $\xi = \eta$. Suppose $\Gamma_{y_0}$ is a union of all time-space null-geodesics in $\mathbb{R} \times \Omega$ starting on $\gamma_{y_0}(\varepsilon)$. Let $T'(y, \eta)$ be maximal in $q_1$-metric for any $y \in \gamma_{y_0}(\varepsilon)$. Then if $L(q_2, T'(y, \eta), y, \eta) = L(q_1, T'(y, \eta), y, \eta)$ for all $y \in \gamma_{y_0}(\varepsilon)$ and if $q_2$ and $q_1$ are sufficiently close in $\Gamma_{y_0}$ then $q_2 = q_1$ in $\Gamma_{y_0}$.

Note that since $q_1$ and $q_2$ are independent of $x_0$, $q_1 = q_2$ on $\Gamma_{y_0}$ is equivalent to $q_1 = q_2$ on the projection of $\Gamma_{y_0}$ on $\Omega$, where the projection consists of all null-geodesics in $\Omega$ starting at $y \in \gamma_{y_0}(\varepsilon)$. 
There are many works on the rigidity of the Riemannian metric, i.e. the rigidity with respect to the distance $d(x, y)$, where $x, y$ are boundary points and $d(x, y)$ is the length of the geodesics connecting $x$ and $y$ ([P.Stefanov, G.Uhlmann, 1998], [M.Lassas, V.Sharifutdinov, G.Uhlmann, 2003], [G.Eskin, 1998], [P.Stefanov, G.Uhlmann, A.Vasy, 2016], [P.Stefanov, G.Uhlmann, A.Vasy, H.Zhou, 2019]). In recent paper [G.Uhlmann, Yang Yang, H.Zhou, 2020] the boundary rigidity problem for some class of Lorentzian metrics is proven. We also study the case of Lorentzian metric. The main novelty of our talk is that we consider the null-geodesics. We also use here some ideas of [G.Eskin, 1998].
Estimates for the null-geodesics

Let

\[ H_p = \frac{1}{2} \sum_{j,k=0}^{n} q_p^{jk}(x) \xi_j \xi_k = \frac{1}{2} q_p^{00} \xi_0^2 + \sum_{j=1}^{n} q_p^{0j} \xi_j \xi_0 + \frac{1}{2} q_p' \xi \cdot \xi, \]

\[ p = 1, 2, \] be two Hamiltonians. Denote

\[ q = q_1 + \tau(q_2 - q_1), \quad 0 \leq \tau \leq 1. \] \hfill (11)

Let \( x_\tau, \xi_\tau \) be solution of the Hamiltonian system

\[
\frac{dx_\tau}{dt} = q'(x_\tau(t))\xi_\tau(t),
\]

\[
\frac{d\xi_\tau}{dt} = -\frac{1}{2} \frac{\partial q'(x_\tau(t))}{\partial x} \xi_\tau(t) \cdot \xi_\tau(t),
\]

\[ x_\tau(t, y, \eta) \bigg|_{t=0} = y, \quad \xi_\tau(t, y, \eta) \bigg|_{t=0} = \eta, \]

and let

\[
\frac{dx_0^\tau}{dt} = q_\tau^{00}(x)\xi_0 + \sum_{j=1}^{n} q_\tau^{0j} \xi_j, \quad x_0^\tau(0) = 0.
\]
We shall study the behavior of \((x_\tau(t, y, \eta), \xi_\tau(t, y, \eta))\) and \(x^\tau_0\) with respect to \(\tau\). Differentiating (12) in \(\tau\) we get

\[
\frac{d}{dt} \frac{d}{d\tau} x_\tau = \left( q'_2(x_\tau(t)) - q'_1(x_\tau(t)) \right) \xi_\tau(t) + \frac{\partial q'_1(x_\tau(t))}{\partial x} \frac{dx_\tau}{d\tau} \xi_\tau(t)
\]

\[
+ q'_1(x_\tau(t)) \frac{d\xi_\tau}{d\tau} + O(\tau(q'_2 - q'_1)^2) \xi_\tau(t),
\]

\[
\frac{d}{dt} \frac{d}{d\tau} \xi_\tau = - \frac{1}{2} \left( \frac{\partial^2 q'_1(x_\tau) \, dx_\tau}{\partial x^2} \| \xi_\tau \| \cdot \xi_\tau - \frac{\partial}{\partial x} q'_1(x_\tau(t)) \xi_\tau \frac{d\xi_\tau}{d\tau}
\]

\[
- \frac{1}{2} \left( \left( \frac{\partial q'_2}{\partial x} - \frac{\partial q'_1}{\partial x} \right) \xi_\tau \right) \cdot \xi_\tau + \left( O\left( \tau \left( \frac{\partial q'_2}{\partial x} - \frac{\partial q'_1}{\partial x} \right)^2 \right) \right) \xi_\tau \right) \cdot \xi_\tau.
\]
Thus
\[
\frac{d}{dt} \left( \frac{dx_\tau}{d\tau} \right) = Q \left( \frac{dx_\tau}{d\tau} \right) + F
\]  

(14)

where

\[
Q = \left[ \begin{array}{c}
\frac{\partial q'_1}{\partial x} \xi_\tau \\
-\frac{1}{2} \left( \frac{\partial^2 q'_1}{\partial x^2} \xi_\tau \right) \cdot \xi_\tau - \frac{q'_1}{\partial x} \\
- \frac{1}{2} \left( \frac{\partial q'_2}{\partial x} - \frac{\partial q'_1}{\partial x} \right) \xi_\tau + O(\tau (q'_2 - q'_1)^2) \xi_\tau \\
\end{array} \right],
\]

(15)

\[
F = \left[ \begin{array}{c}
(q'_2 - q'_1) \xi_\tau + O(\tau (q'_2 - q'_1)^2) \xi_\tau \\
- \frac{1}{2} \left( \frac{\partial q'_2}{\partial x} - \frac{\partial q'_1}{\partial x} \right) \xi_\tau + O(\tau (\frac{\partial q'_2}{\partial x} - \frac{\partial q'_1}{\partial x})^2) \xi_\tau \cdot \xi_\tau \\
\end{array} \right].
\]

Note that
\[
\frac{dx_\tau}{d\tau} \bigg|_{t=0} = 0, \quad \frac{d\xi_\tau}{d\tau} \bigg|_{t=0} = 0
\]

(16)

since \( x_\tau \big|_{t=0} = y \), \( \xi_\tau \big|_{t=0} = \eta \).
We shall write the solution of the Cauchy problem (14), (16) in the form
\[
\begin{bmatrix}
\frac{dx_\tau}{dt} \\
\frac{d\xi_\tau}{dt} \\
\frac{dd_{\tau}}{dt}
\end{bmatrix} = R(t)F,
\tag{17}
\]
where \( R(t) \) is the solution operator of the equation (14).

If \( N \) is large enough then the following estimate for the solution of the Cauchy problem (14), (16) holds:
\[
\max_{0 \leq t \leq T} e^{-Nt} \left( \left| \frac{dx_\tau}{dt} \right| + \left| \frac{d\xi_\tau}{dt} \right| \right) \leq C_N \int_0^T e^{-Nt} |F(x_\tau(t))| \, dt.
\]

Since \( q_2' - q_1' \) is bounded, \( \tau(q_2' - q_1')^2 \leq C|q_2' - q_1'|. \) Thus \( |F| \leq C|q_2' - q_1'| + C\left| \frac{\partial}{\partial x}(q_2' - q_1') \right|. \) Therefore
\[
\max_{0 \leq t \leq T} e^{-Nt} \left( \left| \frac{dx_\tau}{dt} \right| + \left| \frac{d\xi_\tau}{dt} \right| \right) \leq C_N \sup_\tau \int_0^T e^{-Nt} |(q_2' - q_1')(x_\tau(t))| \, dt
\]
\[
+ C_N \sup_\tau \int_0^T e^{-Nt} \left| \frac{\partial}{\partial x}(q_2' - q_1')(x_\tau(t)) \right| \, dt \tag{18}
\]
To prove the estimate (18) we take the inner product of (14) with 
\[ e^{-2Nt} \left( \frac{dx_{\tau}}{d\tau} \right) \] and integrate it in \( t \) from 0 to \( t_0 \), where \( |x_{\tau}(t_0)| = \max_{0 \leq t \leq T} |x_{\tau}(t)| \). Note that for any \( \varphi \)

\[
\int_0^{t_0} \frac{d\varphi}{dt} e^{-2Nt} \varphi dt = \frac{1}{2} \int_0^{t_0} e^{-2Nt} \frac{d}{dt} \varphi^2 dt \quad (19)
\]

\[
= \frac{1}{2} \varphi^2(t_0) e^{-2Nt_0} + N \int_0^{t_0} e^{-2Nt} \varphi^2 dt
\]

Also we use in the proof of (18) that \( N \) is large such that

\[
\left( (NI - Q) \left( \frac{dx_{\tau}}{d\tau} \right), \left( \frac{dx_{\tau}}{d\tau} \right) \right) > 0, \quad (20)
\]

where \( I \) is the identity operator.
In addition to (18) we shall estimate also \( \frac{d^2x}{d\tau^2}, \frac{d^2\xi}{d\tau^2} \):

Differentiating (14) in \( \tau \) we get

\[
\frac{d}{dt} \left[ \begin{array}{c} \frac{d^2x}{d\tau^2} \\ \frac{d^2\xi}{d\tau^2} \end{array} \right] = Q \left[ \begin{array}{c} \frac{d^2x}{d\tau^2} \\ \frac{d^2\xi}{d\tau^2} \end{array} \right] + \frac{dQ}{d\tau} \left[ \frac{dx}{d\tau} \right] + \frac{dF}{d\tau}. \tag{21} \]

Therefore as in (13) we get

\[
\left[ \begin{array}{c} \frac{d^2x}{d\tau^2} \\ \frac{d^2\xi}{d\tau^2} \end{array} \right] = R(t) \left( \frac{dQ}{d\tau} \left[ \frac{dx}{d\tau} \right] + \frac{dF}{d\tau} \right), \tag{22} \]

where \( R(t) \) is the same as in (17).

Note that (cf. (15))

\[
\frac{dQ}{d\tau} = O \left( \left| \frac{dx}{d\tau} \right| + \left| \frac{d\xi}{d\tau} \right| \right) \tag{23} \]

and

\[
\frac{dF}{d\tau} = \left[ -\left( \left( \frac{\partial q_2'}{\partial x} - \frac{\partial q_1'}{\partial x} \right) + O \left( \left( \frac{\partial q_2'}{\partial x} - \frac{\partial q_1'}{\partial x} \right)^2 \right) \right) \frac{d\xi}{d\tau} \right] \frac{d^2\xi}{d\tau} \]
Since \( \frac{dF}{dT} \) can be estimated as in (18) we get, again using (18):

\[
\max_{0 \leq t \leq T} e^{-2Nt} \left( \left| \frac{d^2 x_T}{dT^2} \right| + \left| \frac{d^2 \xi_T}{dT^2} \right| \right) \leq C_N \int_0^T e^{-2Nt} \left( \left| \frac{dx_T}{dT} \right|^2 + \left| \frac{d\xi_T}{dT} \right|^2 \right) dt
\]

\[+ C_N \left( \int_0^T e^{-Nt} \left| (q'_2 - q'_1)(x_T) \right| dt \right)^2 + C_N \left( \int_0^T e^{-Nt} \left| \frac{\partial}{\partial x} ((q'_2 - q'_1)(x_T)) \right| dt \right)^2. \]

(24)

Note that \( \int_0^T e^{-Nt} |\varphi(t)| dt > \int_0^T e^{-2Nt} |\varphi(t)| dt \geq e^{-NT} \int_0^T e^{-Nt} |\varphi)| dt. \)
Now we shall study the behavior in $\tau$ of

$$\frac{dx_0^\tau}{dt} = q^{00}_\tau(x_\tau(t))\xi_0 + \sum_{j=1}^{n} q^{0j}_\tau(x_\tau(t))\xi_j(t), \quad x_0^\tau(0) = 0. \quad (25)$$

Note that

$$q^{0j}_\tau = q_1^{0j} + \tau(q_2^{0j} - q_1^{0j}), \quad 0 \leq j \leq n. \quad (26)$$

Therefore

$$\frac{d}{dt} \frac{d}{d\tau} x_0^\tau = \sum_{j=0}^{n} ((q_2^{0j} - q_1^{0j})\xi_j + O(\tau(q_2^{0j} - q_1^{0j})^2)\xi_j) + \sum_{j=0}^{n} \frac{\partial q_1^{0j}}{\partial x} \frac{dx^\tau}{d\tau} \xi_j$$

$$+ \sum_{j=1}^{n} q_1^{0j}(x_\tau) \frac{d\xi_j}{d\tau}. \quad (27)$$

Note that $\xi_0 = \eta_0$. 

Thus

\[
\frac{d}{d\tau} x_0^{\tau} = \sum_{j=0}^{n} \int_0^t \left( (q_2^{0j} - q_1^{0j}) + O(\tau (q_2^{0j} - q_1^{0j})^2) \right) \xi_j \, dt' + O\left( \left| \frac{dx_\tau}{d\tau} \right| + \left| \frac{d\xi_\tau}{d\tau} \right| \right).
\]  

(28)

Denote

\[
\| q_2^0 - q_1^0 \|_0 = \sum_{j=0}^{n} \sup_{\tau} \int_0^T |(q_2^{0j} - q_1^{0j})(x_\tau(t))| \, dt.
\]  

(29)

Then

\[
\left| \frac{dx_0^\tau}{d\tau} \right| \leq C \| q_2^0 - q_1^0 \|_0 + \max_{0 \leq t \leq T} \left( \left| \frac{dx_\tau}{d\tau} \right| + \left| \frac{d\xi_\tau}{d\tau} \right| \right).
\]  

(30)
**Lengths of null-geodesics**

The length of time-space null-geodesics $x_0 = x_0(t, y, \eta)$, $x = x(t, y, \eta)$, $0 \leq t \leq T$, where $x(T, y, \eta) \in \partial\Omega$, is equal to

$$L(q, T, y, \eta) = \int_0^T \sqrt{\left(\frac{dx_0}{dt}\right)^2 + \left(\frac{dx(q')}{dt}\right)^2} dt$$

(31)

where $x(q')(t)$ is the solution of Hamiltonian system

$$\frac{dx(q', t, y, \eta)}{dt} = q'(x(t))\xi(t), \quad \frac{d\xi(q', t, y, \eta)}{dt} = -\frac{1}{2} \left(\frac{\partial q'(x)}{\partial x}\xi(t)\right) \cdot \xi(t),$$

$$x\bigg|_{t=0} = y, \quad \xi\bigg|_{t=0} = \eta, \quad 0 \leq t \leq T, \quad \left|\frac{dx}{dt}\right| = \sqrt{\sum_{k=1}^n \left(\frac{dx_k}{dt}\right)^2}.$$

Remind that $q'(x, t) = [g^{jk}(x, t)]_{j,k=1}^n$. Also

$$\frac{dx_0}{dt} = \sum_{j=1}^n q^{0j}(x(t))\xi_j(t) + q^{00}(x(t))\eta_0, \quad x_0(0) = 0.$$
Let \( q = q_1 + \tau(q_2 - q_1), 0 \leq \tau \leq 1 \). We have

\[
\frac{\partial L(q, T, y, \eta)}{d\tau} = \int_0^T \left( \left( \frac{dx_0}{dt} \right)^2 + \left| \frac{dx}{dt} \right|^2 \right)^{-\frac{1}{2}} \left( \frac{dx}{dt}, \frac{d}{d\tau} \frac{dx}{dt} \right) dt
\]

\[
+ \int_0^T \left( \left( \frac{dx_0}{dt} \right)^2 + \left| \frac{dx}{dt} \right|^2 \right)^{-\frac{1}{2}} \left( \frac{dx_0}{dt}, \frac{d}{d\tau} \frac{dx_0}{dt} \right) dt. \tag{32}
\]

Note that

\[
\frac{d}{d\tau} \frac{dx}{dt} = \frac{d}{d\tau} (q' \xi) = \left( (q'_2 - q'_1) + O(\tau(q'_2 - q'_1)^2) \right) \xi
\]

\[
+ \frac{\partial q'_1(x(t))}{\partial x} \frac{dx}{d\tau} \xi + q'_1(x) \frac{d\xi}{d\tau}.
\]
Therefore
\[
\frac{\partial L}{\partial \tau} \bigg|_{\tau=0} = \int_0^T \left( \left( \frac{dx_0}{dt} \right)^2 + \left| \frac{dx}{dt} \right|^2 \right)^{-\frac{1}{2}} \left( q'_1(x(t))\xi(t), (q'_2-q'_1)\xi + \frac{\partial q'_1}{\partial x} \xi \frac{dx}{d\tau} + \frac{dx_0}{dt} \frac{d dx_0}{d\tau} \bigg|_{t=0} \right) dt.
\]

Thus by the Taylor’s formula
\[
L(q_2, T, y, \eta) - L(q_1, T, y, \eta) = \tau \frac{\partial L(q, T, y, \eta)}{\partial \tau} \bigg|_{\tau=0} + G_2, \quad (34)
\]
where
\[
G_2 = \frac{1}{2} \frac{\partial^2}{\partial \tau^2} L(q_1 + \theta(q_2 - q_1), T, y, \eta)(q_2 - q_1)^2, \quad 0 < \theta < 1. \quad (35)
\]

Note that
\[
\tau \frac{\partial L(q_1, y, T, \eta)}{\partial \tau} \bigg|_{\tau=0} = l(\tau(q_2 - q_1)) \quad (36)
\]
is the linear part of \( L(q_2) - L(q_1) \).
Let

$$\|q_2 - q_1\| = \sup_0^T e^{-2Nt} |(q_2' - q_1')(x_\tau(t))| dt$$

$$+ \sup_0^T e^{-2Nt} \left| \frac{\partial}{\partial x} (q_2' - q_1')(x_\tau(t)) \right| dt + \|q_2^0 - q_1^0\|_0,$$  \hspace{1cm} (37)

where $\|q_2^0 - q_1^0\|_0$ is the same as in (29). Since $l(q_2 - q_1)$ is nonzero linear functional bounded in the norm (37) and since the kernel of $l(q_2 - q_1)$ has the co-dimension one, we have

$$|l(q_2 - q_1)| \geq l_0 \|q_2 - q_1\|.$$  \hspace{1cm} (38)
Now estimate $G_2$. Differentiating $L(q_1 + \tau(q_2 - q_1))$ twice in $\tau$ we get

$$G_2 = \int_0^T \left( \left( \frac{dx_0}{dt} \right)^2 + \left| \frac{dx}{dt} \right|^2 \right)^{-\frac{1}{2}} \left[ \left( \frac{dx}{dt}, \frac{d^2 dx}{d\tau^2 dt} \right) + \left( \frac{d}{d\tau} \frac{dx}{dt}, \frac{d}{d\tau} \frac{dx}{dt} \right) \right]$$

$$+ \left( \frac{dx_0}{dt}, \frac{d^2}{d\tau^2} \frac{dx_0}{dt} \right) + \left( \frac{d}{d\tau} \frac{dx_0}{dt}, \frac{d}{d\tau} \frac{dx_0}{dt} \right) \right] dt$$

$$+ C \int_0^T \left( \left( \frac{dx_0}{dt} \right)^2 + \left| \frac{dx}{dt} \right|^2 \right)^{-\frac{3}{2}} \left( \left( \frac{dx}{dt}, \frac{d}{d\tau} \frac{dx}{dt} \right)^2 + \left( \frac{d}{d\tau} \frac{dx_0}{dt}, \frac{d}{d\tau} \frac{dx_0}{dt} \right)^2 \right) dt.$$  

(39)

Estimating the right hand sides in (39) as in (18), (24), (30) we get

$$|G_2| \leq C \int_0^T \left[ \left( \left| \frac{d^2 x}{d\tau^2} \right| + \left| \frac{d^2 \xi}{d\tau^2} \right| + \left| \frac{d^2 x_0}{d\tau^2} \right| \right) + C \left( \left| \frac{dx}{d\tau} \right|^2 + \left| \frac{d\xi}{d\tau} \right|^2 + \left( \frac{dx_0}{d\tau} \right)^2 \right) \right] dt.$$  

(40)
Using (24) and (30) we obtain

$$|G_2| \leq C_N\|q_2 - q_1\|^2.$$  \hfill (41)

Since

$$L(q_2, T, y, \eta) - L(q_1, T, y, \eta) = I(q_2 - q_1) + G_2,$$  \hfill (42)

we have, using (40) and (41),

$$l_0\|q_2 - q_1\| \leq |L(q_2, T, y, \eta) - L(q_1, T, y, \eta)| + C_N\|q_2 - q_1\|^2.$$  \hfill (43)

Therefore

$$l_0\|q_2 - q_1\| \left(1 - \frac{C_N}{l_0}\|q_2 - q_1\|\right) \leq |L(q_2, T, y, \eta) - L(q_1, T, y, \eta)|.$$  \hfill (44)

Assuming that $\|q_2 - q_1\| < \frac{l_0}{2C_N}$ we obtain

$$2l_0\|q_2 - q_1\| \leq |L(q_2, T, y, \eta) - L(q_1, T, y, \eta)|.$$  \hfill (45)
Thus $L(q_2, T, y, \eta) = L(q_1, T, y, \eta)$ implies that $\|q_2 - q_1\| = 0$ for $y \in \gamma_0(\varepsilon)$. It follows from (37) that $\|q_2 - q_1\| = 0$ is equivalent to $\|q'_2 - q'_1\| = 0$ and $\|q^0_2 - q^0_1\| = 0$ for $y \in \gamma_0(\varepsilon)$. In particular, $q'_1(x_0(t)) = q'_2(x_0(t))$, $q^0_1(x_0(t)) = q^0_2(x_0(t))$, where $x_0(t)$ is the null-geodesic in $q'_1$ metric starting at $y_0$ when $t = 0$. Thus $q_1(x_0(t)) = q_2(x_0(t))$.

Let $x'_0(t)$ be a null-geodesic in $\Gamma_{y_0}$ starting at $y' \in \gamma_{y_0}(\varepsilon)$ when $t = 0$ and reaching $\mathbb{R} \times \partial \Omega$ at $t = T'(y', \eta)$. Let $x'_1(t)$ be a null-geodesic in $q_2$ metric having the same initial conditions $(y', \eta)$ as $x'_0(t)$. Then repeating the same proof as above we get that if $L(q_2, T', y', \eta) = L(q_1, T', y', \eta)$ then $q_2(x'_0(t)) = q_1(x'_0(t))$. Since $x'_0(t)$ is arbitrary in $\Gamma_{y_0}$ we obtain that $q_2(x) = q_1(x)$ in $\Gamma_{y_0}$. This completes the proof of Theorem 1.
Now we shall prove a global variant of Theorem 1. Let \( z_0 \) be an arbitrary point of \( \Omega \). Consider the metric \( q_1 \) as in Theorem 1. Let \( z_0(t) \) be the forward null-geodesic starting at \( z_0 \) for \( t = 0 \). It will reach the boundary \( \partial \Omega \) at some point \( z_1 \) when \( t = T_1 > 0 \). If we continue \( z_0(t) \) backward from \( z_0 \) starting at \( t = 0 \) we will reach \( \partial \Omega \) at some point \( z_2 \) at the time \(-T_2\). Thus we will get forward null-geodesic \( z_0(t) \) in \( \Omega \) starting at \( t = -T_2 \) on \( \partial \Omega \) and reaching \( \partial \Omega \) again at \( t = T_1 \).

Let \( \tilde{z}_0(t) = (x_0(t), z_0(t)) \) be the corresponding time-space null-geodesic. Construct a "rectangle" \( \Gamma(z_0)(t) \) as \( \Gamma_{y_0} \) in the proof of Theorem 1. Denote by \( \tilde{z}_0(t) \) the time-space null-geodesic in \( q_2 \) metric having the same data at \( t = -T_2, t = T_1 \) as \( z_0(t) \). Applying the proof of Theorem 1 we get that \( q_2 = q_1 \) in \( \Gamma(z_0) \). Repeating this proof for any "rectangle" of the form \( \Gamma(z_0) \), we get \( q_2 = q_1 \) on a dense set of \( \Omega \). Since \( q_2 \) and \( q_1 \) are continuous we have \( q_2 = q_1 \) in \( \Omega \). Thus the following corollary holds:
Corollary

If $L(q_2, T(y, \eta), y, \eta) = L(q_1, T(y, \eta), y, \eta)$ for all $y \in \partial \Omega$, and if the norm $\|q_2 - q_1\|$ over any $q_1$-null-geodesics on $[0, T(y, \eta)]$ is small enough then $q_2 = q_1$ in $\Omega$. 
THANK YOU VERY MUCH FOR YOUR ATTENTION!