Rigidity for Lorentzian metrics having the same length of null-geodesics

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Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Consider a Lorentzian metric in the cylinder $\mathbb{R} \times \Omega$

$$\sum_{j,k=0}^{n} g_{jk}(x) dx_j dx_k, \tag{1}$$

where $x_0 \in \mathbb{R}, x \in \mathbb{R}^n, x_0$ is the time variable, $x = (x_1, ..., x_n) \in \Omega$. We assume that metric tensor $[g_{jk}(x)]_{j,k=0}^{n}$ is independent of x_0, and the signature of the matrix $[g_{jk}]_{j,k=0}^{n}$ is $(+, -, ..., -)$.
Let $[g^{jk}(x)]_{j,k=0}^{n} = \left([g_{jk}]_{j,k=0}^{n}\right)^{-1}$ be the inverse metric tensor. Denote by $H(x, \xi_0, \xi)$ the Hamiltonian, i.e.

$$H(x, \xi_0, \xi) = \frac{1}{2} \sum_{j,k=0}^{n} g^{jk}(x) \xi_j \xi_k. \tag{2}$$

The Hamiltonian system corresponding to the Hamiltonian $H(x, \xi_0, \xi)$ is

$$\frac{dx}{dt} = \frac{\partial H}{\partial \xi}, \quad \frac{d\xi}{dt} = -\frac{\partial H}{\partial x}, \tag{3}$$

$$\frac{dx_0}{dt} = \frac{\partial H}{\partial \xi_0}, \quad \frac{d\xi_0}{dt} = -\frac{\partial H}{\partial x_0} = 0, \tag{4}$$

since H is independent of x_0. Here $x = (x_1, ..., x_n)$, $\xi = (\xi_1, ..., \xi_n)$ and since $\frac{d\xi_0}{dt} = 0, \xi_0$ is independent of $t, \xi_0(t) = \eta_0$.

The initial conditions for (3), (4) are

$$x(0) = y,\ \xi(0) = \eta,\ \xi_0(t) = \xi_0(0) = \eta_0,\ x_0(0) = 0. \qquad (5)$$

We have

$$\frac{dH(x(t), \xi_0(t), \xi(t))}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial H}{\partial \xi_0}\frac{d\xi_0}{dt}. \qquad (6)$$

Substituting (3), (4) into (6) we get

$$\frac{dH(x(t), \xi_0(t), \xi(t))}{dt} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial \xi} - \frac{\partial H}{\partial \xi}\frac{\partial H}{\partial x} = 0,$$

since $\frac{d\xi_0}{dt} = 0$. Thus $H(x(t), \xi_0(t), \xi(t)) = H(y, \eta_0, \eta), \forall t$.

The solution $(x_0(t), x(t), \xi_0(t), \xi(t))$ of (3), (4) such that $H(y, \eta_0, \eta) = 0$, is called null-bicharacteristic.
The restriction of $(x_0(t), x(t)), \xi_0(t), \xi(t))$ to the x-space is called null-geodesics.
We also call the restriction of $(x_0(t), x(t), \xi_0(t), \xi(t))$ to (x_0, x) space the time-space null-geodesics.
The null-bicharacteristic or null-geodesic are called forward if $\frac{dx_0}{dt} > 0$. It follows from (4) that the null-bicharacteristic is forward if $\frac{\partial H}{\partial \xi_0} > 0$.
Note that when $\hat{x} = (x_0, x)$ and if $s(\hat{x})$ is a diffeomorphism of $\mathbb{R} \times \mathbb{R}^n$, $s(\hat{x}) = \hat{x}$ for $|\hat{x}|$ large, then the inverse metric tensor has the form

$$\left(\frac{\partial s}{\partial \hat{x}}\right)^T g(\hat{x}) \frac{\partial s(\hat{x})}{\partial \hat{x}} \tag{7}$$

in new coordinates.

We can use the change of coordinates to simplify the inverse metric tensor. In particular, we can make the change of variables such that

$$\hat{g}^{00} = 1, \hat{g}^{0j} = \hat{g}^{j0} = 0,\ 1 \leq j \leq n. \tag{8}$$

Then the Hamiltonian system, after simplifying notations, will have the form

$$\frac{dx_0}{dt} = 1,\ \frac{dx}{dt} = \hat{g}\xi,\ \frac{d\xi}{dt} = -\frac{1}{2}\frac{\partial \hat{g}}{\partial x}\xi \cdot \xi, \tag{9}$$

where $\hat{g} = [\hat{g}^{jk}]_{j,k=1}^{n}$.

Denote by $L(q, T, y, \eta)$ the integral over the time-space geodesics $x_0 = x_0(t, y, \eta), x = x(t, y, \eta)$ starting at $(0, y, \eta)$ at $t = 0$ and ending when $t = T$ at the point $x_0(T), x_T = x(T, y, \eta)$. We have that the length of this time-space geodesic is

$$L(q, T, y, \eta) = \int_0^T \sqrt{\Big(\frac{dx_0}{dt}\Big)^2 + |x_t|^2} dt, \quad |x_t|^2 = \sum_{k=1}^{n} \Big|\frac{dx_k}{dt}\Big|^2. \quad (10)$$

Let $y_0 \in \partial\Omega$ and $\gamma_{y_0}(\varepsilon)$ be an ε-neighborhood of y_0 in $\partial\Omega$. Denote by Γ_{y_0} the union of all time-space null-geodesics in $\mathbb{R} \times \Omega$ starting on $\gamma_{y_0}(\varepsilon)$. Consider all null-bicharacteristics $x = x(t, y, \eta), \xi = \xi(t, y, \eta)$ where $t \geq 0, y \in \gamma_{y_0}(\varepsilon), \eta$ is fixed. Let $T = T(y, \eta)$ be such that $x_T = x(T, y, \eta) \in \partial\Omega$. We shall call such $T(y, \eta)$ maximal.

Consider the time-space forward null-geodesic $x_0 = x_0(t, y, \eta), x = x(t, y, \eta)$ that enters the cylinder $\mathbb{R} \times \Omega$ at point $(0, y)$ at the time $t = 0$, stays in $\mathbb{R} \times \Omega$ for $0 < t < T$ and reaches again $\mathbb{R} \times \partial\Omega$ at point $x_0(T), x_T = x(T, y, \eta)$ at the time $t = T$.

The main result of the talk is the following theorem:

Theorem

Let $\sum_{j,k=0}^{n} q_{pjk}(x)dx_j dx_k$ be two metrics, $p = 1, 2,$ and let $(x_0(q_p), x(q_p)))$ be the time-space null-geodesics with the same initial conditions $x_0 = 0, x = y$ and $\xi = \eta$. Suppose Γ_{y_0} Is a union of all time-space null-geodesics in $\mathbb{R} \times \Omega$ starting on $\gamma_{y_0}(\varepsilon)$. Let $T'(y, \eta)$ be maximal in q_1-metric for any $y \in \gamma_{y_0}(\varepsilon)$. Then if $L(q_2, T'(y, \eta), y, \eta) = L(q_1, T'(y, \eta), y, \eta)$ for all $y \in \gamma_{y_0}(\varepsilon)$ and if q_2 and q_1 are sufficiently close in Γ_{y_0} then $q_2 = q_1$ in Γ_{y_0}.

Note that since q_1 and q_2 are independent of x_0, $q_1 = q_2$ on Γ_{y_0} is equivalent to $q_1 = q_2$ on the projection of Γ_{y_0} on Ω, where the projection consists of all null-geodesics in Ω starting at $y \in \gamma_{y_0}(\varepsilon)$.

There are many works on the rigidity of the Riemannian metric, i.e. the rigidity with respect to the distance $d(x, y)$, where x, y are boundary points and $d(x, y)$ is the length of the geodesics connecting x and y ([P.Stefanov, G.Uhlmann, 1998], [M.Lassas, V.Sharifutdinov, G.Uhlmann, 2003], [G.Eskin, 1998], [P.Stefanov, G.Uhlmann, A.Vasy, 2016], [P.Stefanov, G.Uhlmann, A.Vasy, H.Zhou, 2019]). In recent paper [G.Uhlmann, Yang Yang, H.Zhou, 2020] the boundary rigidity problem for some class of Lorentzian metrics is proven. We also study the case of Lorentzian metric. The main novelty of our talk is that we consider the null-geodesics. We also use here some ideas of [G.Eskin, 1998].

Estimates for the null-geodesics

Let

$$H_p = \frac{1}{2}\sum_{j,k=0}^{n} q_p^{jk}(x)\xi_j\xi_k = \frac{1}{2}q_p^{00}\xi_0^2 + \sum_{j=1}^{n} q_p^{0j}\xi_j\xi_0 + \frac{1}{2}q_p'\xi\cdot\xi,$$

$p = 1, 2$, be two Hamiltonians. Denote

$$q = q_1 + \tau(q_2 - q_1),\ 0 \le \tau \le 1. \tag{11}$$

Let x_τ, ξ_τ be solution of the Hamiltonian system

$$\begin{aligned} \frac{dx_\tau}{dt} &= q'(x_\tau(t))\xi_\tau(t), \\ \frac{d\xi_\tau}{dt} &= -\frac{1}{2}\frac{\partial q'(x_\tau(t))}{\partial x}\xi_\tau(t)\cdot\xi_\tau(t), \\ x_\tau(t,y,\eta)\Big|_{t=0} &= y, \quad \xi_\tau(t,y,\eta)\Big|_{t=0} = \eta, \end{aligned} \tag{12}$$

and let

$$\frac{dx_0^\tau}{dt} = q_\tau^{00}(x)\xi_0 + \sum_{j=1}^{n} q_\tau^{0j}\xi_j, \quad x_0^\tau(0) = 0.$$

We shall study the behavior of $(x_\tau(t,y,\eta),\xi_\tau(t,y,\eta)$ and x_0^τ with respect to τ. Differentiating (12) in τ we get

$$\frac{d}{dt}\frac{d}{d\tau}x_\tau = \Big(q_2'(x_\tau(t)) - q_1'(x_\tau(t))\Big)\xi_\tau(t) + \frac{\partial q_1'(x_\tau(t))}{\partial x}\frac{dx_\tau}{d\tau}\xi_\tau(t) \tag{13}$$

$$+q_1'(x_\tau(t))\frac{d\xi_\tau}{d\tau} + O(\tau(q_2'-q_1')^2)\xi_\tau(t),$$

$$\frac{d}{dt}\frac{d}{d\tau}\xi_\tau = -\frac{1}{2}\Big(\frac{\partial^2 q_1'(x_\tau)}{\partial x^2}\frac{dx_\tau}{d\tau}\xi_\tau\Big)\cdot\xi_\tau - \frac{\partial}{\partial x}q_1'(x_\tau(t))\xi_\tau\frac{d\xi_\tau}{d\tau}$$
$$-\frac{1}{2}\Big(\Big(\frac{\partial q_2'}{\partial x} - \frac{\partial q_1'}{\partial x}\Big)\xi_\tau\Big)\cdot\xi_\tau + \Big(O\Big(\tau\Big(\frac{\partial q_2'}{\partial x} - \frac{\partial q_1'}{\partial x}\Big)^2\Big)\xi_\tau\Big)\cdot\xi_\tau.$$

Thus
$$\frac{d}{dt}\begin{pmatrix}\frac{dx_\tau}{d\tau}\\ \frac{d\xi_\tau}{d\tau}\end{pmatrix} = Q\begin{pmatrix}\frac{dx_\tau}{d\tau}\\ \frac{d\xi_\tau}{d\tau}\end{pmatrix} + F \tag{14}$$
where
$$Q = \begin{bmatrix} \frac{\partial q_1'}{\partial x}\xi_\tau & q_1' \\ -\frac{1}{2}\big(\frac{\partial^2 q_1'}{\partial x^2}\xi_\tau\big)\cdot\xi_\tau & -\frac{\partial q_1'(x_\tau)\xi_\tau}{\partial x}\end{bmatrix}, \tag{15}$$
$$F = \begin{bmatrix} (q_2' - q_1')\xi_\tau + O(\tau(q_2' - q_1')^2)\xi_\tau \\ -\frac{1}{2}\big(\big(\frac{\partial q_2'}{\partial x} - \frac{\partial q_1'}{\partial x}\big)\xi_\tau\big)\xi_\tau + \big(O\big(\tau\big(\frac{\partial q_2'}{\partial x} - \frac{\partial q_1'}{\partial x}\big)^2\big)\xi_\tau\big)\cdot\xi_\tau\end{bmatrix}.$$
Note that
$$\frac{dx_\tau}{d\tau}\Big|_{t=0} = 0, \quad \frac{d\xi_\tau}{d\tau}\Big|_{t=0} = 0 \tag{16}$$
since $x_\tau\big|_{t=0} = y, \quad \xi_\tau\big|_{t=0} = \eta$.

We shall write the solution of the Cauchy problem (14), (16) in the form

$$\begin{bmatrix} \frac{dx_\tau}{d\tau} \\ \frac{d\xi_\tau}{d\tau} \end{bmatrix} = R(t)F, \tag{17}$$

where $R(t)$ is the solution operator of the equation (14).
If N is large enough then the following estimate for the solution of the Cauchy problem (14), (16) holds:

$$\max_{0\le t\le T} e^{-Nt}\Big(\Big|\frac{dx_\tau}{d\tau}\Big| + \Big|\frac{d\xi_\tau}{d\tau}\Big|\Big) \le C_N \int\limits_0^T e^{-Nt}|F(x_\tau(t))|dt.$$

Since $q_2' - q_1'$ is bounded, $\tau(q_2' - q_1')^2 \le C|q_2' - q_1'|$. Thus $|F| \le C|q_2' - q_1'| + C\big|\frac{\partial}{\partial x}(q_2' - q_1')\big|$. Therefore

$$\max_{0\le t\le T} e^{-Nt}\Big(\Big|\frac{dx_\tau}{d\tau}\Big| + \Big|\frac{d\xi_\tau}{d\tau}\Big|\Big) \le C_N \sup_\tau \int\limits_0^T e^{-Nt}\big|(q_2'-q_1')(x_\tau(t))\big|dt$$

$$+ C_N \sup_\tau \int\limits_0^T e^{-Nt}\Big|\frac{\partial(q_2' - q_1')}{\partial x}(x_\tau(t))\Big|dt \tag{18}$$

To prove the estimate (18) we take the inner product of (14) with $e^{-2Nt}\begin{pmatrix}\frac{dx_\tau}{d\tau}\\ \frac{d\xi_\tau}{d\tau}\end{pmatrix}$ and integrate it in t from 0 to t_0, where $|x_\tau(t_0)| = \max\limits_{0\le t\le T}|x_\tau(t)|$. Note that for any φ

$$\int\limits_0^{t_0}\frac{d\varphi}{dt}e^{-2Nt}\varphi dt = \frac{1}{2}\int\limits_0^{t_0}e^{-2Nt}\frac{d}{dt}\varphi^2 dt \tag{19}$$

$$= \frac{1}{2}\varphi^2(t_0)e^{-2Nt_0} + N\int\limits_0^{t_0}e^{-2Nt}\varphi^2 dt$$

Also we use in the proof of (18) that N is large such that

$$\left((NI-Q)\begin{pmatrix}\frac{dx_\tau}{d\tau}\\ \frac{d\xi_\tau}{d\tau}\end{pmatrix},\begin{pmatrix}\frac{dx_\tau}{d\tau}\\ \frac{d\xi_\tau}{d\tau}\end{pmatrix}\right) > 0, \tag{20}$$

where I is the identity operator. $\square$

In addition to (18) we shall estimate also $\frac{d^2 x_\tau}{d\tau^2}, \frac{d^2 \xi_\tau}{d\tau^2}$:
Differentiating (14) in τ we get

$$\frac{d}{dt}\begin{bmatrix}\frac{d^2 x_\tau}{d\tau^2}\\ \frac{d^2 \xi_\tau}{d\tau^2}\end{bmatrix} = Q\begin{bmatrix}\frac{d^2 x_\tau}{d\tau^2}\\ \frac{d^2 \xi_\tau}{d\tau^2}\end{bmatrix} + \frac{dQ}{d\tau}\begin{bmatrix}\frac{dx_\tau}{d\tau}\\ \frac{d\xi_\tau}{d\tau}\end{bmatrix} + \frac{dF}{d\tau}. \tag{21}$$

Therefore as in (13) we get

$$\begin{bmatrix}\frac{d^2 x_\tau}{d\tau^2}\\ \frac{d^2 \xi_\tau}{d\tau^2}\end{bmatrix} = R(t)\Big(\frac{dQ}{d\tau}\begin{bmatrix}\frac{dx_\tau}{d\tau}\\ \frac{d\xi_\tau}{d\tau}\end{bmatrix} + \frac{dF}{d\tau}\Big), \tag{22}$$

where $R(t)$ is the same as in (17).
Note that (cf. (15))

$$\frac{dQ}{d\tau} = O\Big(\Big|\frac{dx_\tau}{d\tau}\Big| + \Big|\frac{d\xi_\tau}{d\tau}\Big|\Big) \tag{23}$$

and

$$\frac{dF}{d\tau} = \begin{bmatrix}\Big((q_2' - q_1') + O\big((q_2' - q_1')^2\big)\Big)\frac{d\xi_\tau}{d\tau}\\ -\Big(\Big(\frac{\partial q_2'}{\partial x} - \frac{\partial q_1'}{\partial x}\Big) + O\Big(\frac{\partial q_2'}{\partial x} - \frac{\partial q_1'}{\partial x}\Big)^2\Big)\xi_\tau\frac{d\xi_\tau}{d\tau}\end{bmatrix}$$

Since $\frac{dF}{d\tau}$ can be estimated as in (18) we get, again using (18):

$$\max_{0\le t\le T} e^{-2Nt}\Big(\Big|\frac{d^2x_\tau}{d\tau^2}\Big|+\Big|\frac{d^2\xi_\tau}{d\tau^2}\Big|\Big) \le C_N\int\limits_0^T e^{-2Nt}\Big(\Big|\frac{dx_\tau}{d\tau}\Big|^2+\Big|\frac{d\xi_\tau}{d\tau}\Big|^2\Big)dt$$
$$+C_N\Big(\int\limits_0^T e^{-Nt}|(q_2'-q_1')(x_\tau)|dt\Big)^2+C_N\Big(\int\limits_0^T e^{-Nt}\Big|\frac{\partial}{\partial x}((q_2'-q_1')(x_\tau))\Big|dt\Big)^2. \tag{24}$$

Note that $\int\limits_0^T e^{-Nt}|\varphi(t)|dt > \int\limits_0^T e^{-2Nt}|\varphi(t)|dt \ge e^{-NT}\int\limits_0^T e^{-Nt}|\varphi)|dt$.

Now we shall study the behavior in τ of

$$\frac{dx_0^\tau}{dt} = q_\tau^{00}(x_\tau(t))\xi_0 + \sum_{j=1}^{n} q^{0j}(x_\tau(t))\xi_j(t), \quad x_0^\tau(0) = 0. \tag{25}$$

Note that

$$q_\tau^{0j} = q_1^{0j} + \tau(q_2^{0j} - q_1^{0j}), \quad 0 \le j \le n. \tag{26}$$

Therefore

$$\frac{d}{dt}\frac{d}{d\tau}x_0^\tau = \sum_{j=0}^{n}\left((q_2^{0j}-q_1^{0j})\xi_j+O\left(\tau(q_2^{0j}-q_1^{0j})^2\right)\xi_j\right)+\sum_{j=0}^{n}\frac{\partial q_1^{0j}}{\partial x}\frac{dx^\tau}{d\tau}\xi_j + \sum_{j=1}^{n} q_1^{0j}(x_\tau)\frac{d\xi_j}{d\tau}. \tag{27}$$

Note that $\xi_0 = \eta_0$.

Thus

$$\frac{d}{d\tau}x_0^\tau = \sum_{j=0}^{n} \int\limits_0^t \Big((q_2^{0j} - q_1^{0j}) + O\big(\tau (q_2^{0j} - q_1^{0j})^2\big)\Big)\xi_j dt' + O\Big(\Big|\frac{dx_\tau}{d\tau}\Big| + \Big|\frac{d\xi_\tau}{d\tau}\Big|\Big). \tag{28}$$

Denote

$$\|q_2^0 - q_1^0\|_0 = \sum_{j=0}^{n} \sup_\tau \int\limits_0^T |(q_2^{0j} - q_1^{0j})(x_\tau(t))| dt. \tag{29}$$

Then

$$\Big|\frac{dx_0^\tau}{d\tau}\Big| \le C\|q_2^0 - q_1^0\|_0 + \max_{0\le t\le T}\Big(\Big|\frac{dx_\tau}{d\tau}\Big| + \Big|\frac{d\xi_\tau}{d\tau}\Big|\Big). \tag{30}$$

Lengths of null-geodesics

The length of time-space null-geodesics $x_0 = x_0(t, y, \eta),\ x = x(t, y, \eta)$, $0 \le t \le T$, where $x(T, y, \eta) \in \partial\Omega$, is equal to

$$L(q, T, y, \eta) = \int_0^T \sqrt{\Big(\frac{dx_0}{dt}\Big)^2 + \Big(\frac{dx(q')}{dt}\Big)^2}\,dt \qquad (31)$$

where $x(q')(t)$ is the solution of Hamiltonian system

$$\frac{dx(q', t, y, \eta)}{dt} = q'(x(t))\xi(t), \ \ \frac{d\xi(q', t, y, \eta)}{dt} = -\frac{1}{2}\Big(\frac{\partial q'(x)}{\partial x}\xi(t)\Big)\cdot\xi(t),$$

$$x\Big|_{t=0} = y,\ \xi\Big|_{t=0} = \eta,\ 0 \le t \le T, \quad \Big|\frac{dx}{dt}\Big| = \sqrt{\sum_{k=1}^{n}\big(\frac{dx_k}{dt}\big)^2}.$$

Remind that $q'(x, t) = [g^{jk}(x, t)]_{j,k=1}^{n}$. Also

$$\frac{dx_0}{dt} = \sum_{j=1}^{n} q^{0j}(x(t))\xi_j(t) + q^{00}(x(t))\eta_0,\ x_0(0) = 0.$$

Let $q = q_1 + \tau(q_2 - q_1), 0 \le \tau \le 1$. We have

$$\frac{\partial L(q,T,y,\eta)}{d\tau} = \int_0^T \left(\left(\frac{dx_0}{dt}\right)^2 + \left|\frac{dx}{dt}\right|^2\right)^{-\frac{1}{2}} \left(\frac{dx}{dt}, \frac{d}{d\tau}\frac{dx}{dt}\right) dt$$
$$+ \int_0^T \left(\left(\frac{dx_0}{dt}\right)^2 + \left|\frac{dx}{dt}\right|^2\right)^{-\frac{1}{2}} \left(\frac{dx_0}{dt}, \frac{d}{d\tau}\frac{dx_0}{dt}\right) dt. \quad (32)$$

Note that

$$\frac{d}{d\tau}\frac{dx}{dt} = \frac{d}{d\tau}(q'\xi) = \Big((q_2' - q_1') + O\big(\tau(q_2' - q_1')^2\big)\Big)\xi$$
$$+ \frac{\partial q_1'(x(t))}{\partial x}\frac{dx}{d\tau}\xi + q_1'(x)\frac{d\xi}{d\tau}.$$

Therefore

$$\frac{\partial L}{\partial \tau}\Big|_{\tau=0} = \int\limits_0^T \left(\left(\frac{dx_0}{dt}\right)^2 + \left|\frac{dx}{dt}\right|^2\right)^{-\frac{1}{2}} \Big(q_1'(x(t))\xi(t), (q_2' - q_1')\xi + \frac{\partial q_1'}{\partial x}\xi\frac{dx}{d\tau} +$$

$$+ \left(\frac{dx_0}{dt}, \frac{d}{d\tau}\frac{dx_0}{dt}\Big|_{t=0}\right)\Big)dt. \quad (33)$$

Thus by the Taylor's formula

$$L(q_2, T, y, \eta) - L(q_1, T, y, \eta) = \tau\frac{\partial L(q, T, y, \eta)}{\partial \tau}\Big|_{\tau=0} + G_2, \quad (34)$$

where

$$G_2 = \frac{1}{2}\frac{\partial^2}{\partial \tau^2} L(q_1 + \theta(q_2 - q_1), T, y, \eta)(q_2 - q_1)^2,\ 0 < \theta < 1. \quad (35)$$

Note that

$$\tau\frac{\partial L(q_1, y, T, \eta)}{\partial \tau}\Big|_{\tau=0} = l(\tau(q_2 - q_1)) \quad (36)$$

is the linear part of $L(q_2) - L(q_1)$.

Let

$$\|q_2 - q_1\| = \sup_\tau \int_0^T e^{-2Nt} |(q_2' - q_1')(x_\tau(t)| dt$$

$$+ \sup_\tau \int_0^T e^{-2Nt} \left| \frac{\partial}{\partial x}(q_2' - q_1')(x_\tau(t)) \right| dt + \|q_2^0 - q_1^0\|_0, \quad (37)$$

where $\|q_2^0 - q_1^0\|_0$ is the same as in (29). Since $l(q_2 - q_1)$ is nonzero linear functional bounded in the norm (37) and since the kernel of $l(q_2 - q_1)$ has the co-dimension one, we have

$$|l(q_2 - q_1)| \geq l_0 \|q_2 - q_1\|. \quad (38)$$

Now estimate G_2. Differentiating $L(q_1+\tau(q_2-q_1))$ twice in τ we get

$$G_2=\int_0^T\Big(\Big(\frac{dx_0}{dt}\Big)^2+\Big|\frac{dx}{dt}\Big|^2\Big)^{-\frac{1}{2}}\Big[\Big(\frac{dx}{dt},\frac{d^2}{d\tau^2}\frac{dx}{dt}\Big)+\Big(\frac{d}{d\tau}\frac{dx}{dt},\frac{d}{d\tau}\frac{dx}{dt}\Big) \tag{39}$$

$$+\Big(\frac{dx_0}{dt},\frac{d^2}{d\tau^2}\frac{dx_0}{dt}\Big)+\Big(\frac{d}{d\tau}\frac{dx_0}{dt},\frac{d}{d\tau}\frac{dx_0}{dt}\Big)\Big]dt$$

$$+C\int_0^T\Big(\Big(\frac{dx_0}{dt}\Big)^2+\Big|\frac{dx}{dt}\Big|^2\Big)^{-\frac{3}{2}}\Big(\Big(\frac{dx}{dt},\frac{d}{d\tau}\frac{dx}{dt}\Big)^2+\Big(\frac{dx_0}{dt},\frac{d}{d\tau}\frac{dx_0}{dt}\Big)^2\Big)dt.$$

Estimating the right hand sides in (39) as in (18), (24), (30) we get

$$|G_2|\le C\int_0^T\Big[\Big(\Big|\frac{d^2x}{d\tau^2}\Big|+\Big|\frac{d^2\xi}{d\tau^2}\Big|+\Big|\frac{d^2x_0}{d\tau^2}\Big|\Big)+C\Big(\Big|\frac{dx}{d\tau}\Big|^2+\Big|\frac{d\xi}{d\tau}\Big|^2+\Big(\frac{dx_0}{d\tau}\Big)^2\Big)\Big]dt. \tag{40}$$

Using (24) and (30) we obtain

$$|G_2| \leq C_N \|q_2 - q_1\|^2. \tag{41}$$

Since

$$L(q_2, T, y, \eta) - L(q_1, T, y, \eta) = l(q_2 - q_1) + G_2, \tag{42}$$

we have, using (40) and (41),

$$l_0\|q_2 - q_1\| \leq |L(q_2, T, y, \eta) - L(q_1, T, y, \eta)| + C_N\|q_2 - q_1\|^2. \tag{43}$$

, Therefore

$$l_0\|q_2 - q_1\|\Big(1 - \frac{C_N}{l_0}\|q_2 - q_1\|\Big) \leq |L(q_2, T, y, \eta) - L(q_1, T, y, \eta)|. \tag{44}$$

Assuming that $\|q_2 - q_1\| < \frac{l_0}{2C_N}$ we obtain

$$2l_0\|q_2 - q_1\| \leq |L(q_2, T, y, \eta) - L(q_1, T, y, \eta)|. \tag{45}$$

Thus $L(q_2, T, y, \eta) = L(q_1, T, y, \eta)$ implies that $\|q_2 - q_1\| = 0$ for $y \in \gamma_0(\varepsilon)$. It follows from (37) that $\|q_2 - q_1\| = 0$ is equivalent to $\|q_2' - q_1'\| = 0$ and $\|q_2^0 - q_1^0\| = 0$ for $y \in \gamma_0(\varepsilon)$. In particular, $q_1'(x_0(t)) = q_2'(x_0(t), q_1^0(x_0(t)) = q_2^0(x_0(t))$, where $x_0(t)$ is the null-geodesic in q_1' metric starting at y_0 when $t = 0$. Thus $q_1(x_0(t)) = q_2(x_0(t))$.

Let $x_0'(t)$ be a null-geodesic in Γ_{y_0} starting at $y' \in \gamma_{y_0}(\varepsilon)$ when $t = 0$ and reaching $\mathbb{R} \times \partial\Omega$ at $t = T'(y', \eta)$. Let $x_1'(t)$ be a null-geodesic in q_2 metric having the same initial conditions (y', η) as $x_0'(t)$. Then repeating the same proof as above we get that if $L(q_2, T', y', \eta) = L(q_1, T', y', \eta)$ then $q_2(x_0'(t)) = q_1(x_0'(t))$. Since $x_0'(t)$ is arbitrary in Γ_{y_0} we obtain that $q_2(x) = q_1(x)$ in Γ_{y_0}.

This completes the proof of Theorem 1.

Now we shall prove a global variant of Theorem 1. Let z_0 be an arbitrary point of Ω. Consider the metric q_1 as in Theorem 1. Let $z_0(t)$ be the forward null-geodesic starting at z_0 for $t = 0$. It will reach the boundary $\partial\Omega$ at some point z_1 when $t = T_1 > 0$. If we continue $z_0(t)$ backward from z_0 starting at $t = 0$ we will reach $\partial\Omega$ at some point z_2 at the time $-T_2$. Thus we will get forward null-geodesic $z_0(t)$ in Ω starting at $t = -T_2$ on $\partial\Omega$ and reaching $\partial\Omega$ again at $t = T_1$.

Let $\hat{z}_0(t) = (x_0(t), z_0(t))$ be the corresponding time-space null-geodesic. Construct a "rectangle" $\Gamma(z_0)(t)$ as Γ_{y_0} in the proof of Theorem 1. Denote by $\tilde{z}_0(t)$ the time-space null-geodesic in q_2 metric having the same data at $t = -T_2, t = T_1$ as $z_0(t)$. Applying the proof of Theorem 1 we get that $q_2 = q_1$ in $\Gamma(z_0)$. Repeating this proof for any "rectangle" of the form $\Gamma(z_0)$, we get $q_2 = q_1$ on a dense set of Ω. Since q_2 and q_1 are continuous we have $q_2 = q_1$ in Ω. Thus the following corollary holds:

Corollary

If $L(q_2, T(y,\eta), y, \eta) = L(q_1, T(y,\eta), y, \eta)$ *for all* $y \in \partial\Omega$*, and if the norm* $\|q_2 - q_1\|$ *over any* q_1*-null-geodesics on* $[0, T(y,\eta)]$ *is small enough then* $q_2 = q_1$ *in* Ω*.*

THANK YOU VERY MUCH FOR YOUR ATTENTION!