Remarks on the determination of the Lorentzian metric by the lengths of geodesics

Gregory Eskin
University of California Los Angeles

Zoom Inverse Problems Seminar, November 3, 2022

Introduction

Consider a Lorentzian metric

$$\sum_{j,k=0}^{n} g_{jk}(x)dx_j dx_k \tag{1}$$

in $\mathbb{R} \times \mathbb{R}^n$, where $x_0 \in \mathbb{R}$ is the time variable, $x = (x_1, ..., x_n)$ are the space variables, the metric $[g_{jk}]_{j,k=0}^n$ is independent of the time variable x_0 and the signature of the matrix $[g_{jk}]_{j,k=0}^n$ is $(+, -, ..., -)$. Let

$$H(x, \xi_0, \xi) = \frac{1}{2} \sum_{j,k=0}^{n} g^{jk}(x)\xi_j \xi_k \tag{2}$$

be the corresponding Hamiltonian where $[g^{jk}(x)]_{j,k=0}^n$ is the inverse to $[g_{jk}(x)]_{j,k=0}^n$.

Consider the Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial H}{\partial \xi}, \ \frac{d\xi}{dt} = -\frac{\partial H}{\partial x}, \tag{3}$$

$$x = (x_1, ..., x_n), \ \xi = (\xi_1, ..., \xi_n),$$

$$\frac{dx_0}{dt} = \frac{\partial H}{\partial \xi_0}, \ \frac{d\xi_0}{dt} = -\frac{\partial H}{\partial x_0} = 0, \tag{4}$$

with initial conditions

$$x_0(0) = 0, \ x(0) = y, \ \xi(0) = \eta, \ \xi_0(0) = \eta_0. \tag{5}$$

We have

$$\frac{dH(x(t), \xi_0(t), \xi(t))}{dt} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial \xi} + \frac{\partial H}{\partial \xi}\Big(-\frac{\partial H}{\partial x}\Big) + \frac{\partial H}{\partial \xi_0}\frac{d\xi_0}{dt} = 0. \tag{6}$$

Therefore

$$H(x(t), \xi_0(t), \xi(t)) = H(y, \eta_0, \eta), \quad \forall t. \tag{7}$$

The solution $x_0 = x_0(t), x = x(t), \xi_0 = \xi_0(t), \xi = \xi(t)$ of (3), (4) is called a bicharacteristic and its projection on (x_0, x) space-time is called geodesic. When $H(y, \eta_0, \eta) = 0$ the curve

$$x_0 = x_0(t), x = x(t), \xi_0 = \xi_0(t), \xi = \xi(t)$$

is called null-bicharacteristic and it's projection on (x_0, x) space is called null space-time geodesic.
The case of null space-time geodesics was studied in [1] (Eskin, ArXiv: 2208.01842). In the present paper we consider the case when (7) is not zero.
Inverse problems of the recovery of the Riemannian metric from the lengths of geodesics were considered in works of Stefanov and Uhlmann, Stefanov, Uhlmann and Vasy, and others. In some of these papers the emphasis was on the class of simple Riemannian metric. In paper of Uhlmann, Yang Yang and H.Zhow a subclass of Lorentzian metrics was considered.

Now we briefly describe the content of the paper.
The main tool of our approach is the formula (8) below. Using this formula we prove that if the lengths of geodesics starting at any point $(0, y, \eta_0, \eta)$ are equal then the Lorentzian metrics are equal.
In particular, we can reproduce some of the results of Stefanov and Yang Yang (see Corollary).
In the end of the paper we prove the rigidity property for the Lorentzian metrics. Note that our proof is short and it does not require that the metric is simple.

The determination of the metric

The following formula holds (see [Eskin, Commun. Math. Phys. 207, 817-830, 2010], formula (3.9)):

$$(g(\xi_0,\xi),(\xi_0,\xi)) = \Big(g^{-1}\Big(\frac{dx_0}{dt},\frac{dx}{dt}\Big),\Big(\frac{dx_0}{dt},\frac{dx}{dt}\Big)\Big), \tag{8}$$

where $g = [g^{jk}]_{j,k=0}^n$ is the same as in (2), $(g(\xi_0,\xi))_j = \sum_{k=0}^n g^{jk}\xi_k$, $j = 0, ..., n$, g^{-1} is the inverse of g.
To check (8) we use that (see (3), (4))

$$\Big(\frac{dx_0}{dt},\frac{dx}{dt}\Big) = g(\xi_0,\xi). \tag{9}$$

Thus

$$(\xi_0,\xi) = g^{-1}\Big(\frac{dx_0}{dt},\frac{dx}{dt}\Big). \tag{10}$$

Therefore

$$(g(\xi_0,\xi),(\xi_0,\xi)) = \Big(g\,g^{-1}\Big(\frac{dx_0}{dt},\frac{dx}{dt}\Big),g^{-1}\Big(\frac{dx_0}{dt},\frac{dx}{dt}\Big)\Big) \tag{11}$$

$$=\Big(\Big(\frac{dx_0}{dt},\frac{dx}{dt}\Big),g^{-1}\Big(\frac{dx_0}{dt},\frac{dx}{dt}\Big)\Big)$$

Rewrite (8) using (7). We get

$$\sum_{j,k=0}^{n} g_{jk}(x)\frac{dx_j}{dt}\frac{dx_k}{dt} = \sum_{j,k=0}^{n} g^{jk}(x)\xi_j\xi_k \tag{12}$$
$$=2H(x(t),\xi_0(t),\xi(t)) = 2H(y,\eta_0,\eta).$$

We assume that

$$H(y,\eta_0,\eta) > 0. \tag{13}$$

The case $H(y,\eta_0,\eta) = 0$ was considered in [1].
Denote by Σ^+ the set where $H(y,\eta_0,\eta) > 0$.
Therefore on Σ^+ we have

$$\sum_{j,k=0}^{n} g_{jk}(x)\frac{dx_j}{dt}\frac{dx_k}{dt} > 0.$$

Let $x_0 = x_0(t, y, \hat{\eta}), x = x(t, y, \hat{\eta})$ be the space-time geodesic starting at $(0, y, \hat{\eta})$ when $t = 0$. Here $\hat{\eta} = (\eta_0, \eta)$.
Denoting $\hat{x} = (x_0, x), \hat{y} = (0, y)$, we can rewrite the space-time geodesic in a short form

$$\hat{x} = \hat{x}(t, \hat{y}, \hat{\eta}).$$

Let $R(g, T, \hat{y}, \hat{\eta})$ be the length of the geodesic in g^{-1} metric, i.e.

$$R(g, T, \hat{y}, \hat{\eta}) = \int_0^T \sqrt{\sum_{j,k=0}^{n} g_{jk}(x) \frac{dx_j}{dt} \frac{dx_k}{dt}} dt. \tag{14}$$

Using (12) we get

$$\sum_{j,k=0}^{n} g_{jk}(x) \frac{dx_j}{dt} \frac{dx_k}{dt} = 2H(y, \eta_0, \eta) = \sum_{j,k=0}^{n} g^{jk}(y) \eta_j \eta_k. \tag{15}$$

Therefore

$$R(g, T, \hat{y}, \hat{\eta}) = \int_0^T \sqrt{\sum_{j,k=0}^{n} g^{jk}(y) \eta_j \eta_k} \, dt = \sqrt{2H(y, \eta_0, \eta)} T. \tag{16}$$

Theorem

Suppose g_1 and g_2 are two inverse metric tensors in $\mathbb{R}\times\mathbb{R}^n$. Suppose integrals $R(g_1, T, \hat{y}, \hat{\eta})$ and $R(g_2, T, \hat{y}, \hat{\eta})$ are equal for all $\hat{y}, \hat{\eta}$. Then the metrics g_1 and g_2 are also equal.

Proof. Since $R(g_1, T, \hat{y}, \hat{\eta}) = R(g_2, T, \hat{y}, \hat{\eta})$ we have from (16) that

$$\Big(\sum_{j,k=0}^{n} g_1^{jk}(y)\eta_j\eta_k\Big)^{\frac{1}{2}} T = \Big(\sum_{j,k=0}^{n} g_2^{jk}(y)\eta_j\eta_k\Big)^{\frac{1}{2}} T. \tag{17}$$

Therefore

$$g_1(y) = g_2(y).$$

Remark 1. Let V_1, V_2 be two sets. If $R(g_1, T, \hat{y}, \hat{\eta}) = R(g_2, T, \hat{y}, \hat{\eta})$ for $y \in V_1, \hat{\eta} \in V_2$ then (17) holds for $y \in V_1, \hat{\eta} \in V_2$.

Corollary

Let $\mathbb{R} \times \Omega$ be a cylinder in $\mathbb{R} \times \mathbb{R}^n$. We assume that Ω is a smooth boundary domain. As in (16) we have

$$R(g, T, \hat{y}, \hat{\eta}) = \int_0^T \sqrt{\sum_{j,k=0}^{n} g^{jk}(y)\eta_j\eta_k}\, dt = \sqrt{2H(y, \eta_0, \eta)}\, T, \quad (18)$$

i.e. $R(g, T, \hat{y}, \hat{\eta})$ is the length of space-time geodesic starting at $(0, y, \hat{\eta})$.
Suppose (18) is given for any y near $\partial\Omega$ and $\hat{\eta}$ fixed.
Differentiating $R(g, T, y, \hat{\eta})$ in y we can determine $[\frac{\partial^\alpha g_{jk}(y)}{\partial y^\alpha}]_{j,k=0}^n$ for any α, $y \in \partial\Omega$, $\hat{\eta}$ is fixed.

Remark 2. When the condition (13) is not satisfied, i.e. when

$$\sum_{j,k=0}^{n} g^{jk}(y)\eta_j\eta_k = 0, \tag{19}$$

the bicharacteristic (3), (4) is the null-bicharacteristic and its projection on (x_0, x)-space is the space-time null-geodesic. This case was studied in our previous work [1]. Instead of (14), which is zero, we used there the Euclidean length of the space-time null-geodesic.

Remark 3. Consider the particular case: the case of the Riemannian metric $[g_{jk}]_{j,k=1}^n$, i.e. $g_{00} = g_{j0} = g_{0j} = 0$ for $j = 1, ..., n$.
So (12) takes the form:

$$\sum_{j,k=1}^{n} g_{jk}(x)\frac{dx_j}{dt}\frac{dx_k}{dt} = \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k \tag{20}$$
$$=2H'(x(t),\xi(t)) = 2H'(y,\eta),$$

where $H'(y,\eta) = \frac{1}{2}\sum_{j,k=1}^{n} g^{jk}(y)\eta_j\eta_k$. We assume that (20) is positive.
This case was studied in many works.

Let $R(g', T, y, \eta)$ be the length of the geodesic starting at $x = y, \xi = \eta$ when $t = 0$. We have

$$R(g', T, y, \eta) = \int_0^T \sqrt{\sum_{j,k=1}^{n} g_{jk}(x(t)) \frac{dx_j}{dt} \frac{dx_k}{dt}}\, dt = \int_0^T \sqrt{\sum_{j,k=1}^{n} g^{jk}(y) \eta_j \eta_k}\, dt,$$

where $g' = [g^{jk}]_{j,k=1}^{n}$.
Using (12) we again get that

$$R(g', T, y, \eta) = \sqrt{2H'}\, T.$$

Therefore formula (20) allows to recover $g'(y)$.

The rigidity of the Lorentzian metric

Let $g^{(0)}$ and $g^{(1)}$ be two metric in $\mathbb{R} \times \Omega$.
Consider a $g^{(0)}$ geodesic, starting at $y \in \partial\Omega$ when $t = 0$ and ending at $x_T \in \partial\Omega$ when $t = T$, and consider a $g^{(1)}$ geodesic, also starting at $y \in \partial\Omega$ when $t = 0$ and ending at the same point $x_T \in \partial\Omega$ when $t = T$. The rigidity property means that if the lengths of such two geodesics are equal and they are close enough in some norm then these geodesics are equal.
Let

$$g_\tau = g_0 + \tau(g_1 - g_0), \quad 0 \leq \tau \leq 1. \tag{21}$$

Let $\hat{x} = \hat{x}_r(t, \hat{y}, \hat{\eta}_\tau)$ be the equation of the geodesic corresponding to $g_\tau, \hat{x} = (x_0, x)$. Let $\hat{\eta}_0$ be such that $x_0(T, \hat{y}, \hat{\eta}_0) = x_T$ and let $\hat{\eta}_1$ be such that $x_1(T, \hat{y}, \hat{\eta}_1) = x_T$.

Consider the family of $\hat{x}_\tau(\tau, y, \eta_\tau)$ such that $\hat{x}_\tau$ is a geodesic of g_τ metric and

$$\hat{x}_\tau(T, \hat{y}, \hat{\eta}_\tau) = x_T \quad \text{for} \quad 0 \leq \tau \leq 1. \tag{22}$$

We have, differentiating in τ,

$$\frac{d\hat{x}_\tau}{d\tau}(T, \hat{y}, \hat{\eta}_\tau) + \frac{\partial \hat{x}_\tau}{\partial \eta} \frac{d\hat{\eta}_\tau}{d\tau} = 0. \tag{23}$$

Therefore

$$\frac{d\hat{\eta}_\tau}{d\tau} = -\frac{1}{\frac{\partial \hat{x}_\tau}{\partial \eta}} \frac{d\hat{x}_\tau}{d\tau}. \tag{24}$$

The length of geodesic $\hat{x}_\tau$ is

$$R(g_\tau, T, \hat{y}, \hat{\eta}_\tau) = \int_0^T \sqrt{\sum_{j,k=0}^{n} g_{jk} \frac{dx_j}{dt} \frac{dx_k}{dt}} dt$$

$$= \int_0^T \sqrt{\sum_{j,k=0}^{n} g^{jk}(x_\tau(t))\xi_j(t)\xi_k(t)} dt = \sqrt{2H_\tau(y, \hat{\eta}_\tau)}\, T, \quad (25)$$

where

$$H_\tau(y, \hat{\eta}_\tau) = \frac{1}{2} \sum_{j,k=0}^{n} g_\tau^{jk}(y)\eta_{j\tau}\eta_{k\tau}. \quad (26)$$

Note that

$$\frac{dR(g_\tau, T, \hat{y}, \hat{\eta}_\tau)}{d\tau} = (2H_\tau(y, \hat{\eta}_\tau))^{-\frac{1}{2}} \Big(\big((g_1 - g_0)\hat{\eta}_\tau, \hat{\eta}_\tau\big) + 2\Big(g_\tau \hat{\eta}_\tau, \frac{\partial \hat{\eta}_\tau}{\partial \tau}\Big)\Big) \quad (27)$$

and

$$\frac{d\hat{\eta}_\tau}{d\tau} = (-1)\Big(\frac{\partial \hat{x}_\tau}{\partial \eta}\Big)^{-1} \frac{dx_\tau(T, \hat{y}, \hat{\eta}_\tau)}{d\tau}. \quad (28)$$

We have

$$\frac{dR(g_\tau, T, \hat{y}, \hat{\eta}_\tau)}{d\tau}\Big|_{\tau=0} = R_1 + R_2, \tag{29}$$

where

$$R_1 = (2H(y, \hat{\eta}_0)^{-\frac{1}{2}}((g_1 - g_0)\hat{\eta}_0, \hat{\eta}_0), \tag{30}$$

$$R_2 = (2H(y, \hat{\eta}_0)^{-\frac{1}{2}}\left(2g_0\hat{\eta}_\tau, \frac{\partial\hat{\eta}_\tau}{\partial\tau}\right)\Big|_{\tau=0} \tag{31}$$

Note that $\left(g_0\hat{\eta}_\tau, \frac{\partial\hat{\eta}_\tau}{\partial\tau}\right)$ has the form analogous to (3.3) in [1]. Estimating $\frac{dx_\tau}{d\tau}$ analogously to (2.8) in [1] we get

$$\left|\frac{d\hat{\eta}_\tau}{d\tau}\right| \leq C\sup_\tau \int_0^T |(g_1 - g_0)(\hat{x}_\tau(t)\Big|dt + C\sup_\tau \int_0^T \left|\frac{\partial}{\partial x}(g_1 - g_0)(\hat{x}_\tau)\right|dt. \tag{32}$$

Denote

$$|||(g_1-g_0)||| = \sup_{\tau} \int_0^T |(g_1-g_0)(\hat{x}_\tau)|dt + \sup_{\tau} \int_0^T \left| \left(\frac{\partial g_1}{\partial x} - \frac{\partial g_0}{\partial x} \right)(\hat{x}_\tau) \right| dt \tag{33}$$

and denote

$$\|g_1 - g_0\| = \sup_y |g_1(y) - g_0(y)|. \tag{34}$$

Choose $w = \frac{g_1-g_0}{\|g_1-g_0\|}$ such that

$$R_1 = l_0 \|g_1 - g_0\|,$$

where

$$l_0 = (2H(y, \hat{\eta}_0))^{-\frac{1}{2}} (w\hat{\eta}_0, \hat{\eta}_0) > 0. \tag{35}$$

Analogously to (3.3) in [1]

$$R_2 = (2H(y, \hat{\eta}_0))^{-\frac{1}{2}} \left(2g_0\hat{\eta}_1, \frac{\partial \hat{\eta}_1}{\partial \tau}\Big|_{\tau=0} \right) = \alpha(g_1 - g_0), \tag{36}$$

where $\alpha(g_1 - g_0)$ is a linear functional.

Denote

$$w_1 = \frac{g_1 - g_0}{|||g_1 - g_0|||}, \tag{37}$$

where the norm $|||g_1 - g_0|||$ is the same as in (33).
We choose w_1 without changing w such that

$$\alpha(g_1 - g_0) = l_1 |||g_1 - g_0|||, \tag{38}$$

where $l_1 = \alpha(w_1) > 0$.
Therefore we get

$$R_1 + R_2 = l_0 \|g_1 - g_0\| + l_1 |||g_1 - g_0||| > 0. \tag{39}$$

Finally, we estimate the remainder. Analogously to (3.10) in [1],

$$\begin{aligned} |\Delta| = R(g_2, T, y, \eta_1) - R(g_0, T, y, \eta_0) - R_1 - R_2| \\ \leq C_1 \|g_1 - g_0\|^2 + C_2 |||g_1 - g_0|||^2. \end{aligned} \tag{40}$$

If $\|g_1 - g_0\|$ and $|||g_1 - g_0|||$ are small enough, more precisely, if

$$\|g_1 - g_0\| < \frac{1}{2l_0 C_1}, \tag{41}$$

and

$$|||g_1 - g_0||| < \frac{1}{2l_1 C_2}, \tag{42}$$

then

$$\frac{1}{2} l_0 \|g_1 - g_0\| \leq |\Delta| \tag{43}$$

and

$$\frac{1}{2} l_1 |||g_1 - g_0||| \leq |\Delta|. \tag{44}$$

From any of the last inequalities we get that $|\Delta| = 0$ implies that $g_1 - g_0 = 0$.

Thus we proved the following theorem:

Theorem

Let $R(g_1, T, y, \hat{\eta}_1) = R(g_0, T, y, \hat{\eta}_0)$*, where* $\hat{\eta}_1$ *and* $\hat{\eta}_0$ *are such that* $x_0(T, y, \hat{\eta}_0) = x_T, x_1(T, y, \hat{\eta}_1) = x_T$*, i.e. the geodesics in both metrics have the same length. If metrics* g_0 *and* g_1 *are close in norms (33) and (34) then* $g_0 = g_1$*.*

THANK YOU VERY MUCH FOR YOUR ATTENTION!