Hawking radiation from acoustic black holes in two space dimensions

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Black holes

Lorentzian metric

$$ds^2 = \sum_{j,k=0}^{n} g_{jk}(x) dx_j dx_k, \tag{1}$$

where x_0 is the time variable, $x = (x_1, ..., x_n)$ are space variables.

The quadratic form (1) has the signature $(+, -, ..., -)$.

The inverse metric tensor is

$$\left[g^{jk}(x)\right]_{j,k=0}^{n} = \left(\left[g_{jk}(x)\right]\right)^{-1}.$$

Wave equation

$$\Box_g u = \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}} \frac{\partial}{\partial x_j}\left(\sqrt{(-1)^n g}\, g^{jk}(x) \frac{\partial u(x_0, x)}{\partial x_k}\right) = 0, \tag{2}$$

$$g(x) = \det[g_{jk}]_{j,k=0}^{n}.$$

Example 1. Schwarzschield metric
(in Cartezian coordinates)

$$ds^2 = \left(1 - \frac{2m}{R}\right) dx_0^2 - \sum_{j=1}^{3} dx_j^2 - \frac{4m}{R} dx_0 dR - \frac{2m}{R}(dR)^2$$

where $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Black hole is a region B in R^n such that any disturbance (signal) in B can not escape $\overline{B}$, i.e. if $u(x_0, x)$ is a solution of a wave equation $\square_g u = 0$ and $\operatorname{supp} u(t_0, x) \subset \overline{B}$ for some t_0, then $\operatorname{supp} u(x_0, x) \subset \overline{B}$ for all $x_0 > t_0$.

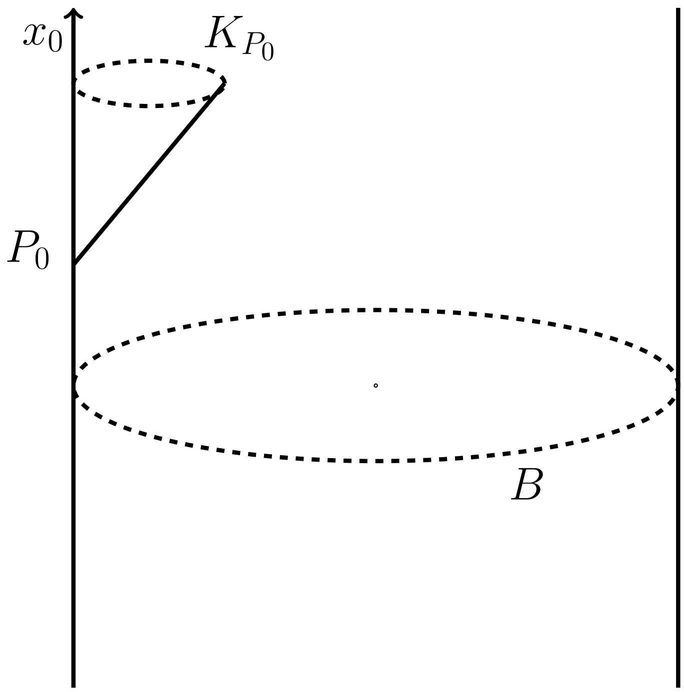

Fig. 1. A black hole B.
B is a characteristic surface of the wave equation $\square_g u = 0$:

$$\sum_{j,k=0}^{n} g^{jk}(x) S_{x_j} S_{x_k} = 0 \quad \text{when} \quad S(x) = 0,$$

$\{S(x) = 0\}$ is the boundary of B,

For any $P_0 \in \partial B \times \mathbb{R}$, let K_{P_0} be the forward cone of time-like rays. Thus $\overline{K_{P_0}}$ is the half-cone of influence of point P_0.

When B is a black hole then $\overline{K_{P_0}} \subset \overline{B} \times \mathbb{R}$ for each P_0.

For example, $\{R < 2m\}$ is the black hole in the case of Schwarzschield metric.The black hole is determined by the metric,

There are two main classes of black holes:

1) Black holes of the general relativity when the metric is a solution of Einstein's equations. For example, $\{R < 2m\}$ is the black hole in the case of Schwarzchield metric.

2) An analogue black hole when the metric is not a solution of the Einstein's equation.

The physical meaning of the wave equation $\Delta_g u = 0$ corresponding to an analogue metric is the wave propagation in a moving medium.

We will study an example of analogue metric: the rotating acoustic metric.

Let $v = (v_1, v_2)$ be the velocity of the fluid in a vortex, $(v_1, v_2) = \frac{A}{|x|}\hat{x} + \frac{B}{|x|}\hat{\theta}$, where $\hat{x} = \frac{(x_1,x_2)}{|x|}$, $\hat{\theta} = \frac{(-x_2,x_1)}{|x|}$, A is a radial velocity of the flow, B is an angular velocity.

The acoustic metric is

$$(3)\quad ds^2 = (1 - v_1^2 - v_2^2)dx_0^2 + \sum_{j=1}^{2} v_j dx_0 dx_j - dx_1^2 - dx_2^2.$$

The corresponding wave equation

$$\Box_g u = 0$$

describes the acoustic waves in the rotating fluid. Symbol of $\Box_g$ (or Hamiltonian) in the case of acoustic metric has the following form in the polar coordinates (ρ, φ):

$$(4)\quad H(\rho, \varphi, \xi_0, \xi_\rho, \xi_\varphi) = \Big(\xi_0 + \frac{A}{\rho}\xi_\rho + \frac{B}{\rho^2}\xi_\varphi\Big)^2 - \xi_\rho^2 - \frac{1}{\rho^2}\xi_\varphi^2$$

where $(\xi_0, \xi_\rho, \xi_\varphi)$ are dual coordinates to (x_0, ρ, φ).

We assume that $A < 0$, $B \neq 0$ are constant. The curve $\rho + |A| = 0$ is a characteristic curve.

Thus, the domain $\{\rho < |A|\}$ is the black hole.

Acoustic black holes (or, in general, analogue black holes,) are quite common.

Consider, for example, a general acoustic metric when the fluid flow has the form

$$v = \frac{A(\rho,\varphi)}{\rho}\hat{x} + \frac{B(\rho,\varphi)}{\rho}\hat{\theta}, \quad \rho = |x|,$$

when radial and angular components of the velocity depend on ρ and φ, $A(\rho,\varphi) < 0$, $B(\rho,\varphi) \neq 0$ for all ρ,φ.

Let $\frac{A^2}{\rho^2} + \frac{B^2}{\rho^2} = 1$ be a smooth curve. This curve is called the ergosphere. Then there exists a smooth curve $\{c(r,\varphi) = 0\}$ inside the ergosphere that is a black hole.

As the recent observations of galaxies have shown, the gravitational black holes are also not a rare phenomenon. The astrophysicists detected many black holes that are moving and quite often merge. The merging of two black holes leads to the release of a huge amount of energy that creates gravitational waves. The LIGO (laser interferometer gravitational waves laboratory) detected this gravitational waves on the Earth. In 2017 the Nobel prize in physics was awarded to the people involved in LIGO project.

Quantum effects
(Second quantization)

In classical physics the black holes are the regions that do not emit any signals or particles. Therefore it was a remarkable discovery of S. Hawking that when quantum effects are added the black hole emits particles.

We shall describe the second quantization for the acoustic wave equation $\square_g u = 0$.

First, we will construct a basis of solutions of the initial value problem for $\square_g u = 0$.

Let $f_k^+(x_0, x)$ be the solution of the wave equation with the initial data

$$f_k^+(x_0,x)\Big|_{x_0=0} = \gamma_k e^{i\rho\eta_\rho + im\varphi}, \quad \frac{\partial f_k^+}{\partial x_0}\Big|_{x_0=0} = i\lambda_0^-(k)\gamma_k e^{i\rho\eta_\rho + im\varphi}, \tag{5}$$

where

$$k = (\eta_\rho, m),\ \eta_\rho \in \mathbb{R}^1,\ \eta_\varphi = m \in \mathbb{Z},\ \gamma_k = \frac{1}{\sqrt{\rho}\big(\eta_\rho^2 + a^2\big)^{\frac{1}{4}}\sqrt{2(2\pi)^2}},$$

$$\lambda_0^-(k) = -\frac{A}{\rho}\eta_\rho - \frac{B\eta_\varphi}{\rho^2} - \sqrt{\eta_\rho^2 + a^2}, \quad a \ \text{ is arbitrary.}$$

Denote

$$f_{-k}^-(x_0,x) = \overline{f_k^+(x_0,x)}. \tag{6}$$

Introduce the Klein-Gordon inner product

$$(7)\quad < f,h >= i\int\limits_{x_0=t} |g|^{\frac{1}{2}}\sum_{j=0}^{2} g^{0j}\Big(\overline{f}\frac{\partial h}{\partial x_j}-\frac{\partial \overline{f}}{\partial x_j}h\Big)dx_1dx_2$$

where $g = \det[g_{jk}]^2_{j,k=0}$.

Note that if f and h are solutions of $\Box_g u = 0$ then $< f,h >$ is independent of t.

Computing the KG inner products we get

$$< f_k^+, f_{k'}^+ >= \delta(\eta_\rho - \eta'_\rho)\delta_{m,m'}, \quad k = (\eta_\rho, m),\ \ k' = (\eta'_\rho, m')$$

Analogously,

$$< f_{-k}^-, f_{-k'}^- >= -\delta(\eta_\rho - \eta'_\rho)\delta_{m,m'},\ \ < f_k^+, f_{k'}^- >= 0,\ \ \forall k,\ \forall k'.$$

Therefore $\{f_k^+, f_{-k'}^-\}$ form an "orthogonal" basis of solution of the wave equation.

Any solution $C(x_0,\rho,\varphi)$ of $\Box_g u = 0$ can be expanded in the basis $\{f_k^+, f_{-k}^-\}$:

$$(8)\ \ C = \sum_{m=-\infty}^{+\infty}\int\limits_{-\infty}^{\infty}\big(C^+(k)f_k^+(x_0,x)+C^-(k)f_{-k}^-(x_0,x)\big)d\eta_\rho,$$

$k = (\eta_\rho, m),\ \ C^+(k) = \langle f_k^+, C\rangle,\ \ C^-(k) = -\langle f_{-k}^-, C\rangle.$

We shall call any solution $C(x_0,\rho,\varphi)$ of wave equation $\Box_g u = 0$ a wave packet.

Let Φ be the field operator. Expanding Φ in the basis $\{f_k^+, f_{-k}^-\}$ we get

$$(9) \quad \Phi = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} (\alpha_k^+ f_k^+(x_0, x) + \alpha_{-k}^- f_{-k}^-(x_0, x)) d\eta_\rho.$$

Operators $\alpha_k^+, \alpha_{-k}^-$ are called annihilation and creation operators, respectively:

$$\alpha_k^+ =< f_k^+, \Phi >, \quad \alpha_{-k}^- = - < f_{-k}^-, \Phi > .$$

Operators $\alpha_k^+, \alpha_{-k}^-$ satisfy the following commutation relation:

$$[\alpha_k^+, \alpha_{-k'}^-] = \delta(\eta_\rho - \eta_\rho')\delta_{mm'} I,$$

where I is the identity operator,

$$[\alpha_k^+, \alpha_{k'}^+] = 0, [\alpha_{-k}^-, \alpha_{-k'}^-] = 0.$$

Number of particles operator

Let C be a wave packet. The number of particles operator, created by the wave packet C, is

$$N(C) = < C, \Phi >^* < C, \Phi > . \tag{10}$$

In-vacuum state

In-vacuum state $|0\rangle$ is defined by the conditions

$$\alpha_k^+ |0\rangle = 0 \text{ for all } \quad k,$$

i.e. in-vacuum state annihilates all annihilation operators.

The average number of particles created by the wave packet C in in-vacuum state is:

$$\langle 0|N(C)|0\rangle = \langle 0|(C, \Phi)^* < C, \Phi > |0\rangle. \tag{11}$$

If C is an arbitrary wave packet with initial data having compact supports outside the black hole $\{\rho < |A|\}$, the average (11) gives some particles created by C. Such particles have no relation to the Hawking radiation. To get particles related to the Hawking radiation one has (following S.Hawking and others) take the limit of $\langle 0|N(C)|0\rangle$ when the time $x_0 = T$ tends to $-\infty$. A rigorous work in this direction for the Schwartzschield metric was done by Fredenhagen and Haag.

We use a different approach: First, we construct a special wave packet C_0 of the form that is singular when $\rho = |A|$:

$$(12)\quad \theta(\rho - |A|)\frac{1}{\sqrt{\rho}}(\rho - |A|)^{\varepsilon} e^{-a(\rho - |A|)} \cdot \exp\left(-ix_0\eta_0 + i\xi_0|A| \ln|\rho - |A|| + m'\varphi\right),$$

where $\xi_0|A| = \left(\eta_0 - \frac{Bm}{|A|^2}\right)|A|$.

Note the role of parameter a in (12):

When $a \to \infty$ the support of (12) tends to the boundary of the black hole. The limit of $\langle 0|N(C_0)|0>$ when $a \to \infty$ contains particles related to the Hawking radiation. Thus the limit when $a \to \infty$ replaces the limit when $T \to -\infty$.

Let C_n be the normalized wave packet:

$$(13)\qquad C_n = \frac{C_0}{< C_0, C_0 >^{\frac{1}{2}}}, \quad \text{i.e.} \quad < C_n, C_n >= 1.$$

Note that for (12) $< C_0, C_0 >= \frac{4\pi\xi_0|A|\Gamma(2\varepsilon)}{(2a)^{2\varepsilon}}$.

Theorem 0.1. *We have*

$$(14)\quad \lim_{a\to\infty}\langle 0|N(\hat{C}_n)|0\rangle = \frac{2^{\varepsilon}}{4\pi\Gamma(2\varepsilon)}\int\limits_{-\infty}^{\infty}\frac{1}{2\xi_0|A|}e^{-2\pi\xi_0|A|}$$

$$|\Gamma_1(i\xi_0|A|+\varepsilon)|^2\,|\xi_0|A|+i\varepsilon|^2\,\left|\frac{\eta_\rho}{(\eta_\rho^2+1)^{\frac{1}{4}}}+(\eta_\rho^2+1)^{\frac{1}{4}}\right|^2$$

$$\cdot\, e^{2\xi_0|A|\arg(\eta_\rho+i)}(\eta_\rho^2+1)^{-\varepsilon}d\eta_\rho = I_+ + I_-.$$

where I_+ *is the integral* $\int\limits_0^{\infty}$ *in (14)*

and I_- *is the integral* $\int\limits_{-\infty}^{0}$ *in (14).*

Note that $\arg(\eta_\rho+i)=\sin^{-1}\frac{1}{\sqrt{\eta_\rho^2+1}}$ when $\eta_\rho>0$ and $arg(\eta_\rho+i) = \pi-\sin^{-1}\frac{1}{\sqrt{\eta_\rho^2+1}}$ when $\eta_\rho<0$. Thus, $I_+=O(e^{-\pi\xi_0|A|})$ and $I_-=O\left(\frac{1}{(\xi_0|A|)^{\varepsilon_1}}\right)$.

Therefore I_+ is exponentially decaying when $\xi_0\to+\infty$. I_+ represents the contribution of particles related to the Hawking radiation and I_- is the contribution of particles not related to the Hawking radiation.

We see in (14) that not all particles created when one averages $N(C_n)$ over in-vacuum $|0>$ are related to the Hawking radiation.

We shall introduce a new vacuum state $|\Psi\rangle$ called the Unruh type vacuum state such that $\langle\Psi|N(C_0)|\Psi\rangle$ consists only of particles related to the Hawking radiation.

Unruh type vacuum state

We shall split f_k^+ in two parts

$$f_k^{++}(x_0, \rho, \varphi) = f_k^+ \theta(\xi_\rho), \quad f_k^{+-} = f_k^+(1 - \theta(\eta_\rho)),$$

where $\theta(\eta_\rho) = 1$ for $\eta_\rho > 0$, $\theta(\eta_\rho) = 0$ for $\eta_\rho < 0$.

Analogously, $f_{-k}^{-+} = f_{-k}^- \theta(\eta_\rho)$, $f_{-k}^{--} = f_{-k}^-(1-\theta(\eta_\rho))$.

We split also operators $\alpha_k^+, \alpha_{-k}^-$:

$$\alpha_k^{++} = \alpha_k^+ \theta(\eta_\rho), \quad \alpha_k^{+-} = \alpha_k^+(1 - \theta(\eta_\rho)),$$

$$\alpha_{-k}^{-+} = \alpha_{-k}^- \theta(\eta_\rho), \quad \alpha_{-k}^{--} = \alpha_{-k}^-(1 - \theta(\eta_\rho))$$

The Unruh type vacuum state $|\Psi\rangle$ is defined by the conditions:

$$\alpha_k^{++}|\Psi\rangle = 0, \quad \alpha_{-k}^{--}|\Psi\rangle = 0 \ \text{ for all } \ k.$$

The average number of particles created by the wave packet C is

$$\langle\Psi|N(C)|\Psi\rangle.$$

Theorem 0.2. *Consider the Unruh type vacuum* $|\Psi\rangle$ *instead of in-vacuum* $|0\rangle$*.*

The limit as $a \to \infty$ *of the average number of particles created by the normalized wave packet* $C_n(x_0, \rho, \varphi)$ *is given by formula*

$$\lim_{a\to\infty} \langle\Psi|N(C_n)|\Psi\rangle = \frac{2^{2\varepsilon} e^{-2\pi\xi_0|A|}|\Gamma_1|^2}{2\pi\Gamma(\varepsilon)}$$

$$\cdot \frac{(\xi_0|A|)^2+\varepsilon^2}{\xi_0|A|} \int\limits_{-\infty}^{0} \frac{|\eta_\rho|}{(\eta_\rho^2+1)^{\varepsilon+1}}\ e^{2\xi_0|A|\sin^{-1}\frac{1}{\sqrt{\eta_\rho^2+1}}} d\eta_\rho.$$

Here

$$\Gamma_1(\xi_0|A|) = i\int\limits_0^\infty e^{(i\xi_0|A|+\varepsilon-1)\ln y + i(\varepsilon-1)\frac{\pi}{2} - iy} dy.$$