56 The heat trace asymptotics.

Consider the Laplace-Beltrami operator in $\Omega \subset \mathbf{R}^n$

$$A(x,D)u = -\sum_{j,k=1}^{n} \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x_j}\left(\sqrt{g(x)}g^{jk}(x)\frac{\partial u}{\partial x_k}\right), \tag{56.1}$$

where $([g^{jk}(x)]_{j,k=1}^{n})^{-1}$ is the matrix tensor, $g(x) = (\det[g^{jk}])^{-1}$, $g^{jk}(x) \in C^{\infty}(\overline{\Omega})$. Let $G_D(x.x^{(0)},t)$ be the Dirichlet heat kernel of A, i.e.

$$\frac{\partial G_D(x,x^{(0)},t)}{\partial t} + A(x,D)G_D(x,x^{(0)},t) = 0 \ \text{ for } t>0,\ x\in\Omega,\ x^{(0)}\in\Omega, \tag{56.2}$$

$$G_D(x,x^{(0)},0) = \delta(x-x^{(0)}), \tag{56.3}$$

$$G_D(x,x^{(0)},t)|_{\partial\Omega\times(0,+\infty)} = 0. \tag{56.4}$$

Denote by A_D the operator A with the zero Dirichlet boundary conditions. Since A_D is positively definite:

$$(A_D\varphi,\varphi) \geq C_0\|\varphi\|_0^2, \quad \forall\varphi\in C_0^{\infty}(\Omega)$$

the resolvent $(A_D+\lambda I)^{-1}$ is a bounded operator in $L^2(\Omega)$ for $\lambda\in\mathbf{C}\setminus(-\infty,0)$ and it depends analytically on λ in this region.

As in §47 we can represent $G_D(x,x^{(0)},t)$ in the form

$$G_D(x,x^{(0)},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(A+i(\sigma-i\tau))^{-1}\delta(x-x^{(0)})e^{it(\sigma-i\tau)}d\sigma, \tag{56.5}$$

where $\tau>0$ is arbitrary. We have

$$(A+\lambda I)^{-1} = \mathcal{R} + T_{-m-N-1}, \tag{56.6}$$

where $\mathcal{R}$ is the parametrix (c.f. (55.31)) with $A(x,D)$ having the form (56.1) and ord $T_{-m-N-1} \leq -m-N-1$.

Analogously to §47 one can show that T_{-m-N-1} is an integral operator with continuous kernel $T_{-m-N-1}(x,x^{(0)},(\sigma-i\tau))$ analytic in $\sigma-i\tau$ for $\tau>0$ and satisfying

$$|T_{-m-N-1}(x,x^{(0)},\sigma-i\tau)| \leq C|\sigma-i\tau|^{-N_1-2}, \tag{56.7}$$

where N_1 is large if N is large.

We shall compute $\int_\Omega G_D(x,x,t)dx$.

Let U_j be such that $U_j \cap \partial\Omega = \emptyset$. Then analoguously to (47.20) we have

$$\int_{\mathbf{R}^n}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_j(x)R_j\varphi_j\delta(x-x^{(0)})e^{it(\sigma-i\tau)}dt\right)\Bigg|_{x=x^{(0)}}dx = \frac{1}{t^{\frac{n}{2}}}\sum_{k=0}^{N_1'}\left(\int_{\mathbf{R}^n}\varphi_j(x)c_{2k}(x)dx\right)t^k, \tag{56.8}$$

where $c_{2k+1}(x)=0$ (c.f. §47).

It follows from (47.21) that

$$\int_{\mathbf{R}^n}\varphi_j(x)c_0(x)dx = \frac{1}{(2\sqrt{\pi})^n}\int_{\mathbf{R}^n}\varphi_j(x)\sqrt{g(x)}dx. \tag{56.9}$$

As in §47 the contribution of $T_{-N-m-1}(x,x,\sigma-i\tau)$ to the heat trace will be

$$\int_\Omega\int_{-\infty}^{\infty}T_{N-m-N-1}(x,x,\sigma-i\tau)e^{it(\sigma-i\tau)}d\sigma dx = O(t^{\frac{N_1}{m}}), \tag{56.10}$$

where N_1 is large.

It remains to find the contribution of the neighborhoods U_j such that $U_j\cap\partial\Omega\neq\emptyset$.

If U_j is a neighborhood such that $U_j\cap\partial\Omega\neq 0$ we can introduce in U_j local coordinates $(y',y_n)=s_j(x)$ for $x\in U_j\cap\overline{\Omega}$.

We choose (y',y_n) such that $A(x,D)$ has the following form in (y',y_n) coordinates

$$A_j(y,D_y) = -\frac{1}{\sqrt{g_j(y)}}\frac{\partial}{\partial y_n}\left(\sqrt{g_j(y)}\frac{\partial}{\partial y_n}\right)-\sum_{k,p=1}^{n-1}\frac{1}{\sqrt{g_j(y)}}\frac{\partial}{\partial y_k}\left(\sqrt{g_j(y)}g_j^{pk}(y',y_0)\frac{\partial}{\partial y_p}\right), \tag{56.11}$$

where $g_j(y) = \det([g_j^{pk}]_{p,k=1}^n)^{-1}$. Such coordinates are called semigeodesic coordinates.

Note that $([g_j^{pk}(y',0)]_{p,k=1}^{n-1})^{-1}$ is the Riemannian metric on $\partial\Omega\cap U_j$.

We consider in y-coordinates

$$\psi_j(s_j^{-1}(y))(R_{j0}+R_{j1})\varphi_j(s_j^{-1}(y'))\delta(y-y^{(0)}),\quad y^{(0)}=(y_0',y_{n0})=s_j(x^{(0)}), \tag{56.12}$$

where R_{j0} and R_{j1} are the same as in (55.3).

The contribution of R_{j0} to the heat trace is the same as in (56.8):

$$\frac{1}{t^{\frac{n}{2}}}\sum_{k=s_0}^{N'}\left(\int_{\mathbf{R}^n}\varphi_j(s_j^{-1}(y))c_{2k}^{(j)}(y)dy\right)t^k, \tag{56.13}$$

in particular, the principal term is

$$\frac{1}{t^{\frac{n}{2}}}\frac{1}{(2\sqrt{\pi})^n}\int_{\mathbf{R}^n}\varphi_j(s_j^{-1}(y))\sqrt{g_j(y)}dy. \tag{56.14}$$

Note that $\sqrt{g_j(y)}dy_1\wedge ...\wedge dy_n$ is the realization of the differential form $\sqrt{g(x)}dx_1\wedge ...\wedge dx_n$ in y-coordinates. Therefore

$$\int_{\mathbf{R}^n}\varphi_j(s_j^{-1}(y))\sqrt{g_j(y)}dy=\int_{\Omega}\varphi_j(x)\sqrt{g(x)}dx. \tag{56.15}$$

Now we shall find the contribution of R_{j1} to the heat trace.

Let $A_{j0}(y,\xi)=\xi_n^2+\sigma_j^2(y,\xi',\lambda^{\frac{1}{2}})$, where

$$\sigma_j(y,\xi',\lambda^{\frac{1}{2}})=\sqrt{\sum_{p,r=1}^{n-1}g_j^{pr}(y)\xi_p\xi_r+\lambda}. \tag{56.16}$$

Then $A_{j0}=A_j^+A_j^-$, where $A_j^+=\xi_n+i\sigma_j,\ A_j^-=\xi_n-i\sigma_j$. The principal part of $\psi_jR_{j1}\varphi_j$ has the form

(56.17)

$$-\psi_j(s_j^{-1}(y)R_j^+(y,D_y)\left(\delta(y_n)p'A_{j0}^{(-1)}(y,D_y)\varphi_j(s^{-1}(y^{(0)}))\delta(y-y^{(0)})\right),$$

where

$$\begin{aligned} R_j^+(y,\xi)&=(\xi_n+i\sigma_j(y,\xi',\sqrt{\lambda}))^{-1},\\ A_{j0}^{(-1)}(y,\xi)&=(A_{j0}(y,\xi)+\lambda)^{-1},\\ y^{(0)}&=s_j(x^{(0)}),\quad x^{(0)}\in U_j\cap\Omega. \end{aligned} \tag{56.18}$$

Note that

$$\begin{aligned} v(y')&=p'A_{j0}^{(-1)}\varphi_j(s_j^{-1}(y^{(0)}))\delta(y-y^{(0)})\\ &=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\frac{\varphi_j(s_j^{-1}(y^{(0)})e^{-iy_{n0}\xi_n+i(y'-y_0')\cdot\xi'}}{\xi_n^2+\sigma_j^2(y',0,\xi',\sqrt{\lambda})}d\xi\\ &=\frac{1}{(2\pi)^{n-1}}\int_{\mathbf{R}^{n-1}}\frac{\varphi_j(s_j^{-1}(y^{(0)}))e^{-y_{n0}\sigma_j(y',0,\xi',\lambda^{\frac{1}{2}})+i(y'-y_0')\cdot\xi'}}{2\sigma_j(y',0,\xi',\lambda^{\frac{1}{2}})}d\xi', \end{aligned} \tag{56.19}$$

where we used the Jordan lemma to compute the integral in ξ_n. Again using the Jordan lemma we get

$$p_+R_j^+(v(y')\delta(y_n)) = \frac{1}{(2\pi)^{n-1}}\int_{\mathbf{R}^{n-1}} e^{-y_n\sigma_j(y,\xi',\lambda^{\frac{1}{2}})+iy'\cdot\xi'}\tilde{v}(\xi')d\xi'. \tag{56.20}$$

Note that (56.19), (56.20) are two ψdo in $\mathbf{R}^{n-1}$. Taking the composition of (56.20) and (56.19) we get modulo lower order terms

$$\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n_+}\frac{\varphi_j(s_j^{-1}(y^{(0)}))e^{-y_{n0}\sigma_j(y',0,\xi',\lambda^{\frac{1}{2}})-y_n\sigma_j(y,\xi',\lambda^{\frac{1}{2}}+i(y'-y_0')\cdot\xi'}}{2\sigma_j(y',0,\xi',\lambda^{\frac{1}{2}})}d\xi'. \tag{56.21}$$

Taking $y_{n0}=y_n$ and integrating in y_n we obtain

$$\begin{aligned}&\int_0^\infty\frac{\varphi_j(s_j^{-1}(y',y_n))e^{-y_n(\sigma_j(y',0,\xi',\lambda^{\frac{1}{2}})+\sigma_j(y,y_n,\xi',\lambda^{\frac{1}{2}}))}}{2\sigma_j(y',0,\xi',\lambda^{\frac{1}{2}})}dy_n.\\ &=\frac{\varphi_j^{-1}(s^{-1}(y',0))}{4(\sum_{p,r=1}^{n-1}g_j^{pr}(y',0)\xi_p\xi_n+\lambda)}+O(\sigma_j^{-3}(y',0,\xi',\lambda^{\frac{1}{2}})).\end{aligned} \tag{56.22}$$

Note that (c.f. (47.10), (47.21))

(56.23)

$$\begin{aligned}&\frac{1}{(2\pi)^{n-1}}\int_{\mathbf{R}^{n-1}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sum_{p,r=1}^{n-1}g_j^{pr}(y',0,\xi')+i(\sigma-i\tau)}e^{it(\sigma-i\tau)}d\sigma d\xi'\\ &=\frac{1}{(2\pi)^{n-1}}\int_{\mathbf{R}^{n-1}}e^{-t\sum_{p,r=0}^{n-1}g_j^{pr}(y',0)\xi_p\xi_r}d\xi=\frac{1}{(2\sqrt{\pi})^{n-1}}t^{-\frac{n-1}{2}}\sqrt{g_j(y',0)},\end{aligned}$$

where $g_j(y',0)=\det[g_j^{pr}(y',0)]^{-1}$. Therefore performing the Fourier-Laplace transform of (56.21) in $\lambda=i(\sigma-i\tau)$, taking $y_n=y_{n0},\ y'=y_0'$ and integrating in (y',y_n) we get that the contribution of the principal term in $\psi_jR_{j1}\varphi_j$ to the heat trace is

$$-t^{-\frac{n-1}{2}}\frac{1}{2^{n+1}\pi^{\frac{n-1}{2}}}\int_{\mathbf{R}^{n-1}}\varphi_j(s_j^{-1}(y',0))\sqrt{g_j(y',0)}dy'+O(t^{-\frac{n-1}{2}}). \tag{56.24}$$

Note that

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathbf{R}^n_+}(e^{it(\sigma-i\tau)}\psi_jR_{j1}\varphi_{j1}\delta(y-y^{(0)})|_{y=y^{(0)}}dyd\sigma\\ &\qquad=t^{-\frac{n-1}{2}}\left(\sum_{m=0}^{N'}c_k't^{\frac{k}{2}}+O(t^{\frac{N'+1}{2}})\right),\end{aligned} \tag{56.25}$$

where c'_0 is defined in (56.24).

To prove (56.25) we make in the left hand side of (56.25) the changes of variables $\lambda = \frac{\mu}{t}$, $\xi' = \frac{\eta'}{\sqrt{t}}$, and proceed with all terms of the parametrix (55.20), (55.28) as we did with the principal term. Combining the contributions for all neighborhoods U_j we get the following result:

Theorem 56.1. *The Dirichlet heat trace* $\int_\Omega G_p(x,x,t)dx$ *has the following assymptotics expansion when* $t \rightarrow +0$*:*

$$\int_\Omega G_D(x,x,t)dx = t^{-\frac{n}{2}} \sum_{k=0}^{N} c_k t^{\frac{k}{2}} + O(t^{\frac{N+1-n}{2}}), \tag{56.26}$$

where

$$c_0 = \frac{1}{(\sqrt{2\pi})^n} Vol\ (\Omega), \qquad c_1 = -\frac{1}{2^{n+1}\pi^{\frac{n-1}{2}}} Vol\ (\partial\Omega). \tag{56.27}$$

We used that $\sum'_j \int_{\mathbf{R}^{n-1}} \varphi_j(s_j^{-1}(y',0)\sqrt{g_j(y',0)}dy' = \text{Vol}\ (\partial\Omega)$, where Vol $\partial\Omega)$ is the volume of the manifold $\partial\Omega$ with respect to the metric induced by (56.1) and $\sum'$ is the summation over all j such that $U_j \cap \partial\Omega \neq \emptyset$.

Now consider the case of the Laplace-Beltrami operator (56.1) with Neumann boundary condition

$$\sum_{j,k=0}^{n} g^{jk}(x)\frac{\partial u}{\partial x_j}\nu_k(x)\left(\sum_{p,r=1}^{n} g^{pr}(x)\nu_p(x)\nu_r(x)\right)^{-\frac{1}{2}}\Bigg|_{\partial\Omega} = 0, \tag{56.28}$$

where $\nu(x) = (\nu_1(x), ..., \nu_n(x))$ is the outward unit normal to $\partial\Omega$ with respect to the Euclidean metric.

Note that in semi-geodesic coordinates (y', y_n) (c.f. (56.11)) the Neumann boundary condition has a simple form

$$\frac{\partial u(y',0)}{\partial y_n} = 0. \tag{56.29}$$

The principal term in R_{j1} has the following form (c.f (56.17)):

$$\psi_j(s_j^{-1}(y))R_j^+(\delta(y_n)\sigma_j^{-1}p'\frac{\partial}{\partial y_n}A_{j0}^{(-1)}\varphi_j(s^{-1}(y^{(0)}))\delta(y-y^{(0)}), \tag{56.30}$$

where $R_j^+, \sigma_j^{-1}, A_j^{(-1)}$ are the same as in (56.17) Analoguously to (56.19)-(56.22) we get that the contribution of the principal term of the R_{j1} in the case of the Neumann boundary condition is

$$t^{-\frac{n}{2}} \frac{1}{2^{n+1}\pi^{\frac{n-1}{2}}} \int_{\mathbf{R}^n} \varphi_j(s_j^{-1}(y',0))\sqrt{g_j(y',0)}dy' + O(t^{-\frac{n-2}{2}}), \tag{56.31}$$

i.e. it has an opposite sign to (56.24).

A theorem analogous to Theorem 56.1 holds:

Theorem 56.2. *Let $G_N(x,y,t)$ be the heat kernel corresponding to the operator (56.1) with the Neumann boundary condition (56.28). Then the heat trace $\int_\Omega G_N(x,x,t)dx$ has the asymptotics of the form*

$$\int_\Omega G_N(x,x,t)dx = t^{-\frac{n}{2}} \left(\sum_{k=0}^{N} d_k t^{\frac{k}{2}} + O(t^{\frac{N+1}{2}} \right),$$

where $d_0 = c_0,\ d_1 = -c_1$ (c.f. (56.27)).