

## 55 The parametrix for the elliptic boundary value problems.

We shall construct the parametrix for the boundary value problem (54.1), (54.2) using the partition of unity  $\{\varphi_j, U_j\}$  in  $\bar{\Omega}$ . When  $U_j \cap \partial\Omega = \emptyset$  we construct the parametrix as in §41.

If  $U_j \cap \partial\Omega \neq \emptyset$  we introduce a local system of coordinates  $(y', y_n)$  in  $U_j$  (c.f. §54). Let  $A_j(y', y_n, D', D_n)$ ,  $B_{jk}(y', y_n, D', D_n)$  be the operators  $A(x, D)$ ,  $B_k(x, D)$  in  $y$ -coordinates in  $\mathbf{R}^n$ . Consider the boundary value problem in  $\mathbf{R}_+^n$ :

$$(55.1) \quad A_j(y', y_n, D', D_n)u(y', y_n) = f(y', y_n), \quad y_n > 0,$$

$$(55.2) \quad B_{jk}(y', y_n, D', D_n)u(y', y_n)|_{y_n=0} = g_k(y'), \quad 1 \leq k \leq m_+.$$

Let  $lf$  be an arbitrary extension of  $f$  to  $\mathbf{R}^n$ . Denote by  $R_{j0}$  the parametrix for  $A_j u = lf$  in  $\mathbf{R}^n$  constructed in §41. We shall look for the parametrix of (55.1), (55.2) in the form

$$(55.3) \quad u = R_{j0}lf + R_{j1}(h_1, \dots, h_{m_+}),$$

where

$$(55.4) \quad h_k(x') = g_k - p' B_{jk} R_{j0} lf,$$

and  $R_{j1}$  is the parametrix of the boundary value problem

$$(55.5) \quad A_j(y', y_n, D', D_n)u(x', x_n) = 0, \quad (y', y_n) \in \mathbf{R}_+^n,$$

$$(55.6) \quad B_{jk}(y', y_n, D', D_n)u(y', y_n)|_{y_n=0} = h_k(y'), \quad 1 \leq k \leq m_+.$$

Now we shall construct  $R_{j1}$ . For the simplicity of the notations we shall drop the subscript  $j$  in (55.5), (55.6) and replace  $(y', y_n)$  by  $(x', x_n)$ .

Let  $A_0, B_{k0}$  be the principal parts of  $A(x', x_n, \xi', \xi_n)$ ,  $B_k(x', x_n, \xi', \xi_n)$ .

Having in mind the applications in §56 we shall assume that  $\Re A_0(x, \xi', \xi_n) > 0$  when  $(\xi', \xi_n) \neq (0, 0)$ .

Let  $\lambda > 0$ . We shall consider  $\lambda$  as part of the principal symbol, i.e.  $A(x, \xi', \xi_n) = A_0 + \lambda + A_1$ , where  $\text{ord } A_1 \leq m - 1$ . Let  $A_0(x, 0, +1) = 1$  and let

$$(55.7) \quad A_0(x, \xi', \xi_n) + \lambda = A_+(x, \xi', \lambda^{\frac{1}{m}}, \xi_n) A_-(x, \xi', \lambda^{\frac{1}{m}}, \xi_n)$$

be the factorization of  $A_0 + \lambda$ , i.e.

$$A_+(x, \xi', \lambda^{\frac{1}{m}}, \xi_n) = \prod_{j=1}^{m+} (\xi_n - \sigma_j^+(x, \xi', \lambda^{\frac{1}{m}})),$$

$$A_-(x, \xi', \lambda^{\frac{1}{m}}, \xi_n) = \prod_{j=1}^{m+} (\xi_n - \sigma_j^-(x, \xi', \lambda^{\frac{1}{m}})),$$

where  $\Im \sigma_j^+ > 0$ ,  $\Im \sigma_j^- < 0$ ,  $\sigma_j^\pm(x, \xi', \lambda^{\frac{1}{m}})$  are homogeneous functions of  $(\xi, \lambda^{\frac{1}{m}})$  of degree 1.

We shall call the symbol  $C^+(x, \xi', \lambda^{\frac{1}{m}}, \xi_n)$  a plus-symbol if  $C^+(x, \xi', \lambda^{\frac{1}{m}}, \xi_n)$  is analytic in  $\xi_n$  for  $\Im \xi_n < 0$ ,  $C^+(x, \xi', \lambda^{\frac{1}{m}}, \xi_n)$  is  $C^\infty$  in all variables and is homogeneous in  $(\xi', \lambda^{\frac{1}{m}}, \xi_n)$ . Analogously  $C^-(x, \xi', \lambda^{\frac{1}{m}}, \xi_n)$  is a minus-symbol if it is analytic in  $\xi_n$  for  $\Im \xi_n > 0$ , smooth and homogeneous. For example,  $A_+$  and  $A_+^{-1}$  are plus-symbols and  $A_-$ ,  $A_-^{-1}$  are minus-symbol. We shall consider in this section only symbols that are rational functions in  $\xi_n$  having no poles when  $\xi_n$  is real.

Any such symbol  $C(x, \xi', \lambda^{\frac{1}{m}}, \xi_n)$  that decays when  $\xi_n \rightarrow \infty$  can be uniquely represented in the form

$$(55.8) \quad C(x, \xi', \lambda^{\frac{1}{m}}, \xi_n) = C^+(x, \xi', \lambda^{\frac{1}{m}}, \xi_n) + C^-(x, \xi', \lambda^{\frac{1}{m}}, \xi_n),$$

where  $C^+$  is a plus-symbol and  $C_-$  is a minus-symbol.

The uniqueness follows from the Liouville's theorem.

The existence follows from the explicit formulas

$$(55.9) \quad C^+ = \Pi^+ C, \quad C^- = \Pi^- C,$$

where

$$\Pi^\pm C = \mp \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{C(x, \xi', \lambda^{\frac{1}{m}}, \eta_n)}{\xi_n - \eta_n \mp i0} d\eta_n,$$

(c.f. Example 5.1)

We can compute the integral  $\Pi^+ C$  using the residues of  $C(x, \xi', \lambda^{\frac{1}{m}}, \eta_n)$  in the half-plane  $\Im \eta_n > 0$ . Analogously  $\Pi^- C$  can be computed by using residues of  $C(x, \xi', \lambda^{\frac{1}{m}}, \eta_n)$  in the half-plane  $\Im \eta_n < 0$ .

We shall call  $\psi do$  plus-operators, or minus-operators, if their symbols are plus-symbols, or minus symbols, respectively.

**Remark 55.1** The main property of plus and minus operators is the following:

If  $C^+$  is a plus-operator and  $\text{supp } u_+ \subset \overline{\mathbf{R}_+^n}$  then  $\text{supp } C^+u_+ \subset \overline{\mathbf{R}_+^n}$ . Analogously if  $C^-$  is a minus-operator and  $\text{supp } u_- \subset \overline{\mathbf{R}_-^n}$  then  $\text{supp } C^-u_- \subset \overline{\mathbf{R}_-^n}$ .

The proof of these properties is the same as the proof of Lemma 46.2.

Therefore if  $C^+ \in S^\alpha$  then  $C^+$  is a bounded operator from  $\mathring{H}_s(\mathbf{R}^n)$  to  $\mathring{H}_{s-\alpha}(\mathbf{R}^n)$ . Note that if  $C^- \in S^\alpha$  then  $p_+C^-lu$  is independent of the extension  $lu$  where  $p_+$  is the restriction to  $\mathbf{R}_+^n$ : If  $l_1u$  and  $l_2u$  are two extensions then  $l_1u - l_2u = u_-$  has support in  $\overline{\mathbf{R}_-^n}$  and therefore  $p_+C^-u_- = 0$ . Therefore  $pC^-lu$  defines a bounded operator from  $H_s(\mathbf{R}_+^n)$  to  $H_{s-\alpha}(\mathbf{R}_+^n)$ .

Let  $\Lambda_-^s$  be the minus-operator with the symbol

$$\Lambda_-(\xi', \xi_n) = \left( \xi_n + i\sqrt{|\xi'|^2 + \lambda^{\frac{2}{m}}} \right)^s.$$

Since  $p_+\Lambda_-^s lu$  does not depend of the extension we have that the norm  $\|u\|_s^+ = \inf \|lu\|_s$  is equivalent to the norm  $\|\Lambda_-^s lu\|_0$ .  $\square$

For any  $N$  we shall construct plus-operators with symbols  $R_p^+(x, \xi', \lambda^{\frac{1}{m}}, \xi_n)$ ,  $\deg R_p^+ = -m_+ - p$ , and the minus-operators with symbols  $A_p^-(x, \xi', \lambda^{\frac{1}{m}}, \xi_n)$ ,  $\deg A_p^- = m_+ - p$ ,  $0 \leq p \leq N$ , such that

$$(55.10) \quad A(x, D', D_n) \sum_{p=0}^N R_p^+(x, D', \lambda^{\frac{1}{m}}, D_n) = \sum_{p=0}^N A_p^-(x, D', \lambda^{\frac{1}{m}}, D_n) + T_{m_+-N-1},$$

$$(55.11) \quad \|T_{m_+-N-1}u\|_{0,s} \leq C\|u\|_{m_+-1,s-N}, \quad \forall s.$$

Since  $A$  is a differential operator the product  $A(x, D)R_p^+(x, D', \lambda^{\frac{1}{m}}, D_n)$  has the form

$$(55.12) \quad A(x, D)R_p^+ = \sum_{j=0}^m B_{jp}(x, D),$$

where

$$(55.13) \quad \begin{aligned} B_{0p}(x, \xi) &= (A_0(x, \xi) + \lambda)R_p^+(x, \xi', \lambda^{\frac{1}{m}}, \xi_n), \\ B_{jp}(x, \xi) &= \sum_{|k|=j} \frac{1}{k!} \frac{\partial^k A(x, \xi)}{\partial \xi^k} D_x^k R_p^+ + A_j(x, \xi)R_p^+, \quad j \geq 1, \end{aligned}$$

$$\text{and } A(x, \xi) = A_0(x, \xi) + \lambda + \sum_{j=1}^m A_j(x, \xi), \quad \text{ord } A_j = m - j.$$

Taking the product of  $A$  and  $\sum_{p=0}^N R_p^+$  and collecting symbols of same degree of homogeneity in  $(\xi', \lambda^{\frac{1}{m}}, \xi_n)$  we get

$$(55.14) \quad (A_0(x, \xi) + \lambda)R_0^+(x, \xi', \lambda^{\frac{1}{m}}, \xi_n) = A_0^-(x, \xi', \lambda^{\frac{1}{m}}, \xi_n),$$

$$(55.15) \quad (A_0(x, \xi) + \lambda)R_p^+(x, \xi', \lambda^{\frac{1}{m}}, \xi_n) + T_p(x, \xi', \lambda^{\frac{1}{m}}, \xi_n) = A_p^-(x, \xi', \lambda^{\frac{1}{m}}, \xi_n), \quad p \geq 1,$$

where  $T_p$  depends on  $R_j^+$ ,  $0 \leq j \leq p - 1$ :

$$(55.16) \quad T_p = \sum_{j+r=p} B_{jr}.$$

Note that  $\deg T_p = m_+ - p$ .

It follows from (55.14) that

$$(55.17) \quad R_0^+ = A_+^{-1}(x, \xi', \lambda^{\frac{1}{m}}, \xi_n), \quad A_0^- = A_-(x, \xi', \lambda^{\frac{1}{m}}, \xi_n).$$

To solve (55.15) multiply it by  $A_-^{-1}$  and use decomposition (55.6)

$$A_-^{-1}T_p = \Pi^+ A_-^{-1}T_p + \Pi^- A_-^{-1}T_p.$$

Therefore we get

$$(55.18) \quad A_+ R_p^+ + \Pi^+ A_-^{-1}T_p = A_-^{-1}A_p^- - \Pi^- A_-^{-1}T_p.$$

Since the decomposition (55.6) is unique we obtain

$$(55.19) \quad R_p^+ = -A_+^{-1}\Pi^+ A_-^{-1}T_p, \quad A_p^- = A_- \Pi^- A_-^{-1}T_p, \quad \forall p \geq 1.$$

Note that the degree of homogeneity in  $(\xi', \lambda^{\frac{1}{m}}, \xi_n)$  decreases when  $p$  increases but the decay in  $\xi_n$  of  $R_p^+$  is  $O(\frac{1}{\xi_n^{m_++1}})$  for all  $p$ . Also  $T_{m_+-N-1}$  is a  $\psi do$  with symbol (55.16) when  $p = N + 1$ . Therefore the estimate (55.11) holds.

We shall look for the parametrix of (55.5) in the form

$$(55.20) \quad u_+ = \sum_{p=0}^N R_p^+(x, D', \lambda^{\frac{1}{m}}, D_n) \left( \sum_{k=1}^{m_+} c_k(x') \delta^{(k-1)}(x_n) \right), \quad x_n > 0.$$

Let  $p_+$  be the restriction operator to  $\mathbf{R}_+^n$ . Since

$$p_+ \left( \sum_{p=0}^N A_p^- \right) \left( \sum_{k=1}^{m_+} c_k(x') \delta^{(k-1)} \right) = 0$$

we get

$$(55.21) \quad A(x, D)u_+ = T_{m_+-N-1} \left( \sum_{k=1}^{m_+} c_k(x') \delta^{(k-1)}(x_n) \right) \quad \text{for } x_n > 0.$$

Note that

$$(55.22) \quad \|T_{m_+-N-1} \left( \sum_{k=1}^{m_+} c_k \delta^{(k-1)} \right)\|_s^+ \leq C \sum_{k=1}^{m_+} [(\Lambda')^{s+m_++k-N-\frac{3}{2}} c_k]_0,$$

where  $\Lambda' = (|\xi'|^2 + 1)^{\frac{1}{2}}$ .

To prove (55.22) note that the symbol  $T_{m_+-N-1}(x, \xi', \lambda^{\frac{1}{m}}, \xi_n) \xi_n^{k-1}$  is a rational function of  $\xi_n$ . Take  $M > s$ , where  $M$  is an integer. Then  $\Lambda_-^M(\xi', \xi_n) T_{m_+-N-1} \xi_n^{k-1} = P_k(\xi', \xi_n) + r_k(\xi', \xi_n)$ , where  $P_k$  is a polynomial in  $\xi_n$  and  $r_k(\xi', \xi_n) = O(\frac{1}{\xi_n})$ . Note that  $p_+ \Lambda_-^{s-M} P_k(c_k \delta) = 0$  since  $\Lambda_-^{s-M} P_k$  is a minus-symbol. Denoting  $v_k = T_{m_+-N-1}(c_k \delta^{(k-1)})$  we get

$$(55.23) \quad \|v_k\|_s^+ \leq \|\Lambda_-^s v_k\|_0 = \|\Lambda_-^{s-M} \Lambda_-^M v_k\|_0 = \|\Lambda_-^{s-M} r_k(x, D', \lambda^{\frac{1}{m}}, D_n)(c_k \delta)\|_0.$$

Now repeating the proof of Theorem 40.1 we get (55.22).  $\square$

Substitute (55.20) into the boundary conditions (55.6). Let  $p'$  be the restriction operator to  $x_n = 0$ . Let  $D(x, \xi', \lambda^{\frac{1}{m}}, \xi_n)$  be a symbol rational in  $\xi_n$ ,  $D = O(\frac{1}{\xi_n})$ . Then  $p' D(c(x') \delta(x_n)) = \lim_{x_n \rightarrow +0} D(c \delta(x_n)) = d(x', D', \lambda^{\frac{1}{m}}) c_k(x')$ , where

$$(55.24) \quad d = \frac{1}{2\pi} \int_{\gamma_+} D(x', 0, \xi', \lambda^{\frac{1}{m}}, \xi_n) d\xi_n,$$

$\gamma_+$  is a closed contour in  $\Im \xi_n > 0$ . This follows from the Jordan lemma.

We shall denote  $\Pi' D = \frac{1}{2\pi} \int_{\gamma_+} D d\xi_n$ . Note that  $\Pi' D_+ = 0$  if  $D_+ = O(\frac{1}{\xi_n^2})$  and  $D_+$  is a plus-symbol. We have

$$(55.25) \quad p' B_j(x'_1, 0, D', D_n) u_k(x) = b_{jk}(x', D', \lambda^{\frac{1}{m}}) c_k(x'),$$

where  $u_k = R^+ D_n^{k-1} (c_k \delta)$ ,  $R^+ = \sum_{p=0}^N R_p^+$ ,

$$(55.26) \quad b_{jk}(x', \xi', \lambda^{\frac{1}{m}}) = \Pi' B_{jk}(x', 0, \xi', \lambda^{\frac{1}{m}}, \xi_n), \quad 1 \leq j, k \leq m_+,$$

and  $B_{jk}$  is the symbol of the composition of  $B_j$  and  $R^+ D_n^{k-1}$ .

Let  $b_{jk0}(x', \xi', \lambda^{\frac{1}{m}})$  be the principal part of  $b_{jk}$ . Since the boundary value problem (55.1), (55.2) is elliptic we have  $\det \|b_{jk0}(x', \xi')\|_{j,k=1}^{m_+} \neq 0$ ,  $\forall(x', \xi')$ ,  $\xi, \xi' \neq 0$  (c.f. (53.12)). Therefore as in §41 we can construct a matrix pseudodifferential operator  $[d_{kj}(x', D', \lambda^{\frac{1}{m}})]$  such that

$$(55.27) \quad [b_{jk}][d_{kp}] = I + [t_{jp}], \quad \text{ord } t_{jk} < -N, \quad 1 \leq j, k \leq m_+.$$

Therefore (55.20), where

$$(55.28) \quad c_k(x') = \sum_{p=1}^{m_+} d_{kp}(x', D', \lambda^{\frac{1}{m}}) h_p, \quad 1 \leq k \leq m_+,$$

is the parametrix of the boundary value problem (55.5), (55.6). Restoring the subscript  $j$  in (55.5), (55.6) we shall denote this parametrix by  $R_{j1}$ . Then (55.3), (55.4) is the parametrix of boundary value problem (55.1), (55.2), in the local coordinates in  $U_j$ , when  $U_j \cap \partial\Omega \neq \emptyset$ . We have (c.f. (55.22), (55.27))

$$(55.29) \quad p_+ A_j R_{j1}(h_1, \dots, h_{m_+}) = T_{-N}(h_1, \dots, h_{m_+}),$$

$$(55.30) \quad p' B_{jk} R_{j1}(h_1, \dots, h_{m_+}) = h_k + T_{-N}^{(k)}(h_1, \dots, h_{m_+}), \quad 1 \leq k \leq m_+,$$

where  $\text{ord } T_{-N}^{(k)} \leq -N$ ,  $\text{ord } T_{-N} \leq -N$ .

Now construct the parametrix of the boundary value problem (54.1), (54.2). Let  $\{\varphi_j, U_j\}$  be a partition of unity,  $\psi_j(x) \in C_0^\infty(U_j)$  and  $\text{supp } (1 - \psi_j) \cap \text{supp } \varphi_j = \emptyset$ .

Denote  $F = (f(x), g_1(x'), \dots, g_{m_+}(x'))$ . We look for the parametrix in the form

$$(55.31) \quad \mathcal{R}F = \sum_j \psi_j R_j \varphi_j F.$$

When  $\text{supp } U_j \cap \partial\Omega = \emptyset$  we assume that  $R_j$  is the parametrix for  $Au = \varphi_j f$  in  $\mathbf{R}^n$ . When  $U_j \cap \partial\Omega \neq \emptyset$  we use a local system of coordinates in  $U_j$ , and  $R_j$  is the parametrix for (55.1), (55.2) in these local coordinates, i.e.  $R_j$  has the form (55.3), (55.4). If  $U_j \cap \partial\Omega = \emptyset$  then  $A(x, D)R_j = I + T_{-N}$ . Therefore

$$A\psi_j R_j \varphi_j f = \psi_j A R_j \varphi_j f + [A, \psi_j] R_j \varphi_j f = \varphi_j f + \psi_j T_{-N} \varphi_j f + [A, \psi_j] R_j f,$$

where  $[A, \psi_j]$  is the commutator of  $A$  and  $\psi_j$ . Since  $\psi_j = 1$  in a neighborhood of  $\text{supp } \varphi_j$  we have that  $\text{ord } [A, \psi_j] R_j \varphi_j = -\infty$  by the pseudo-local property of  $\psi$ . Therefore  $A\psi_j R_j \varphi_j f = \varphi_j f + T_{-N}^{(1)} f$ ,  $\text{ord } T_{-N}^{(1)} \leq -N$ . If  $U_j$  is such that  $U_j \cap \partial\Omega \neq \emptyset$  then  $R_j = R_{j0} + R_{j1}$ . For  $R_{j0}$  we have as above  $A_j \psi_j R_{j0} \varphi_j F = \varphi_j f + T_{-N}^{(1)} f$  in the local system of coordinates. Note that  $R_{j1}$  has the pseudo-locality property in the tangential variables  $(y_1, \dots, y_{n-1})$ . Therefore  $\text{ord } [A_j, \psi_j] R_{j1} \varphi_j F = T_{-\infty} F$ . Using (55.29), (55.30) we get

$$(55.32) \quad p_\Omega A \mathcal{R} F = f + T_{-N} F,$$

$$(55.33) \quad p_{\partial\Omega} B_k \mathcal{R} F = g_k + T_{-N}^{(k)} F,$$

where  $\text{ord } T_{-N} \leq -N$ ,  $\text{ord } T_{-N}^{(k)} \leq -N$ .