54 Elliptic boundary value problems in a smooth bounded domain.

Let Ω be a bounded domain with smooth boundary $\partial\Omega$. Consider the boundary value problem

$$A(x,D)u(x) = f(x), \qquad x \in \Omega, \tag{54.1}$$

$$B_j(x,D)u(x)|_{\partial\Omega} = g_j(x'), \quad x' \in \partial\Omega, \ \ 1 \le j \le m_+, \tag{54.2}$$

where $A(x,0)$ is an elliptic operator of order $m = 2m_+$, $B_j(x,D)$ are differential operators of order m_j, $1 \le j \le m_+$.

Denote by $A_0(x,D), B_{j0}(x,D)$ the principal terms of $A(x,D), B_j(x,D)$, $1 \le j \le m_+$. We have $A_0(x,\xi) \ne 0$, $\forall x \in \overline{\Omega}, \xi \ne 0$.

Denote by $\mathcal{H}_{(s)}(\Omega)$ the space of the right-hand sides of (54.1), (54.2):
(54.3)

$$\mathcal{H}_{(s)} = H_{s-m}(\Omega) \times \Pi_{j=1}^{m_+} H_{s-m_j-\frac{1}{2}}(\partial\Omega), \quad \text{where} \ \ s > \max_{1\le j\le m_+}(m_j + \frac{1}{2}).$$

The operator $\mathcal{A}u = (A(x,D)u|_\Omega, B_j u|_{\partial\Omega}, 1 \le j \le m_+)$ is bounded from $H_s(\Omega)$ to $\mathcal{H}_{(s)}(\Omega)$. We shall study the condition for the operator $\mathcal{A}$ to be Fredholm.

Let $x_0 \in \Omega$ and let U_0 be a neighborhood of x_0, such that $\overline{U}_0 \cap \partial\Omega = \emptyset$. Let $\psi(x) \in C_0^\infty(U_0)$, $|\psi(x)| \le 1$.

Lemma 54.1. *For any $\varepsilon > 0$ there exists U_0 such that*

$$\psi(x)A(x,D)u = \psi(x)(\hat{A}_0(x_0,D) + K + T)u, \tag{54.4}$$

where $A_0(x_0,D) = \sum_{|k|=m} a_k(x_0)D^k$, $\hat{A}_0(x_0,\xi) = A_0(x,(1+|\xi|)\frac{\xi}{|\xi|})$,

$$\|Ku\|_{s-m} \le C\varepsilon \|u\|_s, \tag{54.5}$$

ord $T \le m-1$.

Proof: Let $\psi_1(x) \in C_0^\infty(U_0)$, supp $(1-\psi_1(x)) \cap \text{supp}\, \psi(x) = \emptyset$. Let

$$Ku = \sum_{|k|=m} \Lambda^{-(s-m)}\psi_1(x)(a_k(x) - a_k(x_0))\Lambda^{(s-m)}D^k u, \tag{54.6}$$

where $\Lambda(D)$ is a ψdo with symbol $\Lambda(\xi) = (1 + |\xi|^2)^{\frac{1}{2}}$.

Let $\varepsilon = \max_{x,k} |\psi_1(x)|\ |a_k(x) - a_k(x_0)|$.

We have

$$\begin{aligned}(54.7)\qquad \|Ku\|_{s-m} &\leq \sum_{|k|=m} C\|\psi_1(x)(a_k(x) - a_k(x_0))\Lambda^{s-m}\hat{D}^k u\|_0 \\ &\leq C\varepsilon \sum_{|k|=m} \|\Lambda^{s-m} D^k u\|_0 \leq C_1\varepsilon\|u\|_s.\end{aligned}$$

We used in (54.7) that $||\psi_1(x)(a_k(x) - a_k(x_0))v_0||_0 \leq \varepsilon\|v\|_0$.

Commuting Λ^{s-m} and $\psi_1(x)(a_k(x)-a_k(x_0))$ we get that $T = \psi_2(x)(A(x,D)-\hat{A}_0(x_0,D) - K)$ has order $\leq m-1$, where $\psi_2(x) \in C_0^\infty(U_0)$, supp $(1-\psi_2)\cap$ supp $\psi_1 = \emptyset$. $\square$

Take any $x_0 \in \partial\Omega$ and let U_1 be a neighborhood of x_0. Introduce in U_1 a local system of coordinates $y = s_1(x)$ such that $y_n = 0$ is the equation of $\partial\Omega \cap U_1$, $y_n \geq 0$ when $x \in U_1$ and $(y_0', 0) = s(x_0)$. Let $A^{(1)}(y,\eta), B_j^{(1)}(y,\eta)$ be the symbols of $A(x,D), B_j(x,D)$ in y-coordinates. Denote by $\mathcal{A}^{(1)}$ the operator

$$\mathcal{A}^{(1)}u = (A^{(1)}(y,D_y)u(y)|y_n > 0, B_j^{(1)}(y,D_y)u|_{y_n=0, 1\leq j\leq m_+})$$

and $A_0^{(1)}, B_{j0}^{(1)}$ are the principal parts of $A^{(1)}, B_1^{(1)}$. Let $\hat{\mathcal{A}}_0^{(1)}$ be the operator $\mathcal{A}_0^{(1)}$ when the coefficients are "frozen" at the point $(y_0', 0)$ and ξ' is replaced by $(1 + |\xi'|)\frac{\xi'}{|\xi'|}$. Let

$$\varepsilon = \max\{|\psi_1(y)||a_k^{(1)}(y', y_n) - a_k^{(1)}(y_0', 0)|, |\psi_1(y', 0)||b_{jk}^{(1)}(y', 0) - b_{jk}^{(1)}(y_0', 0)|\},$$

where $\psi_1(x) \in C_0^\infty(U_1)$, $\psi_1(y) = 1$ near $(y_0', 0)$, $a_k^{(1)}, b_{jk}^{(1)}$ are coefficients of $A_0^{(1)}, B_{j0}^{(1)}$. The maximum is taken over all y and k and j.

Denote by $\psi(y)$ a $C_0^\infty(U_0)$ function such that supp $\psi\cap$supp $(1-\psi_1) = \emptyset$.

Lemma 54.2. *For any $\varepsilon > 0$ one can choose U_1 and $\psi_1(y) \in C_0^\infty(U_1)$ such that*

$$(54.8)\qquad \psi(y)\mathcal{A}^{(1)}u = \psi(y)(\hat{\mathcal{A}}_0^{(1)} + K_1 + T_1),$$

where $|||K_1 u|||_{(s)} \leq C\varepsilon\|u\|_s$, ord $T_1 \leq$ ord $\hat{\mathcal{A}}_0^{(1)} - 1$.

Here $|||K_1 u|||_{(s)}$ is the norm in $\mathcal{H}_{(s)}$.

The proof of Lemma 54.2 is the same as the proof of Lemma 54.1 with two adjustments:

1) We use $\Lambda_-^{s-m}(\xi) = (\xi_n + i(1+|\xi'|^2)^{\frac{1}{2}})^{s-m}$ instead of $\Lambda^{s-m}(\xi) = (1+|\xi|^2)^{\frac{1}{2}})^{\frac{s-m}{2}}$ in the definition of K_1 (c.f. (54.6)). Operator $\Lambda_-^{s-m}(D', D_n)$ has the property that $p_+\Lambda_-^{s-m} l_1 u = p_+\Lambda_-^{s-m} l_2 u$, where p_+ is the restriction to the half-space $y_n > 0$ and $l_1 u, l_2 u$ are two arbitrary extensions of u from $\mathbf{R}_+^n$ to $\mathbf{R}^n$.

The proof of this property is similar to the proof of Lemma 46.2: Since $l_1 u - l_2 u = 0$ for $y_n < 0$ we get that $\Lambda_-^{s-m}(l_1 u - l_2 u) = 0$ for $y_n < 0$. Therefore the norm in $H_{s-m}(\mathbf{R}_+^n)$ is equivalent to $\|p_+\Lambda_-^{s-m} l u\|_0$ (c.f. §55).

2) The second adjustment is the use of $(\Lambda')^{s-m_j} = (1+|\xi'|^2)^{\frac{s-m_j}{2}}$ instead of $\Lambda^{s-m}(\xi) = (1+|\xi|^2)^{\frac{s-m}{2}}$ when dealing with $\psi(y',0)B_j^{(1)}(y',0,D)u|_{y_n=0}$.

Theorem 54.3. *Assume that $A(x,D)$ is elliptic in $\overline{\Omega}$. Assume that for any $x_0 \in \partial\Omega$ the condition (53.12) is satisfied for $\hat{\mathcal{A}}_0^{(1)}$. Suppose $s > \max_{1\le j\le m_+}(m_j + \frac{1}{2})$. Then operator $\mathcal{A}$ is Fredholm operator from $H_s(\Omega)$ to $\mathcal{H}_{(s)}(\Omega)$. The following estimate holds:*

$$\|u\|_s^+ \le C(\|f\|_{s-m}^+ + \sum_{j=1}^{m_+} [g_j]_{s-m_j-\frac{1}{2}} + \|u\|_{s-1}^+), \quad \forall s > \max_j(m_j + \frac{1}{2}).$$

Proof: Let x_0 be any interior point of Ω. Denote by R_0 the pseudodifferential operator with symbol $(\hat{A}_0(x_0,\xi))^{-1}$. It follows from Lemma 54.1 that $\psi(x)A(x,D)u = \psi(\hat{A}_0(x_0,D) + K + T)$. Therefore $\psi(x)A(x,D)R_0 = \psi(I + KR_0 + TR_0)$. We choose $\varepsilon > 0$ in Lemma 54.1 such that $|||KR_0||| < 1$ where $|||KR_0|||$ is the norm of the operator KR_0 in $H_{s-m}(\mathbf{R}^n)$. Then we have

$$\psi(x)A(x,D)R_0(I+KR_0)^{-1} = \psi(I+T_1), \tag{54.9}$$

where $T_1 = TR_0(I+KR_0)^{-1}$, ord $T_1 \le -1$.

Consider now an arbitrarypoint $x_1 \in \partial\Omega$. It follows from Lemma 54.2 that the representation (54.8) holds. By Theorem 53.1 there exists $R^{(0)}$ such that $\hat{\mathcal{A}}_0^{(0)} R^{(0)} = I$. Choose U_1 in Lemma 54.2 such that $|||K_1 R^{(0)}||| < 1$. Then

$$\psi\mathcal{A}R^{(0)}(I + K_1 R^{(0)})^{-1} = \psi(I+T_1), \tag{54.10}$$

where $T_1 = TR^{(0)}(I+K_1R^{(0)})^{-1}$, ord $T_1 \le -1$. Therefore we have an open cover of $\overline{\Omega}$ by neighborhoods of the form U_0 when $x_0 \notin \partial\Omega$ and by neighborhoods of the form U_1 when $x_0 \in \partial\Omega$. Choose a finite subcover $\{U_j\}_{j=1}^N$ and let $\varphi_j(x),\ \varphi_j(x) \in C_0^\infty(U_j)$, be the corresponding partition of unity. Denote by R the following operator bounded from $\mathcal{H}_{(s)}$ to $H_s(\Omega)$:

$$RF = \sum_{j=1}^{N} \psi_j R_j \varphi_j F, \tag{54.11}$$

where $F = (f, g_1, .., g_{m_+}) \in \mathcal{H}_{(s)},\ \ \varphi_j F = (\varphi_j, p'\varphi_j g_1, ..., p'\varphi_j g_{m_+}),\ \ \psi_j(x) \in C_0^\infty(U_j),\ \mathrm{supp}\ \varphi_j \cap \mathrm{supp}\ (1-\psi_j) = \emptyset$ and R_j has a form $R_0(I+KR_0)^{-1}$ (c.f. (54.9)) in the case when $U_j \cap \partial\Omega = \emptyset$ and R_j has a form $R^{(0)}(I+K_1R^{(0)})^{-1}$ (c.f. (54.10)) in the local coordinates in U_j when $U_j \cap \partial\Omega \neq \emptyset$. We have

$$\mathcal{A}RF = \mathcal{A}\sum_{j=1}^{N} \psi_j R_j \varphi_j F = \sum_{j=1}^{N} (\psi_j \mathcal{A} + \mathcal{A}_j) R_j \varphi_j F,$$

where $\mathrm{ord}\ \mathcal{A}_j \leq \mathrm{ord}\ \mathcal{A} - 1$. Since $\psi_j \mathcal{A} R_j = \psi_j(I + T_j), \mathrm{ord}\ T_j \leq -1$ and since $\psi_j\varphi_j = \varphi_j$ we get

$$\mathcal{A}RF = F + TF,$$

where $T = \sum_j(\psi_j T_j \varphi_j + \mathcal{A}_j R_j \varphi_j,\ \ \mathrm{ord}\ T \leq -1$. Since $\mathrm{ord}\ T \leq -1$ the operator T is compact (c.f. §42). Therefore R is the right regularizer of $\mathcal{A}$. Analogously, we can construct the left regularizer in the form:

$$R^{(1)}F = \sum_{j=1}^{N} \varphi_j R_j^{(1)} \psi_j F, \tag{54.12}$$

where $\varphi_j(x), \psi_j(x)$ are the same as in (54.11), $\mathbf{R}_j^{(1)}$ has the form $(I+KR_0)^{-1}R_0$ in the case $U_j \cap \partial\Omega = \emptyset$ (c.f. (54.9)), and $R_j^{(1)}$ has the form $(I+K_1R^{(0)})^{-1}R^{(0)}$ in the case of $U_j \cap \partial\Omega \neq \emptyset$. We get

$$R^{(1)}\mathcal{A}u = \sum_{j=1}^{N} \varphi_j R_j^{(0)} \psi_j \mathcal{A}u = \sum_{j=1}^{N} \varphi_j R_j^{(1)} \psi_j (\mathcal{A}_j^{(0)} + K_j + T_j).$$

Since $\psi_j(\mathcal{A}_j^{(0)} + K_j) = (\mathcal{A}_j^{(0)} + K_j)\psi_j + T_{j1}$, where $\mathrm{ord}\ T_{j1} \leq \mathrm{ord}\ \mathcal{A} - 1$, and since $R_j^{(1)}(\mathcal{A}_j^{(0)} + K_j) = I$ we get

$$R^{(1)}\mathcal{A}u = u + T^{(1)}u, \tag{54.13}$$

where $\mathrm{ord}\ T^{(1)} \leq -1$, i.e. $T^{(1)}$ is compact.

Since $\mathcal{A}$ has the right and the left regularizers it is Fredholm.

Remark 54.1 Theorem 54.3 was proven under the assumption that the condition (53.12) is satisfied for each $x_0 \in \partial\Omega$ in some local system of coordinates near x_0. We shall call the boundary conditions (54.2) that satisfy (53.12) for all $x_0 \in \partial\Omega$ the elliptic boundary conditions. The condition (53.12) is also called Shapiro-Lopatinsky condition, or the coercivity condition. We shall show now that if (53.12) is satisfied at $x_0 \in \partial\Omega$ in some local system of coordinates $y = s(x)$ then it is satisfied in any other local system of coordinates $y^{(1)} = s^{(1)}(x)$.

Denote by $y^{(1)} = t(y)$ the transformation $y^{(1)} = t(y) = s^{(1)}(s^{-1}(y))$. Let $\frac{Dt(y)}{Dy}$ be the Jacobi matrix. Since $y_n^{(1)} = t_n(y',0) = 0$ we have that $\frac{\partial t_n(y',0)}{\partial y_k} = 0,\ 1 \le k \le n-1$. Let $y^{(0)} = s(x_0), y_0^{(1)} = s^{(1)}(x_0)$. Denote by $[b_{jk}(y',0)]_{j,k=1}^n$ the matrix $\left[\left(\frac{Dt(y)}{Dy}\right)^*\right]^{-1}$. If $A_0(y,\xi), B_{j0}(y,\xi)$ are the principal symbols of $A(x,D), B_j(x,D)$ in y-coordinates and $A_0^{(1)}(\hat{y},\eta), B_{j0}^{(1)}(\hat{y},\eta)$ are the principal symbols in $y^{(1)}$-coordinates then (c.f. §45)

$$A_0^{(1)}(t(y),[b_{jk}]\xi) = A_0(y,\xi), \quad B_{j0}^{(1)}(t(y),[b_{jk}]\xi) = B_{j0}(y,\xi). \tag{54.14}$$

We have

$$\eta_j = \sum_{k=1}^{n-1} b_{jk}(y^{(0)})\xi_k, \quad 1 \le j \le n-1, \tag{54.15}$$

$$\eta_n = \sum_{k=1}^{n} b_{nk}(y^{(0)})\xi_k, \quad \text{where} \quad b_{nn}(y^{(0)}) > 0. \tag{54.16}$$

Analogously to (53.1) we have

$$A_+(y^{(0)},\xi',\xi_n) = \Pi_{j=1}^{m_+}(\xi_n - \lambda_j^+(y^{(0)},\xi')),$$

$$A_+^{(1)}(y^{(1)},\eta',\eta_n) = \Pi_{j=1}^{m_+}(\eta_n - \lambda_{j1}^+(y^{(1)},\eta')),$$

where $\Im\lambda_j > 0,\ \Im\lambda_{j1} > 0$. Therefore

$$A_+(y',\xi) = \Pi_{j=1}^{m_+}(b_{nn}(y^{(0)})\xi_n + \sum_{k=1}^{n-1} b_{nk}\xi_k - \lambda_{j1}^+(y^{(1)}, b'(y^{(0)})\xi')),$$

where b' is the $(n-1)\times(n-1)$ matrix in (54.15). Analogously to (53.11) we have

$$(54.17)\qquad b_{jk}^{(1)}(y^{(1)},\eta') = \frac{1}{2\pi i}\int_{\gamma_+}\frac{B_{j0}^{(1)}(y_0^{(1)},\eta',z^{(1)})(z^{(1)})^{k-1}}{A_+^{(1)}(y_0^{(1)},\eta',z^{(1)})}dz^{(1)}$$

Making the change of variables $z^{(1)} = b_{nn}z + \sum_{k=1}^{n-1} b_{nk}\xi_k$, $\eta' = b'\xi'$, we get that

$$b_{jk}^{(1)}(y^{(1)},b'\xi') \equiv b_{nn}b_{jk}(y^{(0)},\xi').$$

Therefore the condition (53.12) is satisfied in $y^{(1)} = s^{(1)}(x)$ ccordinates if it is satisfied in $y = s(x)$ coordinates.

Now we shall give a more invariant description of the condition (53.12).

For any $x' \in \partial\Omega$ denote by $n(x')$ the unit interior normal to $\partial\Omega$ at x'. Let Ω_ε be the subdomain of Ω consisting of points $x' + x_n n(x')$ where $0 \leq x_n < \varepsilon$, i.e. Ω_ε is diffeomorphic to $\partial\Omega \times [0,\varepsilon)$. In a neighborhood U_0 of $x' \in \partial\Omega$ we introduce local coordinates (y', y_n), where $y_n = x_n$ and y' are local coordinates in $U_0 \cap \partial\Omega$. Operators $A(x,D), B_j(x,D)$ have the principal terms $A_0(y', y_n, D_{y'}, D_{y_n}), B_{j0}(y', y_n, D_{y'}, D_{y_n})$ in these local coordinates. Let $(y^{(1)}, y_n)$ be another local system of coordinates in U_1, $x' \in U_1$, where $y_n = x_n$ and $y^{(1)}$ are local coordinates in $U_1 \cap \partial\Omega$. We have

$$y^{(1)} = t'(y') \quad\text{and}\quad y_n' = y_n.$$

The principal symbols of $A(x,D), B_j(x,D)$ in $(y^{(1)}, y_n)$ and (y', y_n) coordinates are related by formula (54.14) where $t(y) = (t'(y'), y_n)$, and (54.16) reduces to $\eta_n = \xi_n$. Therefore operators $A(x,D), B_j(x,D)$ determined a family of symbols $A_0(y', \xi_{y'}', 0, \xi_n)$, $B_{j0}(y', \xi_{y'}, 0, \xi_n)$, $1 \leq j \leq m_+$ depending smoothly on $(y', \xi_{y'}') \in T_0^*(\partial\Omega)$.

Consider the family of boundary value problems on the half-line $y_n > 0$ depending smoothly on $(y', \eta_y') \in T_0^*(\partial\Omega)$

$$(54.18)\qquad A_0(y', \eta_{y'}', 0, D_{y_n})u(y_n) = 0, \quad y_n > 0,$$

$$(54.19)\qquad B_{j0}(y', \eta_{y'}', 0, D_{y_n})u(y_n)|_{y_n=0} = h_j 0, \quad h_j \in \mathbf{C},\ 1 \leq j \leq m_+.$$

The condition (53.12), $\forall x_0 \in \partial\Omega$, i.e. the condition that the boundary value problem (54.1), (54.2) is elliptic, is equivalent to the condition that the boundary value problem (54.18), (54.19) in $L_2(\mathbf{R}_+)$ has a unique solution.

Note that the kernel of $A_0(y', \eta'_{y'}, 0, D_{y_n})$ in $L_2(\mathbf{R}^1)$ is m_+-dimensional for any $(y', \eta'_{y'}) \in T_0^*(\partial\Omega)$ and the number of boundary conditions (54.19) is also m_+. Therefore the solvability of (54.18), (54.19) is equivalent to the uniqueness in $L_2(\mathbf{R}_+)$ of the solution of (54.18), (54.19), with $h_j = 0,\ 1 \le j \le m_+$.