Part VII

The elliptic boundary value problems and parametrices.

52 Pseudodifferential operators on a manifold.

Let M be a n-dimentional C^∞ manifold. This means that M is a topoligical space such that any point $x_0 \in M$ has a neighborhood U_0 that is homeomorphic to a neighborhood $B_0 \subset \mathbf{R}^n$. Let $y = s_0(x)$ be the map of U_0 onto B_0. We shall call $y = (y_1, ..., y_n)$ local coordinates in U_0. If U_j and U_k are two neighborhoods in M such that $U_j \cap U_k \neq \emptyset$ and if $y^{(j)} = s_j(x),\ y^{(k)} = s_k(x)$ are local coordinates in U_j and U_k respectively, then $y^{(j)} = s_{jk}(y^{(k)}) = s_j(s_k^{-1}(y^{(k)}))$ is the map of $s_k^{-1}(U_j \cap U_k) \subset \mathbf{R}^n$ onto $s_j(U_j \cap U_k) \in \mathbf{R}^n$.

The manifold is called C^∞ manifold if $s_{jk}(y^{(k)}) \in C^\infty$ and the Jacobian $\det \frac{Ds_{jk}(y^{(k)})}{Dy^{(k)}} \neq 0$. We shall assume in this section that M is compact. A function $\varphi(x)$ on M is called C^∞ function if $\varphi(s_j^{-1}(y^j)) \in C^\infty$ for any neighborhood U_j.

By the definition the vector bundle V on M is a topological space with the following properties: There exists a continuous projection p of V onto M and for each $x_0 \in M$ there is a neighborhood U_0 such that the preimage $p^{-1}(U_0)$ is homeomorphic to $U_0 \times \mathbf{R}^m$ for some $m \geq 1$: $\Phi_0(p^{-1}(U_0)) = U_0 \times \mathbf{R}^m$. Moreover, if U_j and U_k are two such neihghborhoods, $U_j \cap U_k \neq \emptyset$ and Φ_j and Φ_k are corresponding maps on $U_j \times \mathbf{R}^m$ and $U_k \times \mathbf{R}^m$ respectively, then for any $(x, \xi) \in (U_j \cap U_k) \times \mathbf{R}^m$ we have

$$(y, \eta) = \Phi_j \Phi_k^{-1}(x, \xi), \tag{52.1}$$

where $y = x, \eta = c_{jk}(x)\xi$, where $c_{jk}(x)$ is a linear invertible map in $\mathbf{R}^m$. Matrices $c_{jk}(x)$ are called the transition matrices of the bundle V. If one uses local coordinates $y^{(j)} = s_j(x),\ x \in U_j$, and $y^{(k)} = s_k(x),\ x \in U_k$, then we get a map of $s_k^{-1}(U_j \cap U_k) \times \mathbf{R}^m$ onto $s_j^{-1}(U_j \cap U_k) \times \mathbf{R}^m$ of the form

$$y^{(j)} = s_{jk}(y^{(k)}),\ \ \eta^{(j)} = \hat{c}_{jk}((y^{(k)})\eta^{(k)}, \tag{52.2}$$

where $s_{jk}(y^{(k)}) = s_j(s_k^{-1}(y^{(k)}))$, $\hat{c}_{jk}(y^{(k)}) = c_{jk}(s_k^{-1}(y^{(k)}) \in C^\infty$. We call $p^{-1}(x_0) = \xi_x$ the fiber at point $x \in M$ and we shall write the elements of V in the form (x, ξ_x).

An example of a vector bundle is the cotangent bundle $T^*(M)$, where $m = n$ and the transition matrices $\hat{c}_{jk}(y^{(k)})$ (c.f. §45) have the form

$$\hat{c}_{jk}(y^{(k)}) = \left(\left(\frac{Ds_{jk}(y_k)}{Dy^{(k)}}\right)^T\right)^{-1}. \tag{52.3}$$

Since M is assumed to be compact there exists a finite cover of M by neighborhood $U_j, 1 \le j \le N$.

Let $\{\varphi_j(x), U_j\}_{j=1}^N$ be a partition of unity, i.e supp $\varphi_j \subset U_j$, $\sum_{j=1}^N \varphi_j = 1$ on M and $\varphi_j \in C^\infty(M)$. For any $u(x) \in C^\infty(M)$ we introduce a Sobolev norm

$$\|u\|_s^2 = \sum_{j=1}^N \|\varphi_j(s_j^{-1}(y^{(j)}))u(s_j^{-1}(y^{(j)}))\|_s^2, \tag{52.4}$$

where $\|\varphi_j(s_j^{-1}(y^{(j)}))u(s_j^{-1}(y^{(j)}))\|_s$ is the norm in $H_s(\mathbf{R}^n)$.

We denote by $H_s(M)$ the closure of $C^\infty(M)$ in the norm (52.4). It is easy to prove (c.f. §13) that if we take another partition of unity and another set of local coordinates we get a norm equivalent to (52.4). We say that a linear operator T has an order $\le \alpha$ if

$$\|Tu\|_s \le C\|u\|_{s+\alpha}, \quad u \in C^\infty(M), \tag{52.5}$$

for any $s \in \mathbf{R}$.

We say that A is a pseudodifferential operator (*ψdo*) on M if the following two conditions hold:

a) If $\psi(x) \in C^\infty(M)$, $\varphi(x) \in C^\infty(M)$ and supp $\psi \cap$supp $\varphi = \emptyset$ then $\psi A \varphi$ is an operator of order $-\infty$.

b) If supp $\varphi(x)$ and supp $\psi(x)$ belong to the same neighborhood U_0 then for any N

$$\psi A \varphi u = \sum_{j=0}^N \psi A^{(j)} \varphi u + \psi T_{\alpha_N - 1} \varphi u,$$

where ord $T_\alpha \le \alpha - N - 1$ and $A^{(j)}$ are *ψdo* operators in local coordinates $y^{(0)} = s_0^{-1}(x), x \in U$. We assume that the symbol of $A^{(j)}$ is

$A^{(j)}(y^{(0)},\eta^{(0)})(1-\chi(\eta^{(0)})$ in local coordinates, $A^{(j)}(y^{(0)},\eta^{(0)})$ is homogeneous in $\eta^{(0)}$ of degree $\alpha-j$ and C^∞ when $\eta^{(0)}\neq 0$. We shall call $A^{(j)}(y^{(0)},\eta^{(0)})$ the principal symbol of A in local coordinates in U_0.

Now we shall define the principal symbol of ψdo on M. If $U_1\cap U_2\neq\emptyset$ and if $A_p^{(0)}, p=1,2$, are principal symbols of A in $U_p, p=1,2$, then we have (c.f. §45):

$$\psi_1(s_1^{-1}(y^{(1)})A_1^{(0)}(y^{(1)},\xi^{(1)})\varphi_1(s_1^{-1}(y^{(1)}))$$
$$=\psi_2(s_2^{-1}(y^{(2)})A_2^{(0)}(y^{(2)},\xi^{(2)})\varphi_2(s_2^{-1}(y^{(2)}))$$

in $U_1\cap U_2$, where

$$y^{(2)}=s_{12}(y^{(1)}),\quad \xi^{(2)}=\left(\left(\frac{Ds_{12}(y^{(1)})}{Dy^{(1)}}\right)^T\right)^{-1}\xi^{(1)},\quad s_{12}(y^{(1)})=s_2(s_1^{-1}(y^{(1)})).$$

Therefore operator $A^{(0)}$ determines a function $A_0(x,\xi_x)$ on $T_0^*(M)=\{(x,\xi_x)\in T^*(M),\xi_x\neq 0\}$. The symbol $A_j^{(0)}(y^{(j)},\xi^{(j)})$ is the realization of this function in local coordinates in U_j.

We shall call $A^{(0)}(x,\xi_x)$ the principal symbol of ψdo on M. Note that $A^{(0)}(x,\xi_x)$ is homogeneous in ξ_x of degree α. Vice versa, given such $A^{(0)}(x,\xi_x)$ we can costruct a ψdo on M whose principal symbol is $A^{(0)}(x,\xi_x)$. Operator A has the form $Au=\sum_{j=1}^N\psi_jA_j^{(0)}\varphi_ju$, where $A_j^{(0)}$ is a ψdo in $\mathbf{R}^n$ with symbol $A_j^{(0)}(y^{(j)},\xi^{(j)})(1-\chi(\xi^{(j)}),\ A_j^{(0)}(y^{(j)},\xi^{(j)})$ is the realization of $A^{(0)}(x,\xi_x)$ in the local system of coordinates $y^{(j)}$ in $U_j,\ \psi_j(x)\in C_0^\infty(U_j),\ \text{supp }(1-\psi_j)\cap\text{supp }\varphi_j=\emptyset,\ \sum_{j=1}^N\varphi_j=1$.

It follows from Theorems 40.1, 40.2 and §45 that A is an operator of order α. Also if A and B are two ψdo with principal symbols $A^{(0)}(x,\xi_x),\ B^{(0)}(x,\xi_x)$, $\deg_{\xi_x}A^{(0)}=\alpha,\ \deg_{\xi_x}B^{(0)}=\beta$ then

$$AB=C+T,$$

where $\text{ord } T\leq\alpha+\beta-1$ and C is a ψdo with the principal symbol $C^{(0)}(x,\xi_x)=A^{(0)}(x,\xi_x)B^{(0)}(x,\xi_x)$.

Theorem 52.1. *Let A be elliptic ψdo on M, i.e. $A^{(0)}(x,\xi_x)\neq 0,\ \forall(x,\xi_x)\in T_0^*(M)$. Then A is a Fredholm operator from $H_s(M)$ to $H_{s-\alpha}(M)$ for $\forall s$.*

Proof: Denote by R the ψdo with the principal symbol $R^{(0)}(x,\xi_x) = (A^{(0)}(x,\xi_x))^{-1}$. Then $AR = I + T^{(1)}$, $RA = I + T^{(2)}$ where $\operatorname{ord} T^{(p)} \leq -1,\ p = 1,2$.

It follows from §42 that any operator of negative order is compact in $H_s(M)$ for $\forall s$. Therefore R is the left and the right regularizer of A and therefore A is Fredholm.

53 Boundary value problem for the elliptic operators in the half-space.

Let $\mathbf{R}_+^n = \{(x',x_n) : x_n > 0, x' \in \mathbf{R}^{n-1}\}$. Let $A_0(\xi',\xi_n)$ be a homogeneous polynomial of degree m. A_0 is elliptic if $A_0(\xi',\xi_n) \neq 0$ for all $(\xi',\xi_n) \neq (0,0)$. Assuming that $A_0(0,+1) = 1$ we have $A_0(\xi',\xi_n) = \Pi_{j=1}^m(\xi_n - \lambda_j(\xi'))$, where $\deg_{\xi'} \lambda_j(\xi') = 1$ and $\Im \lambda_j(\xi') \neq 0$ when $\xi' \neq 0$. Denote

$$(53.1) \qquad \begin{aligned} A_+(\xi',\xi_n) &= \Pi_{j=1}^{m_+}(\xi_n - \lambda_j^+(\xi')), \\ A_-(\xi',\xi_n) &= \Pi_{j=1}^{m_-}(\xi_n - \lambda_j^-(\xi')), \end{aligned}$$

where $\Im\lambda_j^+(\xi')) > 0,\ 1 \leq j \leq m_+,\ \Im\lambda_j^-(\xi')) < 0,\ 1 \leq j \leq m_-,\ m_+ + m_- = m$. Note that $\lambda^\pm$ are continuous in ξ', $\xi' \neq 0$. If $n \geq 3$ then $\Im\lambda_j^+(\xi') > 0$ for all $\xi' \in \mathbf{R}^{n-1}\backslash\{0\}$ since $\mathbf{R}^{n-1}\backslash\{0\}$ is connected. Therefore m_+ (and analogously m_-) does not depend on ξ'. Since $A_0(-\xi',-\xi_n) = (-1)^m A_0(\xi',\xi_n)$ we have that $\lambda_j^+(-\xi') = -\lambda_j^+(\xi') = \lambda_{k(j)}^-(\xi')$ for some $k(j)$. Therefore $m_+ = m_-$ and $m = m_+ + m_-$ is even. If $n = 2$ this is not always true. For example, $A_0(\xi_1,\xi_2) = \xi_1 + i\xi_2$ is an elliptic polynomial. We shall assume in this and the following sections that $\deg A_0(\xi)$ is even also when $n = 2$, and that $m_+ = m_- = \frac{m}{2}$.

Since $A_0(\xi',\xi_n)$ vanishes when $(\xi',\xi_n) = (0,0)$ we replace it by $\hat{A}_0(\xi',\xi_n) = A_0((1+|\xi|)\frac{\xi'}{|\xi|},\xi_n)$.

Note that $\hat{A}_0(\xi',\xi_n) - A_0(\xi',\xi) = A_1(\xi',\xi_n)$ where $\operatorname{ord} A_1(\xi) \leq m-1$ and $C_1(1+|\xi|)^m \leq |\hat{A}_0(\xi)| \leq C_2(1+|\xi|)^m$. Also fo any homogeneous function $C(\xi',\xi_n)$ we shall denote by $\hat{C}(\xi',\xi_n) = C((1+|\xi'|)\frac{\xi'}{|\xi'|},\xi_n)$.

Let $B_{j0}(\xi',\xi_n)$ be homogeneous polynomials of degree m_j and let

$$\hat{B}_{j0}(\xi'\xi_n) = B_{j0}((1+|\xi'|)\frac{\xi'}{|\xi'|},\xi_n).$$

We consider the following boundary value problem in $\mathbf{R}_+^n$:

$$\hat{A}_0(D', D_n)u(x', x_n) = f(x', x_n), \quad x_n > 0, \tag{53.2}$$

$$\hat{B}_{j0}(D', D_n)\, u(x', x_n)|_{x_n=0} = g_j(x'), \ \ 1 \leq j \leq m_+ = \frac{m}{2}, \tag{53.3}$$

where $f \in H_{s-m}(\mathbf{R}_+^n)$, $g_j(x') \in H_{s-m_j-\frac{1}{2}}(\mathbf{R}^{n-1})$, $s > \max_{1\leq j\leq m_+}(m_j - \frac{1}{2})$. We will look for the solution $u \in H_s(\mathbf{R}_+^n)$. Denote

$$\mathcal{H}_{(s)} = H_{s-m}(\mathbf{R}_+^n) \times \Pi_{j=1}^{m_+} H_{s-m_j-\frac{1}{2}}(\mathbf{R}^{n-1}).$$

Operator $\mathcal{A}_0 u = (\hat{A}_0 u, \hat{B}_{j0}\, u|_{x_n=0}, 1 \leq j \leq m_+)$ is bounded from $H_s(\mathbf{R}_+^n)$ to $\mathcal{H}_{(s)}$.

We shall find the conditions when $\mathcal{A}$ is invertible.

Let $lf \in H_{s-m}(\mathbf{R}^n)$ be an extension of $f(x', x_n)$ such that

$$\|lf\|_{s-m} \leq C\|f\|_{s-m}^+,$$

where $\|f\|_{s-m}^+$ is the norm in $H_{s-m}(\mathbf{R}_+^n)$ (c.f. §33). Denote

$$u_0(x) = F^{-1}\frac{\tilde{lf}(\xi)}{\hat{A}_0(\xi', \xi_n)}. \tag{53.4}$$

Then $\hat{A}_0(D', D_n)u_0 = f$ in $\mathbf{R}_+^n$. Denote

$$v(x) = u(x) - u_0(x).$$

Then $v(x)$ satisfies

$$\hat{A}_0(D', D_n)v(x', x_n) = 0, \ \ x_n > 0, \tag{53.5}$$

$$\hat{B}_{j0}(D', D_n)\, v(x', x_n)|_{x_n=0} = h_j(x'), \ \ 1 \leq j \leq m_+, \tag{53.6}$$

where $h_j(x') = g_j(x') - \hat{B}_{j0}\, u_0|_{x_n=0} \in H_{s-m_j-\frac{1}{2}}(\mathbf{R}^{n-1})$. Performing the Fourier transform in x' we get

$$\hat{A}_0(\xi', D_n)\tilde{v}(\xi', x_n) = 0, \tag{53.7}$$

$$(53.8) \qquad \hat{B}_{j0}(\xi', D_n)\, \tilde{v}(\xi', x_n)|_{x_n=0} = \tilde{h}_j(\xi').$$

Note that (53.7), (53.8) is the boundary value problem on $\mathbf{R}^1_+$ for the ordinary differential equation. The general solution of (53.7) belonging to $H_s(\mathbf{R}_+)$ has the form

(53.9)
$$\tilde{v}(\xi', x_n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sum_{k=1}^{m_+} c_k(\xi')\xi_n^{k-1}e^{ix_n\xi_n}}{\hat{A}_+(\xi',\xi_n)} d\xi_n = \int_{\gamma_+} \sum_{k=1}^{m_+} \frac{z^{k-1}c_k(\xi')e^{ix_nz}}{\hat{A}_+(\xi', z)} dz,$$

where γ_+ is a closed simple contour in $\Im z > 0$ enclosing all zeros of $\hat{A}_+(\xi', z)$ (c.f. the Jordan Lemma). If ξ' is such that all $\lambda_j^+(\xi')$ are simple roots, $1 \le j \le m_+ = \frac{m}{2}$, then computing (53.9) by the residues we get

$$\tilde{v}(\xi', x_n) = \sum_{j=1}^{m_+} c_{j1}(\xi')e^{ix_n\hat{\lambda}_j^+(\xi')},$$

where $\hat{\lambda}_j^+(\xi') = \lambda_j^+((1+|\xi'|)\frac{\xi'}{|\xi'|})$. Substituting (53.9) into (53.8) we get the following linear system for $c_k(\xi')$:

$$(53.10) \qquad \sum_{k=1}^{m_+} \hat{b}_{jk}(\xi')c_k(\xi') = \tilde{h}_j(\xi'), \quad 1 \le j \le m_+,$$

where

$$(53.11) \qquad b_{jk}(\xi') = \frac{1}{2\pi i}\int_{\gamma_+} \frac{B_{j0}(\xi', z)z^{k-1}}{A_+(\xi', z)} dz,$$

$\hat{b}_{jk}(\xi') = b_{jk}((1+|\xi'|)\frac{\xi'}{|\xi'|})$. Note that $b_{jk}(\xi')$ are homogeneous functions of degree $m_j + k - m_+$.

We shall assume that

$$(53.12) \qquad \det[d_{jk}(\xi')]_{j,k=1}^{m_+} \ne 0, \quad \forall \xi' \ne 0.$$

Let $[d_{kj}(\xi')]_{k,j=1}^{m_+}$ be the matrix inverse to $[b_{jk}(\xi')]_{j,k=1}^{m_+}$. Then

$$(53.13) \qquad c_k(\xi') = \sum_{j=1}^{m_+} \hat{d}_{kj}(\xi')\tilde{h}_j(\xi'), \quad 1 \le k \le m_+,$$

and (53.9), (53.13) give the unique solution of the boundary value problem (53.7), (53.8). Note that $\deg_{\xi'} d_{kj}(\xi) = -m - k - m_j$. We have

$$\|v\|_s^+ \le C \sum_{j=1}^{m_+} [h_j]_{s-m_j-\frac{1}{2}}, \tag{53.14}$$

where $[h_j]_t$ is the norm in $H_t(\mathbf{R}^{n-1})$. To prove (53.14) choose $N \ge s$, N is an integer. Note that $\Lambda_-^N(\xi', z) = (z + i|\xi'|)^N$ has zeros only outside of the contour γ_+.

We have

$$\tilde{v}(\xi', x_n) = \int_{\gamma_+} \sum_{k=1}^{m_+} \frac{(z + i(|\xi'| + 1))^N z^{k-1} c_k(\xi') e^{ix_n z}}{\hat{\Lambda}_-^N(\xi', z)\hat{A}_+(\xi', z)} dz.$$

Denote by $p_k(\xi', z)$ the remainder of the division of $\Lambda_-^N(\xi', z) z^{k-1}$ by $A_+(\xi, z)$. Note that $p_k(\xi, z)$ is homogeneous in (ξ, z) of degree $N + k - 1 - m_+$ and $p_k(\xi, z)$ is a polynomial in z of degree $\le m_+ - 1$:

$$\Lambda_-^N(\xi', z) z^{k-1} = q_k(\xi', z) A_+(\xi', z) + p_k(\xi', z), \tag{53.15}$$

where q_k is a polynomials in z. Note that

$$\int_{\gamma_+} \frac{\hat{q}_k(\xi', z) c_k(\xi') e^{ix_n z}}{\hat{\Lambda}_-^N(\xi', z)} dz = 0$$

since $\hat{\Lambda}_-^N(\xi', z)$ has no poles inside γ_+. Using he Jordan lemma we get for $x_n > 0$

$$\tilde{v}(\xi', x_n) = \sum_{k=1}^{m_+} \int_{-\infty}^{\infty} \frac{\hat{p}_k(\xi', \xi_n) c_k(\xi') e^{ix_n \xi_n}}{\hat{\Lambda}_-^N(\xi', \xi_n)\hat{A}_+(\xi', \xi_n)} d\xi_n. \tag{53.16}$$

Extend $\tilde{v}(\xi', x_n)$ for $x_n < 0$ by the right-hand side of (53.16) and perform the Fourier transform in x_n. We get

$$(\|v\|_s^+)^2 \le C \sum_{k=1}^{m_+} \int_{\mathbf{R}^n} \frac{(1 + |\xi'| + |\xi_n|)^{2s} |\hat{q}_k(\xi', \xi_n)|^2 |c_k(\xi')|^2}{|\hat{\Lambda}_-^N(\xi', \xi_n)|^2 |\hat{A}_+(\xi', \xi_n)|^2} d\xi' d\xi_n.$$

Making the change of variables $\xi_n = (1 + |\xi'|)\eta_n$ and taking into account that $N \ge s$, we get

$$(\|v\|_s^+)^2 \le C \sum_{k=1}^{m_+} \int_{\mathbf{R}^{n-1}} (1 + |\xi'|)^{2s - 2m_+ + 2k - 1} |c_k(\xi')|^2 d\xi'. \tag{53.17}$$

Since $\deg_{\xi'} d_{kj}(\xi') = m_+ - k - m_j$ we get (53.14) from (53.17) and (53.13). Therefore $u = u_0(x', x_n) + v(x', x_n)$ where u_0 is given by (53.4) is the unique solution in $H_s(\mathbf{R}^n_+)$ of the boundary value problem (53.2)(53.3) and the following estimate holds

$$\|u\|_s^+ \le C\|f\|_{s-m}^+ + C\sum_{j=1}^{m_+}[g_k(x')]_{s-m_j-\frac{1}{2}},$$

where $s > \max_{1\le j\le m_+}(m_j + \frac{1}{2})$. Denote by R_0 the bounded operator from $\mathcal{H}_s$ to $H_s(\mathbf{R}^n_+)$ given by $u = u_0(x', x_0) + v(x', x_n),\ x_n > 0$. We proved the following theorem:

Theorem 53.1. *Let* $s > \max_{1\le j\le m}(m_j + \frac{1}{2})$. *Suppose the condition (53.12) is satisfied. Then* R_0 *is the inverse to the operator* $\mathcal{A}_0$*:*

$$\mathcal{A}_0 R_0 + I_1, \quad R_0\mathcal{A}_0 = I_2,$$

where I_1 *is the identity operator in* $\mathcal{H}_{(s)}$ *and* I_2 *is the identity operator in* $H_s(\mathbf{R}^n_+)$.