49 The finite domain of dependence property.

We shall study now the finite domain of dependence and the finite domain of influence properties of the solutions to the hyperbolic equations. We shall give here only an estimate of the domains of influence and dependence.

Theorem 49.1. *Let $f_+(x,t)$ and $u_+(x,t)$ be the same as in Theorem 48.1. Suppose supp $f_+(x,t) \subset B_{R+a_0|t-t_0|}$, $\forall t > -t_0$, where $B_R = \{x : |x| < R\}, t_0 > 0$. There exists $a > a_0$ such that supp $u_+(x,t) \subset B_{R+a|t+t_0|}$ for $\forall t > -t_0$.*

Proof: Consider the cone

$$K = \{|x|^2 < a^2(t+t_0)^2, \quad t > -t_0, \tag{49.1}$$

where $t_0 > 0$ and $a^2 > a_0^2$ is large. Since $a > a_0$ supp f_+ is contained in K.

We shall define a one-to-one map

$$(y, y_0) = s(x,t) \tag{49.2}$$

of $\overline{K}$ onto the half-space $y_0 \geq -t_0$, $y \in \mathbf{R}^n$ such that $s(x,t) \in C^\infty$ when $(x,t) \neq (0, -t_0)$ and the Jacobi matrix $\frac{D(y,y_0)}{D(x,t)}$ has a norm close to I, i.e.

$$\left\| \frac{D(y,y_0)}{D(x,t)} - I \right\| < C\alpha, \tag{49.3}$$

where α is small.

To describe explicitly the map (49.2) choose spherical coordinates (r, ψ, ω) in $\overline{K}$, where

$$\omega = \frac{x}{|x|} \in S^{n-1}, \quad r = \sqrt{|x|^2 + (t+t_0)^2},$$

$$\cos\psi = \frac{|x|}{\sqrt{|x|^2 + (t+t_0)^2}}, \qquad 0 \leq \psi \leq \frac{\pi}{2} - \alpha, \quad \alpha = \cot a.$$

Analogously we introduce spherical coordinates in the half-space $y_0 \geq -t_0$, $y \in R^n$:

$$\omega' = \frac{y}{|y|}, \quad r' = \sqrt{|y|^2 + (y_0+t_0)^2},$$

$$\cos\psi' = \frac{|y|}{\sqrt{|y|^2 + (y_0+t_0)^2}}, \qquad 0 \leq \psi \leq \frac{\pi}{2}.$$

The change of coordinates (49.2) has the following form in spherical coordinates:

$$\omega' = \omega, \quad r' = r, \quad \psi' = \frac{1}{1 - \frac{2\alpha}{\pi}}\psi. \tag{49.4}$$

Therefore $\frac{D(y,y_0)}{D(x,t)}$ is close to the identity when α is small. Make change of variables (49.2). Then $H(x,t,D_x,D_t)$ has the form $\hat{H}(y,y_0,D_y,D_{y_0}) = H(s^{-1}(y,y_0),(\frac{D(y,y_0)}{D(x,t)})^T(D_y,D_{y_0}))$ (c.f. §45). We assume that $H(x,t,D_x,D_t)$ is independent of (x,t) when $|x|^2+t^2 > R^2$, and is strictly hyperbolic with respect to σ, i.e. $H_0(x,t,\xi,\sigma) = 0$ has m distinct real roots $\sigma_k(x,t,\xi)$, $1 \leq k \leq m$, for all $x,t,\xi \neq 0$. Note that if we change slightly (x,t) and if we make a linear change of variables $(\xi',\sigma') = A(\xi,\sigma)$, where matrix A is close to the identity, then the new polynomial will be still strictly hyperbolic with respect to σ'. Therefore $\hat{H}_0(y,y_0,D_y,D_{y_0})$ will be strictly hyperbolic with respect to D_{y_0}.

Consider the equation

$$\hat{H}(y,y_0,D_y,D_{y_0})\hat{w}_+(y,\hat{y}_0) = \hat{f}_+(y,y_0) \tag{49.5}$$

in the half-space $y_0 < T,\ y \in \mathbf{R}^n$, where $\hat{f}_+(y,y_0)$ is the image of $f_+(x,t) \in \overset{\circ}{H}_{0,s}$ $(\mathbf{R}^{n+1})$ under the map (49.2). Note that supp $\hat{f}_+ \cap (-\infty,T]$ is compact since $a > a_0$. We look for

$$\hat{w}_+(y,y_0) \in H_{m,s-1}(\mathbf{R}^n \times (-\infty,T)),\ \hat{w}_+(y,y_0) = 0 \ \text{ for } y_0 < -t_0.$$

It follows from Theorem 48.1 (c.f. Remark 46.2) that such $\hat{w}_+(y,y_0)$ exists and is unique. Since supp $f_+ \subset K$ there exists $\varepsilon > 0$ small and such that supp $\hat{f}_+(y,y_0) \subset \{y_0 \geq -t_0+\varepsilon, y \in \mathbf{R}^n\}$. By the uniqueness $\hat{w}_+(y,y_0)$ also equal to zero when $y_0 < -t_0+\varepsilon$. Denote by Σ_ε and Σ_T the preimages of the planes $y_0 = -t+\varepsilon$ and $y_0 = T$ under the map (49.2). Let $\Omega_{\varepsilon T}$ be the domain between Σ_ε and Σ_T. Denote by $w(x,t)$ the preimage of $\hat{w}_+(y,y_0)$ restricted to the strip $-t_0+\varepsilon < y_0 < T,\ y \in \mathbf{R}^n$. Then $w(x,t)$ satisfies the equation $Hw = f(x,t)$ in $\Omega_{\varepsilon T}$ and has zero Cauchy data on Σ_ε. Denote by $w_+(x,t)$ the extension of $w(x,t)$ by zero below Σ_ε. Then

$$H(x,t,D_x,D_t)w_+(x,t) = f_+(x,t) \quad \text{for} \quad (x,t) \in \Omega_{-\infty,T}, \tag{49.6}$$

where $(x,t) \in \Omega_{-\infty,T}$ if (x,t) is below Σ_T.

Compare $w_+(x,t)$ with $u_+(x,t)$ that solves $H(x,t,D_x,D_t)u_+ = f_+(x,t)$ in $\mathbf{R}^{n+1}$. Let T_1 be such that the plane $t = T_1$ is below Σ_T, i.e. $t = T_1$ is contained in $\Omega_{-\infty,T}$. Then by the uniqueness part of Theorem 48.1 (c.f. Remark 46.2) we have that $u_+ = w_+(x,t)$ for $t < T_1$. Therefore $u_+(x,t) = 0$ for $a^2|x|^2 > (t+t_0)^2$, $t < T$, where $a > 0$ is large. Since we can choose T (and consequently T_1) arbitrary large we prove that $u_+ = 0$ for $a^2|x|^2 > |t+t_0|^2$, $\forall t > -t_0$.

$\square$

Corollary 49.2 (The domain of influence)**.** *Let $G(x,y,t,t')$ be the forward Green function of the Cauchy problem, i.e.*

$$\begin{aligned}
(49.7) \qquad & H(x,t,D_x,D_t)G(x,y,t,t') = 0 \quad \textit{for} \quad t > t', x \in \mathbf{R}^n, \\
& \left.\frac{\partial^k G(x,y,t,t')}{\partial t^k}\right|_{t=t'} = 0, \quad 0 \le k \le m-2, \\
& \left.\frac{\partial^{m-1} G(x,y,t,t')}{\partial t^{m-1} k}\right|_{t=t'} = \delta(x-y).
\end{aligned}$$

We shall use Theorem 49.1 to show that supp $G(x,y,t,t') \subset \overline{K}_{y,t'}$ where $K_{y,t'}$ is the cone $\{(x,t) : |x-y|^2 < a^2(t-t')^2, t \ge t'\}$ when a^2 is large. Therefore the cone $K_{y,t'}$ is the domain of influence of the point (y,t').

Let $h(x,t) = \frac{(t-t')^{k-1}}{(k-1)!}\delta(x-y)$ for $t \ge t'$ and let

$$(49.8) \qquad G^{(1)}(x,t) = G(x,y,t,t') - h(x,t) \quad \text{for} \quad t \ge t'.$$

We have

$$HG^{(1)} = g, \quad t > t',$$

where $g = -Hh$. Note that $G^{(1)}$ has zero initial data when $t = t'$. Denote $G_+^{(1)} = G^{(1)}$ for $t \ge t'$, $G_+^{(1)} = 0$ for $t < t'$. Analogously let $g_+ = g$ for $t > t'$ and $g_+ = 0$ for $t < t'$. Note that supp $g_+ = \{(x,t) : x = y,\ t \ge t'\}$ and $g_+ \in H_{0,s}(\mathbf{R}^{n+1})$ where $s < -\frac{n}{2} - m$. By the Theorem 48.1 there exists $G_+^{(1)}$ that satisfies $HG_+^{(1)} = g_+$ in $\mathbf{R}^{n+1}$, $G_+^{(1)} = 0$ for $t < t'$ and $e^{-\tau t}G_+^{(1)} \in H_{m,s-1}(\mathbf{R}^{n+1})$ for $\tau > 0$ large. It is enough to prove that supp $G_+^{(1)} \subset K_{y,t'}$. Consider the cone $K_{y,t'+t_0}, t+t_0 \ge 0,\ t_0 > 0$ is arbitrary. Then $K_{y,t+t_0}$ contains the support of g_+ for any $t_0 > 0$. By the Theorem 49.1 $K_{y,t'+t_0} \supset$ supp $G_+^{(1)}(x,t)$ if a is large and since t_0 is arbitrary we get that $\overline{K}_{y,t'} \supset$ supp $G_+^{(1)}(x,t)$. $\square$

Corollary 49.3 (The domain of dependence)**.** *Let $u(x,t)$ be a smooth solution of Cauchy problem (48.1), (48.2) in R_T. Denote by $G^*(x,y,t,t')$ the backward Green function, i.e.*

$$H^*(x,t,D_x,D_t)G^*(x,y,t,t') = 0 \quad \textit{for} \quad t < t', x \in \mathbf{R}^n, \tag{49.9}$$
$$\left.\frac{\partial^k G^*(x,y,t,t')}{\partial t^k}\right|_{t=t'} = 0, \quad 0 \le k \le m-2,$$
$$\left.\frac{\partial^{m-1} G^*(x,y,t,t')}{\partial t^{m-1} k}\right|_{t=t'} = \delta(x-y),$$

where H^* is the adjoint to H.

Analogously to the proof of Theorem 49.1 and Corollary 49.2 we get that supp $G^*(x,y,t,t')$ is contained in the cone $\overline{K_{y,t'}^-} = \{(x,t) : |x-y|^2 < a^2(t-t')^2,\ t \le t'\}$. Integrating by parts in R_T we get

$$\begin{aligned}(Hu, G^*(x,y,t,T))_{R_T} &= (u, H^*G^*)_{R_T} - i[u(x,T), D_t^{m-1}G(x,y,T,T)] \\ &- \sum_{k=0}^{m-1}\left[\frac{\partial^k u(x,0)}{\partial t^k}, C_k(x,0,D_x)G^*(x,y,0,T)\right],\end{aligned} \tag{49.10}$$

where $(\ ,\ )_{R_T}$ is the L^2-inner product in R_T, $[\ ,\ ]$ is the L^2-inner product in $\mathbf{R}^n$, $C_k(x,0,D_x)$ are differential operators of order $\le m-k$. Therefore we have

$$\begin{aligned} i^{m-2}u(y,T) &= (f(x,t), G^*(x,y,t,T))_{R_T} \\ &+ \sum_{k=0}^{m-1}[g_k(x), C_k(x,0,D_x)G^*(x,y,0,T)].\end{aligned} \tag{49.11}$$

Since supp $G^* \subset K_{y,T}^-$ the solution $u(y,t)$ depends on the values of $f(x,t)$ in $K_{y,T}^- \cap \overline{R_T}$ and of the values of $g_k(x)$, $0 \le k \le m-1$, in $K_{y,T}^- \cap \{t=0\}$. Note that for each t the domains of influence and dependence are bounded.

50 Propagation of singularities for the hyperbolic equations.

Consider one-to-one map

$$\hat{x} = \hat{x}(x,\xi), \quad \hat{\xi} = \hat{\xi}(x,\xi) \tag{50.1}$$

of $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ to $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ where $\hat{x}(x,\xi)$ is homogeneous in ξ of degree zero, $\hat{\xi}(x,\xi)$ is homogeneous in ξ of degree one: $\deg_\xi \hat{x}(x,\xi) = 0$, $\deg_\xi \hat{\xi}(x,\xi) = 1$, $\hat{x}(x,\xi), \hat{\xi}(x,\xi)$ are $C^\infty(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ functions. The transformation (50.1) is called canonical if it preserves the differential form $\sum_{j=1}^n dx_j \wedge d\xi_j$, i.e.

$$\sum_{k=1}^n d\hat{x}_k \wedge d\hat{\xi}_k = \sum_{k=1}^n dx_k \wedge d\xi_k, \tag{50.2}$$

where $\hat{x}, \hat{\xi}$ are given by (50.1). We shall give some examples of canonical transformations.

The first example is

$$\hat{x} = a(x), \quad \hat{\xi} = \left(\left(\frac{Da(x)}{Dx}\right)^T\right)^{-1} \xi, \tag{50.3}$$

where $\hat{x} = a(x)$ is a diffeomorphism.

To check that (50.3) is a canonical transformations we shall show that

$$\sum_{j=1}^n \xi_j dx_j = \sum_{j=1}^n \hat{\xi}_j d\hat{x}_j.$$

We have $d\hat{x} = \frac{Da(x)}{Dx} dx$. Therefore $\hat{\xi} \cdot d\hat{x} = \hat{\xi} \cdot \frac{Da(x)}{Dx} dx = \left(\frac{Da}{Dx}\right)^T \hat{\xi} \cdot dx = \xi \cdot dx$ since $\xi = \left(\frac{Da}{Dx}\right)^T \hat{\xi}$. It is clear that $\xi \cdot dx = \hat{\xi} \cdot d\hat{x}$ implies $d\xi \wedge dx = d\hat{\xi} \wedge d\hat{x}$ sincwe $d^2 = 0$.

Let $S(x,\hat{\xi}) \in C^\infty(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$, $S(x,\hat{\xi})$ is homogeneous in $\hat{\xi}$ of degree one, and $\det \frac{\partial^2 S(x,\hat{\xi})}{\partial x \partial \hat{\xi}} \neq 0$, $\forall x \in \mathbf{R}^n$, $\hat{\xi} \in \mathbf{R}^n \setminus \{0\}$. Define the map $(x,\xi) \to (\hat{x},\hat{\xi})$ by formulas

$$\xi = \frac{\partial S(x,\hat{\xi})}{\partial x}, \quad \hat{x} = \frac{\partial S(x,\hat{\xi})}{\partial \hat{\xi}}. \tag{50.4}$$

Assuming that the map $(x,\xi) \to (\hat{x},\hat{\xi})$ is a diffeomorphism we shall show that it is a canonical transform. The function $S(x,\hat{\xi})$ is called the generating function of this canonical transformation.

We have

$$d\xi \wedge dx = \left(\frac{\partial^2 S}{\partial x^2} dx + \frac{\partial^2 S}{\partial \hat{\xi} \partial x} d\hat{\xi}\right) \wedge dx$$

and

$$d\hat{\xi} \wedge d\hat{x} = d\hat{\xi} \wedge \left(\frac{\partial^2 S(x, \hat{\xi})}{\partial x^2} dx + \frac{\partial^2 S(x, \hat{\xi})}{\partial \hat{\xi}^2} d\hat{\xi} \right).$$

Note that $\frac{\partial^2 S(x,\hat{\xi})}{\partial x^2} dx \wedge dx = 0$ and $d\hat{\xi} \wedge \frac{\partial^2 S(x,\hat{\xi})}{\partial \hat{\xi}^2} d\hat{\xi} = 0$. Therefore $d\hat{\xi} \wedge d\hat{x} = d\xi \wedge dx$.

□

The following example of a canonical transformation will be used in this section.

Let $y = (x, x_0),\ \eta = (\xi, \xi_0)$, where $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n,\ x_0 \in \mathbf{R}, \xi_0 \in \mathbf{R}$. Let $P_0(y, \eta)$ be a homogeneous in η symbol, $\deg_\eta P(y, \eta) = m$. We assume that $P(y, \eta) \in C^\infty$ when $\eta \neq 0,\ P(y, \eta)$ is real-valued and $\frac{\partial P(y,\eta)}{\partial \eta} \neq 0$ when $P(y, \eta) = 0,\ \eta \neq 0$.

Such symbols P_0 will be called the symbols of real principal type.

The bicharacteristics of $P_0(y, \eta)$ is the solution of the system

$$\begin{aligned} \frac{dy}{ds} &= \frac{\partial P_0(y(s), \eta(s))}{\partial \eta}, \quad y(0) = y, \\ \frac{d\eta}{ds} &= -\frac{\partial P_0(y(s), \eta(s))}{\partial y}, \quad \eta(0) = \eta,\ s \in \mathbf{R}. \end{aligned} \tag{50.5}$$

Note that

$$\frac{d}{ds} P_0(y(s), \eta(s)) = \frac{\partial P_0(y(s), \eta(s))}{dy} \frac{dy}{ds} + \frac{\partial P_0(y(s), \eta(s))}{\partial \eta} \frac{d\eta}{ds} = 0,$$

when $y(s), \eta(s)$ are the solution of (50.5), i.e. $P_0(y(s), \eta(s)) = const$ along a bicharacteristics. If the initial value (y, η) of (50.5) are such that $P_0(y, \eta) = 0$ then $P_0(y(s), \eta(s)) = 0$.

In this case the bicharacteristics is called the null-bicharacteristics.

We shall show that the map

$$\begin{aligned} \hat{y} &= \hat{y}(s, y, \eta), \\ \hat{\eta} &= \hat{\eta}(s, y, \eta) \end{aligned} \tag{50.6}$$

is a canonical transformation for each $s \in \mathbf{R}$. We have

$$\frac{d}{ds}(d\hat{y} \wedge d\hat{\eta}) = \left(d\frac{d\hat{y}}{ds}\right) \wedge d\hat{\eta} + d\hat{y} \wedge d\frac{d\hat{\eta}}{ds} \tag{50.7}$$
$$= d\frac{\partial P_0(\hat{y}(s), \hat{\eta}(s))}{\partial \eta} \wedge d\hat{\eta} - d\hat{y} \wedge d\frac{\partial P_0(\hat{y}, \hat{\eta})}{\partial y}$$
$$= \left(\frac{\partial^2 P_0}{\partial y \partial \eta} d\hat{y} + \frac{\partial^2 P_0}{\partial \eta^2} d\hat{\eta}\right) \wedge d\hat{\eta} - d\hat{y} \wedge \left(\frac{\partial^2 P_0}{\partial y^2} d\hat{y} + \frac{\partial^2 P_0}{\partial \eta \partial y} d\hat{\eta}\right) = 0,$$

since

$$\frac{\partial^2 P_0}{\partial y^2} d\hat{y} \wedge d\hat{y} = 0, \qquad \frac{\partial^2 P_0}{\partial \eta^2} d\hat{\eta} \wedge d\hat{\eta} = 0.$$

Therefore $d\hat{\eta}(s) \wedge d\hat{y}(s)$ is independent of s, i.e. $d\hat{\eta}(s) \wedge d\hat{y}(s) = d\eta \wedge dy$. $\square$

Let $P(y, \eta) = \sum_{k=0}^{N} P_k(y, \eta)(1 - \chi(\eta)) + P^{(N+1)}(y, \eta)$, where $P_k(y, \eta)$ are homogeneous and smooth when $\eta \neq 0$, $\deg_\eta P_k(y, \eta) = m - k$, N is arbitrary, $P_0(y, \eta)$ is a symbol of real principal type, $P^{N+1}(y, \eta) \in S^{m-N-1}$.

We shall call $P(y, D_y)$ the ψdo of real principal type.

The following theorem holds:

Theorem 50.1. *Let $u(y) \in H_s(\mathbf{R}^{n-1})$ be the solution of $P(y, D_y)u = f$ in $\mathbf{R}^{n+1}$ for some $s \in \mathbf{R}$. Assume that $P(y, D_y)$ is a ψdo of real principal type. If $(y, \eta) \notin WF(u)$ and if the null-bicharacteristics $\hat{y} = \hat{y}(s, y, \eta)$, $\hat{\eta} = \hat{\eta}(s, y, \eta)$, $\hat{y}(0, y, \eta) = y$, $\hat{\eta}(0, y, \eta) = \eta$ does not belong to $WF(f)$ for any $s \in [0, L]$, then $(\hat{y}(s, y, \eta), \ \hat{\eta}(s, y, \eta))$ does not belong to $WF(u)$ when $s \in [0, L]$.*

We will denote by $\gamma(s, y, \eta)$ the null-bicharacteristics given by (50.6).

Proof: Let $P_0(y^{(0)}, \eta^{(0)}) = 0$. Since $P_0(y, \eta)$ is of real principal type there exists η_j such that $\frac{\partial P_0(y^{(0)}, \eta^{(0)})}{\partial \eta_j} \neq 0$, $0 \leq j \leq n$. Assume for definiteness that $\frac{\partial P_0(y^{(0)}, \eta^{(0)})}{\partial \xi_0} > 0$, where $y^{(0)} = (x^{(0)}, x_0^{(0)})$, $\eta^{(0)} = (\xi^{(0)}, \xi_0^{(0)})$. Assume also that $P_0(y^{(0)}, 0, \pm 1) \neq 0$. Therefore $\xi^{(0)} \neq 0$.

By the implicit function theorem there exists a conic neighborhood U_0 of $(y^{(0)}, \eta^{(0)})$ such that

$$P_0(y, \eta) = q_0(y, \eta)(\xi_0 - \lambda_0(y, \xi)), \tag{50.8}$$

where $q_0(y, \eta) \neq 0$ in U_0, $\lambda_0(y, \xi)$ is real-valued, $\xi \neq 0$ in U_0, $\deg_\eta q_0(y, \eta) = m - 1$, $\deg_\xi \lambda_0(y, \xi) = 1$, q_0 and λ_0 are C^∞ in U_0.

We shall write often $\hat{y}(s) = (\hat{x}(s), \hat{x}_0(s),\ \hat{\xi}(s) = (\hat{\xi}(s), \hat{\xi}_0(s))$ instead of $\hat{y}(s,y,\eta), \hat{\eta}(s,y,\eta)$. Since $\hat{y}(s), \hat{\eta}(s)$ is a null-bicharacteristics and $q_0 \neq 0$ in U_0 we have

$$\hat{\xi}_0(s) - \lambda_0(\hat{x}(s), \hat{x}_0(s), \hat{\xi}(s)) = 0.$$

Since $\frac{\partial P_0}{\partial \xi_0} > 0$ in $U(y^{(0)}, \eta^{(0)})$ we can take

$$x_0 = \int_0^s \frac{\partial P_0(\hat{y}(s'), \hat{\eta}(s'))}{\partial \xi_0} ds' + x_0^{(0)} \tag{50.9}$$

as a parameter on $\gamma(x^{(0)}, x_0^{(0)}, \xi^{(0)}, \xi_0^{(0)})$. Note that

$$\frac{\partial P_0(\hat{y}(s), \hat{\eta}(s))}{\partial \xi_0} = \frac{\partial q_0(\hat{y}(s), \hat{\eta}(s))}{\partial \xi_0} \cdot \tag{50.10}$$

$$\cdot \Big(\hat{\xi}_0(s) - \lambda_0(\hat{x}(s), \hat{x}_0(s), \hat{\xi}(s)) + q_0(\hat{y}(s), \hat{\eta}(s)\Big) = q_0(\hat{y}(s), \hat{\eta}(s)).$$

Analogously,

$$\frac{\partial P_0(\hat{y}(s), \hat{\eta}(s))}{\partial \xi} = q_0(\hat{y}(s), \hat{\eta}(s)) \left(- \frac{\partial \lambda_0(\hat{y}(s), \hat{\xi}(s))}{\partial \xi} \right),$$

$$\frac{\partial P_0(\hat{y}(s), \hat{\eta}(s))}{\partial y} = q_0(\hat{y}(s), \hat{\eta}(s)) \left(- \frac{\partial \lambda_0(\hat{y}(s), \hat{\xi}(s))}{\partial y} \right).$$

Therefore the system (50.5) takes the following equivalent form when x_0 is a parameter:

$$\frac{d\hat{x}(x_0)}{dx_0} = - \frac{\partial \lambda_0(\hat{x}(x_0), x_0, \hat{\xi}(x_0))}{\partial \xi}, \quad \hat{x}(x_0^{(0)}) = x^{(0)}, \tag{50.11}$$

$$\frac{d\hat{\xi}(x_0)}{dx_0} = \frac{\partial \lambda_0(\hat{x}(x_0), x_0, \hat{\xi}(x_0))}{\partial x}, \quad \hat{\xi}(x_0^{(0)}) = \xi^{(0)},$$

$$\frac{d\hat{\xi}_0(x_0)}{dx_0} = \frac{\partial \lambda_0(\hat{x}(x_0), x_0, \hat{\xi}(x_0))}{\partial x_0}, \quad \hat{\xi}_0(x_0^{(0)}) = \xi_0^{(0)}. \tag{50.12}$$

Note that

$$\hat{\xi}_0(x_0) = \lambda_0(\hat{x}(x_0), x_0, \hat{\xi}(x_0))$$

and the solution of (50.11) supplemented with equations $x_0 = x_0$ and $\hat{\xi}(x_0) - \lambda_0(\hat{x}(s), x_0, \hat{\xi}(s)) = 0$ is the equation for the null-bicharacteristics of $P_0(y, D_y)$ or $D_{x_0} - \lambda_0(y, D_x)$.

For the simplicity of notations put $x_0^{(0)} = 0$. Let U_1 be a neighborhood of $(x^{(0)}, 0, \eta^{(0)})$ in $U_0 \cap \{x_0 = 0\}$. Let $V(-3\varepsilon, T+3\varepsilon)$ be the union of all null-bicharacteristics starting at $(x, 0, \eta) \in U_1$ and having $-3\varepsilon < x_0 < T + 3\varepsilon$. We assume that T is such that $V(-3\varepsilon, T+3\varepsilon) \subset U_0$ and $WF(f) \cap \overline{V(-3\varepsilon, T+3\varepsilon)} = \emptyset$.

Let $\chi_{2\varepsilon}(x_0) \in C_0^\infty(-3\varepsilon, T+3\varepsilon)$ and $\chi_{2\varepsilon} = 1$ for $-2\varepsilon < x_0 < T + 2\varepsilon$ and let $\chi_\varepsilon(x_0) \in C_0^\infty(-2\varepsilon, T+2\varepsilon)$, $\chi_\varepsilon(x_0) = 1$ for $-\varepsilon < x_0 < T + \varepsilon$.

Let $\varphi(y, \eta) \in C^\infty(U_0)$ and

$$\operatorname{supp} \varphi(y, \eta) \subset V(-2\varepsilon, T+2\varepsilon).$$

We will choose $\varphi(y, \eta)$ such that

$$\varphi(y, D_y) P(y, D_y) u = P(y, D_y) \varphi(y, D_y) u + T_{m-N-1} u + T_\varepsilon u, \tag{50.13}$$

where $T_\varepsilon u = 0$ when $-\varepsilon < x_0 < T + \varepsilon$, ord $T_{m-N-1} \leq m - N - 1$ and N is arbitrary.

We assumed that $P(y, \xi) = (P_0(y, \xi) + P_1(y, \eta) + ... + P_N(y, \eta))(1 - \chi(\eta)) + P^{(N+1)}(y, \eta)$, where P_k are homogeneous and smooth when $\eta \neq 0$, $\deg_\eta P_k = m - k$, $P^{(N+1)}(y, \eta) \in S^{m-N-1}$. We are looking for $\varphi(y, \eta)$ in the form:

$$\varphi(y, \eta) = \sum_{k=0}^{N} \varphi_{-k}(y, \eta)(1 - \chi(\eta)) \chi_\varepsilon(x_0), \tag{50.14}$$

where φ_{-k} are homogeneous in η of degree $-k$, supp $\varphi_{-k} \chi_\varepsilon(x_0) \subset V(-2\varepsilon, T + 2\varepsilon)$. Using Theorem 40.2 and collecting symbols of the same degree of homogeneity in $\eta = (\xi, \xi_0)$ we get: $\varphi_0(y, \eta) P_0(y, \eta) = P_0(y, \eta) \varphi_0(y, \eta)$,

$$\sum_{j=0}^{n} \frac{\partial \varphi_0(y, \eta)}{\partial \xi_j} \frac{\partial P_0(y, \eta)}{\partial x_j} - \sum_{j=0}^{n} \frac{\partial \varphi_0(y, \eta)}{\partial \xi_j} \frac{\partial P_0(y, \eta)}{\partial x_j} = 0, \tag{50.15}$$

(50.16)
$$\sum_{j=0}^{n} \left(\frac{\partial \varphi_{-k}(y, \eta)}{\partial \xi_j} \frac{\partial P_0(y, \eta)}{\partial x_j} - \frac{\partial \varphi_{-k}(y, \eta)}{\partial x_j} \frac{\partial P_0(y, \eta)}{\partial \xi_j} \right) = b_k(y, \eta), \quad 1 \leq k \leq N,$$

where $\deg_\eta b_k(y, \eta) = m - k - 1$, $b_k(y, \eta)$ depends on $\varphi_0, \varphi_{-1}, ..., \varphi_{-k+1}$.

The operator T_ε consists of terms containing derivatives of $\chi_\varepsilon(x_0)$ and therefore $T_\varepsilon = 0$ when $-\varepsilon < x_0 < T+\varepsilon$.

Note that the left-hand side of (50.15) is called the Poisoon bracket of φ_0 and P_0 (c.f. Corollary 40.3). Substituting (50.5) or (50.11), (50.12) into (50.15) we get

$$\frac{d}{dx_0}\varphi_0(\hat{y}(x_0,y,\eta),\hat{\eta}(x_0,y,\eta)) = 0. \tag{50.17}$$

Let $\psi(x,\eta) \in C_0^\infty(U_1)$, $\psi(x^{(0)},\eta^{(0)}) \neq 0$, $\deg_\eta \psi(x,\eta) = 0$. We assume that $\varphi_{-k}(x,x_0,\eta)$ satisfy the following initial conditions when $x_0 = 0$:

$$\begin{gathered}\varphi_0(x,0,\eta) = \psi(x,\eta),\\ \varphi_{-k}(x,0,\eta) = 0, \quad 1 \le k \le N.\end{gathered} \tag{50.18}$$

Then

$$\varphi_0(\hat{x}(x_0,x,\eta),x_0,\hat{\eta}(x_0,x,\eta)) = \psi(x,\eta)$$

for $-2\varepsilon \le x_0 \le T+2\varepsilon$, where $\hat{x},\hat{\eta}$ are the solutions of (50.11), (50.12). Let

$$x = x(x_0,\hat{x},\hat{\eta}), \quad \eta = \eta(x_0,\hat{x},\hat{\eta}) \tag{50.19}$$

be the inverse to (50.11), (50.12). Then

$$\varphi_0(\hat{x},x_0,\hat{\eta}) = \psi(x(x_0,\hat{x},\hat{\eta}),\eta(x_0,\hat{x},\hat{\eta})), \ (\hat{x},x_0,\hat{\eta}) \in V(-2\varepsilon,T+2\varepsilon). \tag{50.20}$$

Using x_0 as parameter we get from (50.16)

$$\frac{d}{dx_0}\varphi_{-k}(\hat{x}(x_0),x_0,\hat{\eta}(x_0)) = b_k(\hat{x}(x_0),x_0,\hat{\eta}(x_0))\left(\frac{\partial P_0(\hat{x}(x_0),x_0,\hat{\eta}(x_0))}{\partial \xi_0}\right)^{-1}.$$

Integrating in x_0 from 0 to x_0 using the initial conditions (50.18) and the inverse map (50.19) we get analogously to (50.20) that $\operatorname{supp}\varphi_{-k}\chi_\varepsilon(x_0) \subset V[-2\varepsilon,T+2\varepsilon)$. Note that $\varphi(x,x_0,\eta)$ is micro-elliptic on $\gamma(x^{(0)},0,\eta^{(0)})$ since $\varphi_0 \neq 0$ and $\deg_\eta \varphi_{-k} \le -k$, $k \ge 1$.

Applying $\varphi(y,D_y)$ to $P(y,D_y)u = f$ and using (50.13) we get

$$P(y,D_y)\varphi(y,D_y)u = \varphi(y,D_y)f - T_{m-N-1}u - T_\varepsilon u. \tag{50.21}$$

Note that $\varphi(y,D_y)f \in C^\infty$ since $\operatorname{supp}\varphi \subset V(-2\varepsilon,T+2\varepsilon)$.

Let $\varphi^{(1)}(y,\eta) \in C^\infty(\mathbf{R}^{n+1} \times (\mathbf{R}^{n+1} \setminus \{0\}))$, $\deg_\eta \varphi^{(1)} = 0$ and supp $\varphi^{(1)}$ is contained in a small conic neighborhood containing U_0. We assume that $\varphi^{(1)} = 1$ in a conic neighborhood of supp $\varphi_{-k}(y,\eta)$, $0 \le k \le N$. Therefore supp $(1-\varphi^{(1)}) \cap \text{supp } \varphi_{-k} \cap \{|\xi|^2 + |\xi_0|^2 = 1\} = \emptyset$, $0 \le k \le N$.

Let Q_1 be the ψdo with symbol $\varphi^{(1)}(y,\eta) q_0^{-1}(y,\eta)(1-\chi(\eta))\chi_{2\varepsilon}(x_0)$. Applying $Q_1(y,D_y)$ to (50.21) we get using Theorem 40.2

$$\begin{aligned} (50.22) \qquad & (D_0 - \lambda_0(y,D_x)(1-\chi(D)))\varphi(y,D_y)u \\ &= C_0(y,D_y)\varphi u + T_{-N}^{(1)} u + T_\varepsilon^{(1)} u + Q_1 \varphi f, \end{aligned}$$

where ord $T_{-N} \le -N-1$, $C_0(y,\xi) \in S^0$, $T_\varepsilon^{(1)} = 0$ when $-\varepsilon \le x_0 \le T + \varepsilon$. Since $\varphi f \in C^\infty$ we have that $g = Q_1 \varphi f \in C^\infty(\mathbf{R}^{n+1})$. We used in (50.22) that the composition of ψdo with symbol $(\xi_0 - \lambda_0)(1 - \varphi^{(1)}(y,\eta)\chi_{2\varepsilon}(x_0))$ and $\varphi(y,D_y)$ is an operator of order $-\infty$. Note that $Q_1 T_\varepsilon = T_\varepsilon^{(1)} + T'_{-N-1}$, where ord $T'_{-N-1} \le -N-1$. Note also that $\chi(D_x)\varphi(y,D_x,D_0) = T_{-\infty}$ where ord $T_{-\infty} = -\infty$ since $\varphi(y,\xi,\xi_0) = 0$ when $|\xi| < \varepsilon|\xi_0|$ for $\varepsilon > 0$ small.

By the Taylor theorem

$$(50.23) \qquad C_0(y,\xi,\xi_0) = C_0(y,\xi,\lambda_0(y,\xi)) + C_{-1}(y,\xi,\xi_0)(\xi_0 - \lambda_0(y,\xi)),$$

where $C_{-1}(y,\xi,\xi_1)(1-\chi(\xi)) \in S^{-1}$.

Denote $\lambda_{-1}(y,\xi) = C_0(y,\xi,\lambda_0(y,\xi))$. Then

$$\begin{aligned} (50.24) \qquad & C_0(y,D_y)\varphi u = \lambda_{-1}(y,D_x)(1-\chi(D))\varphi u \\ & + C_{-1}(y,D,D_0)(D_0 - \lambda_0(y,D)(1-\chi(D)))\varphi u + C_{-1}^{(1)}(y,D_y)\varphi u + T_{-N}^{(2)} u, \end{aligned}$$

where ord $T_{-N}^{(2)} \le -N$, $C_{-1}^{(1)}(y,\eta) \in S^{-1}$. Replacing $(D_0 - \lambda_0(1-\chi(D)))\varphi u$ in (50.24) using the equation (50.22) we get

$$\begin{aligned} & C_0(y,D_y)\varphi u = \lambda_{-1}(1-\chi(D))\varphi u + C_{-1}(C_0 + T_\varepsilon^{(1)} u + T_{-N}^{(1)} u + Q_1 \varphi f) \\ (50.25) \qquad & \qquad + C_{-1}^{(1)} \varphi u + T_{-N}^{(2)} u. \end{aligned}$$

Substituting (50.25) in (50.22) we obtain

$$(50.26) \quad D_0 \varphi u - (\lambda_0 + \lambda_{-1})(1-\chi(D))\varphi u = C_{-1}^{(2)} \varphi u + T_{-N}^{(3)} u + T_\varepsilon^{(2)} u + Q_2 \varphi f,$$

where $C_{-1}^{(2)}(y,\eta) \in S^{-1}$, ord $T_{-N}^{(3)} \le -N$, $Q_2(y,\xi) \in S^{m-1}$, $T_\varepsilon^{(2)} = 0$ when $x_0 \in (-\varepsilon, T+\varepsilon)$.

Now we repeat with $C_{-1}^{(2)}$ the same operations (50.23)-(50.26) that we did with $C_0(y, D_y)$. We get

$$(50.27)\ D_0\varphi u - (\lambda_0(y,D) + \lambda_{-1}(y,D) + \lambda_{-2}(y,D))(1-\chi(D))\varphi u = C_{-2}^{(3)}(y,D_y)\varphi u + T_{-N}^{(3)}u + T_\varepsilon^{(4)}u + Q_3\varphi f,$$

where $C_{-2}^{(3)}(y,D_y) \in S^{-2}(\mathbf{R}^{n+1}\times\mathbf{R}^{n+1})$, $\lambda_{-2}(y,\xi)(1-\chi(\xi)) \in S^{-1}(\mathbf{R}^{n+1}\times\mathbf{R}^n)$, $Q_3 \in S^{m-1}$, ord $T_{-N}^{(4)} \le -N$, $T_\varepsilon^{(3)} = 0$ when $x_0 \in (-\varepsilon, T+\varepsilon)$. After N steps we get

$$(50.28)\qquad D_0\varphi u - \lambda(y,D)(1-\chi(D))\varphi u = T_{-N}u + T_\varepsilon^{(N)}u + Q_N\varphi f,$$

where $\lambda = \lambda_0 + \lambda_{-1} + \dots + \lambda_{-N}$, $\lambda_{-k}(y,\xi)(1-\chi(\xi)) \in S^{-k+1}(\mathbf{R}^{n+1}\times\mathbf{R}^n)$, ord $T_{-N} \le -N$, $Q^{(N)}(y,\eta) \in S^{m-1}$, $T_\varepsilon^{(N)} = 0$ for $x_0 \in (-\varepsilon, T+\varepsilon)$.

Take the restriction of (50.28) to $R_T = \mathbf{R}^n \times (0,T)$. Then $T_\varepsilon^{(N)}u|_{\mathbf{R}_T} = 0$. We assumed that $\varphi u \in C^\infty$ in U_1. Therefore

$$\varphi u|_{x_0=0} = h \in C^\infty.$$

Also the right-hand side of (50.28) belongs to $H_{s+N}(R_T)$. We assumed that $u \in H_s(\mathbf{R}^{n+1})$ and therefore $\varphi u \in H_s(R_T)$. It follows from (50.28) that $\varphi u \in H_{1,s_1}(R_T)$ for some s_1.

This property of solution of differential equations with smooth right-hand side is called partial hypoellipticity (c.f. §37). To prove this note that

$$(50.29)\qquad D_0\varphi u = -\lambda(1-\chi)\varphi u + T_{-N}u + Q_N\varphi f \in H_{s,-1}(R_T),$$

where $H_{s,-1}(R_T)$ is the Sobolev space analogous to $H_{p,s}(R_T)$ when $p \in \mathbf{R}$. Since $D_0\varphi u \in H_{s,-1}(R_T)$ and $\Lambda\varphi u \in H_{s,-1}(R_T)$ we get that $\varphi u \in H_{s+1,-1}(R_T)$. Differentiating (50.29) p times we get $\varphi u \in H_{s+p+1,-p-1}(R_T)$. When $p + s \ge 0$ we get that

$$\varphi u \in H_{1,-p-1}(R_T).$$

Now we can apply Theorem 48.3 (or Theorem 48.2) for $m = 1$ to the initial value problem (50.28) on R_T with $\varphi u|_{x_n=0} = h$, to get that $\varphi u \in H_{s+N}(R_T)$. Since N is arbitrary we get $\varphi u \in C^\infty$.

Remark 50.1. We shall show that the solution of (50.11) exists for all $-\infty < x_0 < +\infty$. Since $\deg_\xi \lambda_0(x,x_0,\xi) = 1$ we have that $|\frac{\partial\lambda_0(x,x_0,\xi)}{\partial\xi}| \le C$ for all $(x, x_0) \in \mathbf{R}^{n+1}, \xi \neq 0$, since coefficients of $P_0(x, \xi)$ are uniformly bounded. Therefore

$$|\hat{x}(x_0) - x| \leq C|x_0 - x_0^{(0)}|. \tag{50.30}$$

Taking the dot product $\hat{\xi}(x_0) \cdot \frac{d\hat{\xi}(x_0)}{dx_0}$ we get

$$|\hat{\xi}(x_0) \cdot \frac{d\hat{\xi}_0}{dx_0}| = |\hat{\xi}(x_0) \cdot \frac{\partial \hat{\lambda}}{\partial x}| \leq C|\hat{\xi}(x_0)|^2.$$

Therefore $\frac{1}{2}\frac{d}{dx_0}|\hat{\xi}(x_0)|^2 \leq C|\hat{\xi}(x_0)|^2$, and we get

$$|\hat{\xi}(x_0)|^2 \leq |\xi^{(0)}|^2 \exp(2C|x_0 - x_1^{(0)}|). \tag{50.31}$$

Estimates (50.30), (50.31) imply that the solution of (50.11), (50.12) exists on $(-\infty, +\infty)$. Therefore we can take T arbitrary assuming that the decomposition $P_0 = q_0(\xi_0 - \lambda_0(x, x, \xi)$ holds for all x_0 as in the case of strictly hyperbolic $P_0(x, t, \xi, \xi_0)$ when $x_0 = t$. Since in the case of arbitrary real principal type operator the null-characteristics starting at $(x^{(0)}, x_0^{(0)}, \xi^{(0)}, \xi_0^{(0)})$ exists for all $s \in \mathbf{R}$ we can continue at $x_0 = T$ using another (if necessary) decomposition of $P_0(y, \eta)$. Therefore the result of Theorem 50.1 holds along the whole null-bicharacteristics $\gamma(x^{(0)}, x_0^{(0)}, \xi^{(0)}, \xi_0^{(0)})$,
$-\infty < s < +\infty$, assuming that the right-hand side $f \in C^\infty$ microlocally in a neighborhood of γ, $-\infty < s < +\infty$ and $u \in C^\infty$ microlocally in a neighborhood of $(x^{(0)}, x_0^{(0)}, \xi^{(0)}, \xi_0^{(0)})$.

Corollary 50.2. Let γ be the null-bicharacteristics as in Theorem 50.1 and suppose that the right-hand side $f \in C^\infty$ microlocally in a neighborhood of γ. If some point $(x^{(1)}, x_0^{(1)}, \xi^{(1)}, \xi_0^{(1)})$ of γ belongs to $WF(u)$ then al points of γ belong to $WF(u)$. This is the consequence of Theorem 50.1: If some point $(x^{(2)}, x_0^{(2)}, \xi^{(2)}, \xi_0^{(2)})$ of γ does not belong to $WF(u)$ then all points of γ, including $(x^{(1)}, x_0^{(1)}, \xi^{(1)}, \xi_0^{(1)})$, will not belong to $WF(u)$. □

Now we shall describe the wave front set $WF(u)$ of the Cauchy problem (48.1), (48.2) in the terms of $WF(f)$ and the wave front sets of initial conditions $g_k(x)$, $0 \leq k \leq m-1$. Since $H_0(x, t, \xi, \sigma)$ has m distinct real roots $\sigma = \lambda_k(x, x_0, \xi)$, $1 \leq k \leq m$, there are m null-bicharacteristics passing through the arbitrary point $(x^{(0)}, t_0, \xi^{(0)}) \in \mathbf{R}^{n+1} \times (\mathbf{R}^n \setminus \{0\})$ with $\xi_0^{(k)} = \lambda_k(x^{(0)}, t_0, \xi^{(0)})$, $1 \leq k \leq m$. Consider first the Cauchy problem (48.1) with zero initial conditions, i.e. $Hu_+ = f_+$ in $\mathbf{R}^{n+1}$, $u_+ \in H_s$ locally in $\mathbf{R}^{n+1}$ for some $s \in \mathbf{R}$, $u_+ = 0$ for $t < 0$.

If $(x^{(0)}, t_0, \xi^{(0)}, \xi_0^{(0)}) \in WF(f_+)$, $t_0 \geq 0$, then, obviously, $(x^{(0)}, t_0, \xi^{(0)}, \xi_0^{(0)}) \in WF(u_+)$. If $\xi_0^{(0)} \neq \lambda_k(x^{(0)}, t_0, \xi^{(0)})$, $1 \leq k \leq m$, then there is no propagation of singularities (Theorem 44.5). If $\xi_0^{(0)} = \lambda_{k_0}(x^{(0)}, t_0, \xi^{(0)})$ for some $1 \leq k_0 \leq m$ then the wave front set may propagate or not.

In the first case $WF(u_+)$ may contain the forward null-bicharacteristics starting at $(x^{(0)}, t_0, \xi_0^{(0)}, \lambda_{k_0})$. In the case of the Cauchy problem (48.1), (48.2) denote $u_+ = u$ for $t \geq 0$, $u_+ = 0$ for $t < 0$. Then $u_+(x,t)$ satisfies the following equation in $\mathbf{R}^{n+1}$

$$H(x, t, D_x, D_t)u_+ = f_+ + \sum_{k=0}^{m-1} b_k(x, 0, D) g_k(x) \delta^{(m-1-k)}(t), \tag{50.32}$$

where ord $b_k \leq m - 1 - k$, $f_+ = f$ for $t > 0$, $f_+ = 0$ for $t < 0$.

Suppose $(x^{(0)}, \xi^{(0)}) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ belongs to $WF(g_{k_0}(x))$ for some k_0, $1 \leq k_0 \leq m - 1$. Then $(x^{(0)}, 0, \xi^{(0)}, \sigma) \in WF(g_{k_0}(x)\delta^{(m-1-k)}(t))$, where σ is arbitrary. In particular, $(x^{(0)}, 0, \xi^{(0)}, \lambda_j(x^{(0)}, 0, \xi^{(0)})$ for each $1 \leq j \leq m$ belongs to $WF(g_{k_0}(x)\delta^{(m-1-k)}(t))$. There are m null-bicharacteristics $\gamma_{jk_0}((x^{(0)}, 0, \xi^{(0)}, \lambda_j(x^{(0)}, 0, \xi^{(0)})$ that start at $(x^{(0)}, 0, \xi^{(0)}, \lambda_j(x^{(0)}, 0, \xi^{(0)}))$, $1 \leq j \leq m$.

The folowing theorem holds:

Theorem 50.2. *Let $u(x,t)$ be the solution of the Cauchy problem (48.1), (48.2). Let the wave front sets $WF(f_+) \subset \mathbf{R}_+^{n+1} \times (\mathbf{R}^{n+1} \setminus \{0\})$ and $WF(g_k(x)) \subset \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$, $0 \leq k \leq m-1$, are given. Denote by $\Sigma(f, g_0, ..., g_{m-1}) \subset \mathbf{R}_+^{n+1} \times (\mathbf{R}^{n+1} \setminus \{0\})$ the set of points satisfying the following condition:*
If $(x^{(0)}, t_0, \xi^{(0)}, \sigma_0) \in WF(f_+)$, $t_0 \geq 0$, $\xi^{(0)} \neq 0$, and $\sigma_0 \neq \lambda_j(x^{(0)}, t_0, \xi^{(0)})$, $1 \leq j \leq m$, then $(x^{(0)}, t_0, \xi^{(0)}, \sigma_0) \in \Sigma$.
If $\sigma_0 = \lambda_{k_0}(x^{(0)}, t_0, \xi^{(0)})$ for some k_0, $1 \leq k_0 \leq m$, then the forward null-bicharacteristics $\gamma_{k_0}^+(x^{(0)}, t_0, \xi^{(0)}, \sigma_0)$, $t \geq t_0$, belongs to Σ.
If $(x^{(0)}, \xi^{(0)}) \in WF(g_k)$, $0 \leq k \leq m-1$, then the union of m forward null-bicharacteristics $\gamma_{jk}^+(x^{(0)}, 0, \xi^{(0)}, \lambda_j(x^{(0)}, 0, \xi^{(0)}))$, $t \geq 0$, $1 \leq j \leq m$, belongs to Σ.
Then the set Σ contains $WF(u), t > 0$.

Proof: Let $\mathcal{P}_0 = (x^{(0)}, t_0, \xi^{(0)}, \lambda_{j_0}(x^{(0)}, t_0, \xi^{(0)})) \notin \Sigma$. Denote by γ_- the backward null-bicharacteristic starting at $(x^{(0)}, t_0, \xi^{(0)}, \sigma^{(0)})$ assuming that $\sigma^{(0)} = \lambda_{j_0}$. Let $(x^{(1)}, 0, \xi^{(1)}, \lambda_{j_0}(x^{(1)}, 0, \xi^{(1)}))$ be the point on γ_- when $t = 0$. Since $\mathcal{P} \notin \Sigma$ we have that $(x^{(1)}, 0, \xi^{(1)}) \notin WF(g_k(x))$ for $0 \leq k \leq m - 1$.

We shall show that $(x^{(1)}, 0, \xi^{(1)}, \lambda_{j_0}(x^{(1)}, 0, \xi^{(1)})) \notin WF(b_k g_k(x)\delta^{(m-k-1)}(t))$ for all $0 \le k \le m-1$.

Let $\beta(x,t) \in C_0^\infty(\mathbf{R}^{n+1})$, $\beta(x^{(0)}, 0) \neq 0$ and $\beta(x,t)$ is zero outside of small neighborhood of $(x^{(0)}, 0)$. Then $F(\beta b_k g_k \delta^{(m-k-1)}(t)) = \sigma^{m-k-1}\beta_1(\xi)$, where $|\beta_1(\xi)| \le C_N(1+|\xi|)^{-N}$, $\forall N$ in a conic neighborhood of $\xi^{(1)}$. Therefore $|\sigma^{m-k-1}\beta_1(\xi)| \le C(1+|\xi|+|\sigma|)^{-N}$, $\forall N$ in a conic neighborhood of $(\xi^{(1)}, \lambda_{j_0}(x^{(1)}, 0, \xi^{(1)}))$. This proves that

$$(x^{(1)}, 0, \xi^{(1)}, \lambda_{j_0}(x^{(1)}, 0, \xi^{(1)})) \notin WF(b_k g_k(x)\delta^{(m-k-1)}(t))$$

for all $0 \le k \le m-1$. Therefore the wave front set of right-hand side of (50.32) does not intersect γ_- for $-\infty < t \le t_0$. Since $u_+ = 0$ for $t < 0$ we get from Theorem 50.1 that $WF(u_+) \cap \gamma_- = \emptyset$ for $t \le t_0$.

Remark 50.2 Consider the Cauchy problem (48.1), (48.2) assuming that $f(x,t) \in C^\infty(\mathbf{R}_+^{n+1})$. Then by the partial hypoellipticity the solution $u(x,t)$ is a smooth function of $t \ge 0$ with distribution values in $\mathbf{R}^n$. We shall describe the wave front set of $u(x,t) \subset \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ for fixed $t > 0$. It follows from Theorems 50.1, 50.2 that if $(x^{(0)}, t_0, \xi^{(0)}, \sigma^{(0)}) \in WF(u)$, $t > 0$, then $\xi^{(0)} \neq 0$ and $\sigma_0 = \lambda(x^{(0)}, t_0, \xi^{(0)})$. We shall show that $WF(u(\cdot, t_0))$ is contained in the set $W(t_0) \subset \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ consisting of $(x^{(0)}, \xi^{(0)})$ such that $(x^{(0)}, t_0, \xi^{(0)}, \sigma^{(0)}) \in WF(u)$ for some $\sigma_0 \in \mathbf{R}^1$.

Proof. Suppose $(x^{(0)}, \xi^{(0)}) \in W(t_0)$, i.e. $(x^{(0)}, t_0, \xi^{(0)}, \sigma^{(0)}) \notin WF(u)$ for all $\sigma \in \mathbf{R}$. Then there exists $\varphi(x) \in C_0^\infty(\mathbf{R}^n)$, $\varphi(x_0) \neq 0$, $\alpha(\xi) \in C^\infty(\mathbf{R}^n \setminus \{0\})$, $\deg_\xi \alpha(\xi) = 0$, $\alpha(\xi^{(0)}) \neq 0$, $\delta > 0$ small such that supp $\varphi(x)\chi(\frac{t-t_0}{\delta})\alpha(\xi) \cap WF(u) = \emptyset$. Therefore $\alpha(D)\varphi(x)\chi(\frac{t-t_0}{\delta})u \in C^\infty(\mathbf{R}^{n+1})$. In particular $\alpha(D)\varphi(x)u(x,t_0) \in C^\infty(\mathbf{R}^n)$, i.e. $(x^{(0)}, \xi^{(0)}) \notin WF(u(\cdot, t_0))$. Therefore $WF(u(\cdot, t_0) \subset W(t_0)$.