48 The Cauchy problem for the strictly hyperbolic equations.

Let $H(x,t,\xi,\sigma)$ be the polynomial of degree m

$$H(x,t,\xi,\sigma) = H_0(x,t,\xi,\sigma) + H_1(x,t,\xi,\sigma),$$

where

$$H_0(x,t,\xi,\sigma) = \sum_{|k|+j=m} a_{jk}(x,t)\xi^k\sigma^j, \quad a_{m0}=1,$$

$$H_1(x,t,\xi,\sigma) = \sum_{|k|+j\leq m-1} a_{jk}(x,t)\xi^k\sigma^j.$$

We assume that $a_{jk}(x,t)\in C^\infty(\mathbf{R}^{n+1})$ and $a_{jk}(x,t)=a_{jk}^\infty$ when $|x|^2+|t|^2\geq R^2$. The polynomial $H(x,t,\xi,\sigma)$ is called strictly hyperbolic if $H_0(x,t,\xi,\sigma)=0$ has m real roots $\sigma_k(x,t,\xi)$ such that $\sigma_j(x,t,\xi)\neq\sigma_k(x,t,\xi)$ when $j\neq k$ for all $(x,t)\in\mathbf{R}^{n+1}$ and $\xi\neq 0$. Note that $\sigma_k(x,t,\xi)$ is a homogeneous function of ξ of degree 1 and $\sigma_k(x,t,\xi)\in C^\infty(\mathbf{R}^{n+1}\times(\mathbf{R}^n\setminus\{0\}))$.

Consider the Cauchy problem

$$H(x,t,D_x,D_t)u(x,t) = f(x,t), \quad (x,t)\in\mathbf{R}_+^{n+1}=\{t>0, x\in\mathbf{R}^n\}, \tag{48.1}$$

where $D_x=-i\frac{\partial}{\partial x}$, $D_t=-i\frac{\partial}{\partial t}$, with the initial conditions

$$\left.\frac{\partial^k u(x,t)}{\partial t^k}\right|_{t=0} = g_k(x), \quad 0\leq k\leq m-1. \tag{48.2}$$

As in the parabolic case denote $v(x,t)=e^{-\tau t}u(x,t)$, $g(x,t)=e^{-\tau t}f(x,t)$. Then $v(x,t)$ satisfies the equation

$$H(x,t,D_x,D_t-i\tau)v(x,t) = g(x,t), \quad (x,t)\in\mathbf{R}_+^{n+1}, \tag{48.3}$$

and

$$\left.\frac{\partial^k v(x,t)}{\partial t^k}\right|_{t=0} = h_k(x), \quad 0\leq k\leq m-1, \tag{48.4}$$

where $h_k(x)$ are linear combinations of $g_j(x)$, $0\leq k\leq n-1$, $0\leq j\leq n-1$. Denote by $H_{p,s}(\mathbf{R}^{n+1})$ the Sobolev space with the norm

$$\|v\|_{p,s}^2 = \int_{\mathbf{R}^{n+1}} (1+|\xi|^2+\sigma^2)^p(1+|\xi|^2)^s|\tilde{u}(\xi,\sigma)|^2 d\xi d\sigma, \tag{48.5}$$

where $p \geq 0$ is an integer, $s \in \mathbf{R}$. The spaces $\overset{\circ}{H}_{p,s}(\mathbf{R}^{n+1})$ and $H^+_{p,s}(\mathbf{R}^{n+1}_+)$ are defined as in the parabolic case.

We first consider the Cauchy problem (48.1), (48.2) (or (48.3), (48.4)) with zero initial data. Let $f_+ = f$ for $t > 0$, $f_+ = 0$ for $t < 0$. Analogously let $u_+ = u$, $v_+ = v(x,t)$, $g_+ = g(x,t)$ for $t > 0$ and $u_+ = v_+ = g_+ = 0$ for $t < 0$.

Theorem 48.1. *Let $\tau \geq \tau_0$, where τ_0 is large, and s is arbitrary. Then for any f_+ such that $e^{-\tau t} f_+ \in \overset{\circ}{H}_{0,s}(\mathbf{R}^{n+1})$ there exists a unique solution u_+ of $H(x,t,D_x,D_t)u_+ = f_+$ in $\mathbf{R}^{n+1}$ such that $e^{-\tau t} u_+ \in \overset{\circ}{H}_{m-1,s}(\mathbf{R}^{n+1})$.*

Proof: Note that

$$H_0(x,t,\xi,\sigma - i\tau) = \Pi_{j=1}^m (\sigma - i\tau - \sigma_j(x,t,\xi)) \neq 0$$

when $\tau \neq 0$. We have

$$\frac{\partial H_0(x,t,\xi,\sigma - i\tau)}{\partial \sigma} = \sum_{k=1}^m q_k(x,t,\xi,\sigma - i\tau),$$

where

$$q_k(x,t,\xi,\sigma - i\tau) = \Pi_{j\neq k}(\sigma - i\tau - \sigma_j(x,t,\xi)), \quad 1 \leq k \leq m. \tag{48.6}$$

Therefore

$$H_0 \overline{\frac{\partial H_0}{\partial \sigma}} = \Pi_{j=1}^m (\sigma - i\tau - \sigma_j) \sum_{k=1}^m \Pi_{j\neq k}(\sigma + i\tau - \sigma_j) \tag{48.7}$$

$$= \sum_{k=1}^m (\sigma - i\tau - \sigma_k) \Pi_{j\neq k}((\sigma - \sigma_j)^2 + \tau^2) = -i\tau Q_0 + Q_1,$$

where

$$Q_0 = \sum_{k=1}^m q_k \overline{q_k}, \quad Q_1 = \sum_{k=1}^m (\sigma - \sigma_k) q_k \overline{q_k} \tag{48.8}$$

and $Q_0(x,t,\xi,\sigma,\tau) > 0$ when $(\xi,\sigma,\tau) \neq (0,0,0)$. Therefore by the homogenuity in (ξ,σ,τ)

$$Q_0(x,t,\xi,\sigma,\tau) \geq C_0(|\xi|^2 + |\sigma|^2 + \tau^2)^{m-1}. \tag{48.9}$$

We shall show that for any $v_+ \in \overset{\circ}{H}_{m,s}(\mathbf{R}^{n+1})$ the following estimates holds:

$$\|(|D_x|^2+|D_t|^2+\tau^2)^{\frac{m-1}{2}}v_+\|_{0,s} \le \frac{C}{\tau}\|H(x,t,D_x,D_t-i\tau)v_+\|_{0,s}, \tag{48.10}$$

where $(|D_x|^2+|D_t|^2+\tau^2)^{\frac{m-1}{2}}$ is the pseudodifferential operator with symbol $\Gamma^{m-1}(\xi,\sigma,\tau)=(|\xi|^2+\sigma^2+\tau^2)^{\frac{m-1}{2}}$.

Note that symbols $\sigma_k(x,t,\xi)$ are not smooth when $\xi=0$. Therefore we will introduce $\hat{\sigma}_k(x,t,\xi)=\sigma_k(x,t,\xi)(1-\chi(\xi))$ and $\hat{q}_k(x,t,\xi,\sigma)=\Pi_{j\neq k}(\sigma-i\tau-\hat{\sigma}_j(x,t,\xi))$, where $\chi(\xi)\in C_0^\infty(\mathbf{R}^n)$, $\chi(\xi)=1$ when $|\xi|<1$.

Consider the L^2 inner product in $\mathbf{R}^{n+1}$:

$$(Hv_+,H_{0\sigma}\Lambda^{2s}v_+)=\sum_{k=1}^m(Hv_+,q_k\Lambda^{2s}v_+), \tag{48.11}$$

where $\Lambda(\xi)=(1+|\xi|^2)^{\frac{1}{2}}$.

It follows from Theorem 40.2 that

$$H(x,t,D_x,D_t-i\tau)=(D_t-i\tau-\hat{\sigma}_k(x,t,D_x))\hat{q}_k(x,t,D_x,D_t-i\tau)+q_{k1}, \quad 1\le k\le m, \tag{48.12}$$

where $\operatorname{ord} q_{k1}\le m-1$, q_{k1} is a differential operator in t and a *ψdo* in $x\in\mathbf{R}^n$,

$$\|q_{k1}v_+\|_{0,s}\le C\|\Gamma^{m-1}(D_x,D_t,\tau)v_+\|_{0,s}. \tag{48.13}$$

Therefore using (48.12) and Theorems 40.2, 43.2 we can represent (48.11) in the form

$$(Hv_+,H_{0\sigma}\Lambda^{2s}v_+)=\sum_{k=1}^m((D_t-i\tau-\hat{\sigma}_k(x,t.D_x))\hat{q}_k\Lambda^s v_+,q_k\Lambda^s v_+) +(Q_2\Lambda^s v_+,\Lambda^s v_+), \tag{48.14}$$

where

$$|(Q_2\Lambda^s v_+,\Lambda^s v_+)|\le C\|\Gamma^{m-1}\Lambda^s v_+\|_0^2. \tag{48.15}$$

Note that the symbol of $D_t-\hat{\sigma}_k(x,t,D_x)$ is real valued. Therefore by Theorem 43.2 we have that $\Im((D_t-\hat{\sigma}_k)\hat{q}_k\Lambda^s v_+,\hat{q}_k\Lambda^s v_+)$ satisfies the estimate of the form (48.15). Therefore taking the imaginary part of (48.14) we get

$$-\Im(Hv_+,H_{0\sigma}\Lambda^{2s}v_+)\ge\tau\sum_{k=1}^m\|\hat{q}_k\Lambda^s v_+\|_0^2-C\|\Gamma^{m-1}\Lambda^s v_+\|_0^2. \tag{48.16}$$

We shall show that

$$\sum_{k=1}^{m} \|\hat{q}_k \Lambda^s v_+\|_0^2 = (\sum_{k=1}^{m} \hat{q}_k^* \hat{q}_k \Lambda^s v_+, \Lambda^s v_+) \tag{48.17}$$
$$\geq C\|\Gamma^{m-1}\Lambda^s v_+\|_0^2 - C_1\|\Gamma^{m-\frac{3}{2}}\Lambda^s v_+\|_0^2.$$

The inequality (48.17) is called the Garding inequality. We have

$$\hat{Q}(x,t,\xi,\sigma,\tau) = \hat{Q} - \frac{C}{2}(\xi|^2 + |\sigma|^2 + \tau^2)^{m-1} + \frac{C}{2}(|\xi|^2 + |\sigma|^2 + \tau^2)^{m-1},$$

where

$$\hat{Q} = \sum_{k=1}^{m} |q_k(x,t,\xi,\sigma - i\tau)|^2 \geq C(\xi|^2 + |\sigma|^2 + \tau^2)^{m-1} \qquad \text{(c.f. (48.9))}.$$

Let

$$B(x,t,\xi,\sigma,\tau) = \sqrt{\hat{Q} - \frac{C}{2}\Gamma^{2(m-1)}(\xi,\sigma,\tau)}.$$

Note that $B > 0$ and $B(x,t,\xi,\sigma,\tau) \in S^{m-1}$. Denote by B^* the adjoint operator to B. By Theorems 40.2 and 43.2 we have

$$\sum_{k=1}^{m} \hat{q}_k^* \hat{q}_k - \frac{C}{2}\Gamma^{2(m-1)} = B^*B + C_2, \tag{48.18}$$

where ord $C_2 \leq 2m-3$, and

$$|(C_2\Lambda^s v_+, \Lambda^s v_+)| \leq C_1\|\Gamma^{m-\frac{3}{2}}\Lambda^s v_+\|_0^2. \tag{48.19}$$

Note that $(B^*B\Lambda^s v_+, \Lambda^s v_+) = \|B\Lambda^s v_+\|^2 \geq 0$. Therefore (48.18), (48.19) imply

$$\tau(\sum_{k=1}^{m} q_k^* q_k \Lambda^s v_+, \Lambda^s v_+) \geq \frac{C}{2}\tau\|\Gamma^{m-1}\Lambda^s v_+\|_0^2 - C_1\tau\|\Gamma^{m-\frac{3}{2}}\Lambda^s v_+\|_0^2$$
$$\geq (\frac{C}{2}\tau - C_1)\|\Gamma^{m-1}\Lambda^s v_+\|_0^2, \tag{48.20}$$

since $\tau\|\Gamma^{m-\frac{3}{2}}\Lambda^s v_+\|_0^2 \leq \|\Gamma^{m-1}\Lambda^s v_+\|_0^2$. Combining (48.14), (48.15), (48.16) and (48.20) we obtain

$$\Im(Hv_+, H_\sigma\Lambda^{2s}v_+) \geq C\|\Gamma^{m-1}\Lambda^s v_+\|_0^2.$$

Since

$$|(Hv_+, H_{0\sigma}\Lambda^{2s}v_+) \leq C\|\Lambda^s Hv_+\|_0 \ \|\Gamma^{m-1}\Lambda^s v_+\|_0^2,$$

we get (48.10).

Now we can prove the uniqueness: Suppose $Hu_+ = 0$ where $e^{-\tau t}u_+ = v_+ \in \overset{\circ}{H}_{m-1,s} (\mathbf{R}^{n+1})$. Since $v_+ \in \overset{\circ}{H}_{m-1,s} (\mathbf{R}^{n+1})$ and $Hv_+ = D_t^m v_+ + \sum_{j=0}^{m-1} b_k(x,t,D_x,\tau)D_t^k v_+ = 0$ we get that

$$D_t^m v_+ = -\sum_{k=0}^{m-1} b_k(x,t,D_x,\tau)D_t^k v_+ \in \overset{\circ}{H}_{0,s-1} (\mathbf{R}^{n+1}). \tag{48.21}$$

Therefore $v_+ \in \overset{\circ}{H}_{m,s-1} (\mathbf{R}^{n+1})$. Applying the inequality (48.10) with $Hv_+ = 0$ and s replaced by $s-1$ we get that $v_+ = 0$.

Now we shall prove the existence of v_+ such that $H(x,t,D_x,D_t - i\tau)v_+ = g_+$ in $\mathbf{R}^{n+1}$ and $v_+ \in \overset{\circ}{H}_{m-1,s} (\mathbf{R}^{n+1})$. We assume that $g_+ \in \overset{\circ}{H}_{0,s} (\mathbf{R}^{n+1})$. We shall use the method of "parabolic" regularization.

Let $H_\varepsilon(x,t,\xi,\sigma - i\tau) = H(x,t,\xi,\sigma - i\tau - i\varepsilon\Lambda(\xi))$, $\varepsilon > 0$. Operator $H(x,t,D_x,D_t - i\tau - \varepsilon\Lambda(D_x))$ is a "parabolic" ψdo in the sence that $H(x,t,\xi,z - i\varepsilon\Lambda(\xi))$ and H^{-1} are analytic when $\Im z < 0$ and

$$C_{1\varepsilon}(|\xi| + |z|)^m \geq |H(x,t,\xi,z - i\varepsilon\Lambda(\xi))| \geq C_\varepsilon(|\xi| + |z|)^m \tag{48.22}$$

for all $\Im z > 0$ large (c.f. class $P_{\alpha,m}^+$ in §46).

Repeating the proof of Theorem 46.3 we get that for any $f_+ \in \overset{\circ}{H}_{0,s} (\mathbf{R}^{n+1})$ there exists $v_\varepsilon \in \overset{\circ}{H}_{m,s} (\mathbf{R}^{n+1})$ such that

$$H(x,t,D_x,D_t - i\tau - i\varepsilon\Lambda(D_x))v_\varepsilon = g_+, \quad (x,t) \in \mathbf{R}^{n+1}. \tag{48.23}$$

Note that the proof of the estimate (48.10) can be repeated for the equation (48.23) without change, and we get

$$\|\Gamma^{m-1}\Lambda^s v_\varepsilon\|_0 \leq C\|\Lambda^s g_+\|_0, \tag{48.24}$$

where the constant C is independent of ε. Therefore $\{v_\varepsilon\}$, $\varepsilon > 0$, is bounded in $\overset{\circ}{H}_{m-1,s} (\mathbf{R}^{n+1})$. It follows from the weak compactness of the bounded set in $\overset{\circ}{H}_{m-1,s} (\mathbf{R}^{n+1})$ that there exists a sequence v_{ε_k} such that v_{ε_k} converges weakly to some $v_+ \in \overset{\circ}{H}_{m-1,s} (\mathbf{R}^{n+1})$. For any $\varphi \in C_0^\infty(\mathbf{R}^{n+1})$ we have $(g_+, \varphi) =$

$(H_{\varepsilon_k} v_{\varepsilon_k}, \varphi) = (v_{\varepsilon_k}, H^*_{\varepsilon_k}\varphi)$. Passing to the limit when $\varepsilon_k \to 0$ we get $(g_+, \varphi) = (v_+, H^*\varphi)$, i.e. $Hv_+ = g_+$ in $\mathbf{R}^{n+1}$. $\square$

Remark 48.1 It follows from Theorem 48.1 that for any $g_+ \in \overset{\circ}{H}_{0,s} (\mathbf{R}^{n+1})$ there exists a unique $v_+ \in \overset{\circ}{H}_{m-1,s} (\mathbf{R}^{n+1})$ such that

$$(D_t - i\tau)^m v_+ + \sum_{j=0}^{m-1} a_j(x,t,D_x)(D_t - i\tau)^j v_+ = g_+(x,t), \tag{48.25}$$

where ord $a_j \leq m - j$. Then it follows from (48.25) that $D_t^m v_+ \in \overset{\circ}{H}_{0,s-1} (\mathbf{R}^{n+1})$. Therefore $v_+ \in \overset{\circ}{H}_{m,s-1} (\mathbf{R}^{n+1})$ and consequently $\frac{\partial^k v_+(x,t)}{\partial t^k}$ are continuous function of t with values in $H_{m+s-k-\frac{3}{2}}(\mathbf{R}^n)$, $0 \leq k \leq m-1$ (c.f. Theorem 13.6). Since $v_+ = 0$ for $t < 0$ we get that $\frac{\partial^k v_+(x,0)}{\partial t^k} = 0$, $0 \leq k \leq m-1$, i.e. $v_+(x,t)$ satisfies zero initial conditions.

One can consider also the Cauchy problem (48.1), (48.2), (or (48.3), (48.4)) with nonzero initial conditions. If $h_k(x) \in H_{m-k+s-\frac{1}{2}}(\mathbf{R}^n)$, $0 \leq k \leq m-1$, there exists $v_0 \in H_{m,s}(\mathbf{R}^{n+1}_+)$ such that $\frac{\partial^k v_0(x,+0)}{\partial t^k} = h_k(x)$, $0 \leq k \leq m-1$ (c.f. Example 13.3). We shall look for the solution of (48.3) in the form $v = v_0 + w$, where $w(x,t)$ satisfies

$$\begin{aligned} H(x,t,D_x,D_t - i\tau)w(x,t) &= g_0, \quad t > 0, \\ g_0 = g(x,t) - Hv_0 &\in H_{0,s}(\mathbf{R}^{n+1}_+). \end{aligned} \tag{48.26}$$

Let $g_+ \in \overset{\circ}{H}_{0,s} (\mathbf{R}^{n+1})$ be the extension of g_0 by zero for $t < 0$. By Theorem 48.1 there exists $w_+ \in \overset{\circ}{H}_{m-1,s} (\mathbf{R}^{n+1})$ such that $Hw_+ = g_+$. Let $w = w_t$ for $t > 0$. It was shown in §33 that $w \in H_{m,s-1}(\mathbf{R}^{n+1}_+)$ and $\frac{\partial^k w(x,0)}{\partial t^k} = 0$, $0 \leq k \leq m-1$. Therefore $v = v_0 + w$ satisfies (48.3) for $t > 0$, $v \in H_{m,s-1}(\mathbf{R}^{n+1}_+)$ and $\frac{\partial^k v(x,0)}{\partial t^k} = h_k(x)$, $0 \leq k \leq m-1$.

Switching back from v_+, g_+ to $u_+ = e^{t\tau}v_+$, $f_+ = e^{t\tau}g_+$ we get the following theorem:

Theorem 48.2. *For any $e^{-\tau t}f \in H_{0,s}(\mathbf{R}^{n+1}_+)$ and any $g_k(x) \in H_{m+s-k-\frac{1}{2}}(\mathbf{R}^n)$ there exists a unique $e^{-\tau t}u(x,t) \in H_{m,s-1}(\mathbf{R}^{n+1}_+)$ such that (48.1), (48.2) hold.*

$\square$

A modification of the proof of Theorem 48.1 gives an existence and uniqueness theorem in $R_T = \mathbf{R}^n \times (0,T)$ with less requirements on smoothness of the initial data.

Denote by $[u,v]$ the L^2-inner product in $\mathbf{R}^n$ when t is fixed and let $[u]_s = [\Lambda^s u]_0$ be the norm in $H_s(\mathbf{R}^n)$. Denote by $C_{m-1,s}$ the space of functions such that $\frac{\partial^k u(x,t)}{\partial t^k}$, $0 \le k \le m-1$, are continuous in $t \in [0,T]$ with values in $H_{m-k-1+s}(\mathbf{R}^n)$. The norm in $C_{m-1,s}$ is $\max_{0\le t\le T}\sum_{k=0}^{m-1}[D_t^k u]_{m-1-k+s}$. Also denote by $L^1[(0,T),H_s(\mathbf{R}^n)]$ the space with the norm $\int_0^T [f(x,t)]_s dt$.

Theorem 48.3. *Let s be arbitrary. For any $g_k(x) \in H_{s+m-1-k}(\mathbf{R}^n)$, $0 \le k \le m-1$, and any $f(x,t) \in L^1[(0,T),H_s(\mathbf{R}^n)]$, there exists a unique solution $u(x,t) \in C_{m-1,s}$ of the Cauchy problem (48.1), (48.2) in R_T such that*

$$\max_{0\le t\le T}\sum_{k=0}^{m}[D_t^k u(x,t)]_{m-k-1+s} \le C\sum_{k=0}^{m-1}[g_k]_{m-1-k+s} + C\int_0^T [f]_s dt. \tag{48.27}$$

Proof: Take any $w(x,t) \in H_{m,s}(R_T)$. Denote $F(x,t) = H(x,t,D_x,D_t - i\tau)w$, $\varphi_k(x) = \frac{\partial^k w(x,0)}{\partial t^k}$, $0 \le k \le m-1$. As in (48.14) we have

$$\begin{aligned}&[H(x,t,D_x,D_t - i\tau)w, H_{0\sigma}\Lambda^{2s}w] \\ &= \sum_{k=1}^{m}[(D_t - i\tau - \hat{\sigma}_k(x,t,D_x)\hat{q}_k\Lambda^s w, \hat{q}_k\Lambda^s w] + [Q_2\Lambda^s w, \Lambda^s w],\end{aligned} \tag{48.28}$$

where

$$|[Q_2\Lambda^s w, \Lambda^s w]| \le C\sum_{k=0}^{m-1}[\Lambda_\tau^{m-1-k}D_t^k w]_s^2 \overset{def}{=} C|[w]|_{m-1,s}^2, \tag{48.29}$$

$\Lambda_\tau = (|\xi|^2+\tau^2+1)^{\frac{1}{2}}$. Integrating (48.28) from 0 to t and taking the imaginary part we get as in (48.16):

$$\begin{aligned}&\sum_{k=1}^{m}[\hat{q}_k(x,t,D_x,D_t - i\tau)w]_s^2 - \sum_{k=1}^{m}[\hat{q}(x,0,D_x,D_t - i\tau)w]_s^2 \\ &+\tau\int_0^t\sum_{k=1}^{m}[\hat{q}_k w]_s^2 dt' \le C\int_0^t |[w]|_{m-1,s}^2 dt' + C\int_0^t [F]_s|[w]|_{m-1,s}dt'.\end{aligned} \tag{48.30}$$

We shall prove a variant of the Garding inequality (c.f. (48.17)):

$$\sum_{k=1}^{m}[\hat{q}_k(x,t,D_x,D_t-i\tau)w]_s^2 \geq C|[w]|_{m-1,s}^2 - \frac{C_1}{\tau}|[w]|_{m-1,s}^2. \tag{48.31}$$

Denote

$$w_k(x,t) = \Lambda_\tau^{m-1-k}\Lambda^s D_t^k w(x,t), \quad 0 \leq k \leq m-1. \tag{48.32}$$

We have

$$\hat{q}_k(x,t,D_x,D_t-i\tau)w = \sum_{j=0}^{m-1} a_{kj}(x,t,D_x,\tau)w_j(x,t),$$

where $a_{kj}(x,t,D_x,\tau)$ are ψdo of order zero. Note that

$$\sum_{k=1}^{m}[\hat{q}_k\Lambda^s w]_0^2 = [\sum_{j,p=0}^{m-1} a_{jp}^{(1)}w_p, w_j], \tag{48.33}$$

where $a_{jp}^{(1)} = \sum_{k=1}^m a_{kj}^* a_{kp},\ a_{kj}^*$ are the adjoint operators to a_{kj}.

Denote by $A(x,t,\xi,\tau)$ the matrix with elements $\sum_{k=1}^m \overline{a_{kj}(x,t,\xi,\tau)}a_{kp}(x,t,\xi,\tau)$.

It follows from (48.8), (48.9) that the matrix $A(x,t,\xi,\tau)$ is positive definite and

$$A(x,t,\xi,\tau) \geq C_0 I.$$

Let $B(x,t,\xi,\tau) = (A(x,t,\xi,\tau) - \frac{C_0}{2}I)^{\frac{1}{2}}$ and let $B(x,t,D_x,\tau)$ be the matrix ψdo with symbol $B(x,t,\xi,\tau)$. Let $\vec{w}(x,t) = (w_0,...,w_{m-1})$. Using Theorems 40.2, 43.2 we get (c.f. (48.18))

$$\sum_{j,p=0}^{m-1}[a_{jp}^{(1)}w_p, w_j] = \frac{C_0}{2}\sum_{k=0}^{m-1}[w_k]_0^2 + [B^*B\vec{w},\vec{w}] + [C_2\vec{w},\vec{w}], \tag{48.34}$$

where ord $C_2 \leq -1$ and

$$|[C_2\vec{w},\vec{w}]| \leq \frac{C}{\tau}\sum_{k=0}^{m-1}[w_k]_0^2. \tag{48.35}$$

Since $[B^*B\vec{w},\vec{w}] = [B\vec{w},B\vec{w}] \geq 0$ and since $\sum_{k=0}^{m-1}[w_k]_0^2 = |[w]|_{m-1,s}^2$ we get (48.31). Taking τ large and combining (48.30) and (48.31) we get

$$|[w(x,t)]|_{m-1,s}^2 \leq C\sum_{k=1}^{m}[\hat{q}(x,0,D_x,D_t-i\tau)w]_s^2+C\left(\int_0^T [F]_s dt\right)\max_{0\leq t\leq T}|[w]|_{m-1,s}. \tag{48.36}$$

Let t_0 be such that

$$|[w(x,t_0)]|_{m-1,s}^2 = \max_{0\leq t\leq T}|[w(x,t)]|_{m-1,s}.$$

Choosing such t_0 in (48.36) and using that

$$\sum_{k=1}^{m}[q_k(x,0,D_x,D_t-i\tau)w]_s^2 \leq C\sum_{k=0}^{m-1}[\varphi_k]_{m-k-1+s}^2 \leq C_1\max_{0\leq t\leq T}|[w(x,t)]|_{m-1,s}^2 \tag{48.37}$$

we get the estimate

$$\max_{0\leq t\leq T}|[w(x,t)]|_{m-1,s} \leq C\sum_{k=0}^{m-1}[\varphi_k]_{m-1-k+s} + C\int_0^T [F]_s dt. \tag{48.38}$$

The proof of the uniqueness of the solution of the Cauchy problem (48.1), (48.2) in R_T is the same as in Theorem 48.1: Let $u \in C_{m-1,s}$ be a solution of $H(x,t,D_x,D_t)u = 0$ in R_T, with zero initial conditions. Then $v = e^{-t\tau}u$ satisfies $H(x,t,D_x,D_t-i\tau)v = 0$ with zero initial conditions and $v \in C_{m-1,s}$.

It follows from (48.25) that $v \in C_{m,s-1}$. Since s in the estimate (48.38) is arbitrary we get $v = 0$.

To prove the existence choose any $F \in H_{0,s+1}(R_T)$. Note that $F \in L^1[(0,T),H_s(\mathbf{R}^n)]$ since $(\int_0^T [F]_s dt)^2 \leq T\|F\|_{0,s}^2$. Choose also any $\varphi_k \in H_{m-k+\frac{1}{2}+s}(\mathbf{R}^n)$, $0 \leq k \leq m-1$. It follows from Theorem 48.2 that there exists $w \in H_{m,s}(R_T)$ such that $Hw = F$, $\frac{\partial^k w(x,0)}{\partial t^k} = \varphi_k(x)$, $0 \leq k \leq m-1$. Note that the estimate (48.38) holds for $w(x,t)$ since $H_{m,s}(R_T) \subset C_{m-1,s}$. Take a sequence $F_n \in H_{0,s+1}(R_T)$ that converges to $f(x,t)$ in the norm $L^1[(0,T),H_s(\mathbf{R}^n)]$. Also let $\varphi_k^{(n)}(x) \in H_{m-k+\frac{1}{2}+s}(\mathbf{R}^n)$ converge to $g_k \in H_{m-1-k+s}(\mathbf{R}^n)$ in $H_{m-k-1+s}(\mathbf{R}^n)$, $0 \leq k \leq m-1$. Let $w_n(x) \in H_{m,s}(R_T)$ be the solution of the Cauchy problem $Hw_n = F_n$ in R_T, $\frac{\partial^k w_n(x,0)}{\partial t^k} = \varphi_k^{(n)}(x)$, $0 \leq k \leq m-1$. It follows from (48.38) that w_n converges in $C_{m-1,s}$ to the solution $u(x,t) \in C_{m-1,s}$ of the Cauchy problem (48.1), (48.2) in R_T.

Note that since $e^{-\tau t}$ and $e^{\tau t}$ are bounded on R_T the norms of $u(x,t)$ and $v(x,t) = e^{-t\tau}u(x,t)$ are equivalent.