46 The Cauchy problem for the parabolic equations.

Consider a differential equation in $\mathbf{R}_+^{n+1} = \{t > 0, x \in \mathbf{R}^n\}$ of the form:

$$\frac{\partial u(x,t)}{\partial t} + A(x,t,D)u(x,t) = f, \quad t > 0, \quad x \in \mathbf{R}^n, \tag{46.1}$$

where $A(x,t,\xi) = A_0(x,t,\xi) + A_1(x,t,\xi),\ A_0(x,t,\xi) = \sum_{|k|=m} a_k(x,t)\xi^k$, $A_1(x,t,\xi) = \sum_{|k|\le m-1} a_k(x,t)\xi^k,\ D = -i\frac{\partial}{\partial x}$.

We assume that $a_k(x,t) \in C^\infty(\mathbf{R}^{n+1}),\ a_k(x,t) = a_k^\infty$ for $|x|^2 + |t|^2 > R^2$. Equation (46.1) is called parabolic if $A(x,t,D)$ is elliptic, moreover, $\Re A_0(x,t,\xi) \ge C|\xi|^m,\ \forall(x,t)$. We shall study the Cauchy problem, i.e. $u(x,t)$ satisfies (46.1) and the initial condition

$$u(x,0) = u_0(x),\ x \in \mathbf{R}^n. \tag{46.2}$$

Let $v(x,t) = e^{-t\tau}u(x,t), \tau > 0$ Then $v(x,t)$ satisfies the equation:

$$\frac{\partial v(x,t)}{\partial t} + \tau v + A(x,t,D)v(x,t) = g, \quad t > 0, \quad x \in \mathbf{R}^n, \tag{46.3}$$

and

$$v(x,0) = u_0(x), \tag{46.4}$$

We shall prove the existence and the uniqueness of the solution of the Cauchy problem (46.3), (46.4) assuming that τ is large.

Introduce Sobolev spaces that are adapted to the form of the parabolic equation (46.1). Let $\Pi_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ be the space of distributions with finite norm

$$\|u\|^2_{\frac{s}{m},s} = \int_{\mathbf{R}^{n+1}} |\tilde{u}(\xi,\tau)|^2 |i\sigma + (|\xi|^2+1)^{\frac{m}{2}}|^{\frac{2s}{m}} d\xi d\sigma, \tag{46.5}$$

where

$$\tilde{u}(\xi,\sigma) = \int_{\mathbf{R}^{n+1}} u(x,t)e^{-ix\cdot\xi - i\sigma t} dxdt.$$

As in \$ 33 we denote by $\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ the subspace of $\Pi_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ consisting of distributions with supports in $\mathbf{R}_+^{n+1} = \{t > 0, x \in \mathbf{R}^n\}$ and denote by

$\Pi_{\frac{s}{m},s,\tau}(\mathbf{R}_+^{n+1})$ the space of restrictions of $u \in \Pi_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ to the half-space $t > 0$ with the norm:

$$\|u\|^+_{\frac{s}{m},s} = \inf_l \|lu\|_{\frac{s}{m},s}, \tag{46.6}$$

where lu is an arbitrary extension of u to $\mathbf{R}^{n+1}$ (c.f. §33).

Denote by $P_{\alpha,m}$ the class of symbols $A(x,t,\xi,\sigma)$ such that A is C^∞ in all variables, $A(x,t,\xi,\sigma) = A(\infty,\xi,\sigma)$ when $|x|^2 + |t|^2 > R^2$ and

$$\left|\frac{\partial^{k_1+k_2+k_3+k_4} A(x,t,\xi,\sigma)}{\partial x^{k_1}\partial t^{k_2}\partial \xi^{k_3}\partial \sigma^{k_4}}\right| \leq C_k |\Lambda_m^+|^{\frac{\alpha}{m}-\frac{k_3}{m}-k_4}, \quad \forall k, \tag{46.7}$$

where

$$\Lambda_m^+ = i\sigma + (|\xi|^2+1)^{\frac{m}{2}}. \tag{46.8}$$

Note that when $m = 1$ P_m^α coincide with S^α (c.f. §40). Note also that the norm (46.5) has the form $\|(\Lambda_m^+)^s u\|_0$, where $\| \ \|_0$ is the norm in $L^2(\mathbf{R}^{n+1})$, and Λ_m^+ is the *ψdo* with the symbol $\Lambda_m^+(\xi,\sigma)$.

Analoguously to the proof of Theorems 40.1 and 40.2 we have

Lemma 46.1. *Pseudodifferential operator A with symbol $A(x,t,\xi,\sigma) \in S_m^\alpha$ is bounded from $\Pi_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ to $\Pi_{\frac{s-\alpha}{m},s-\alpha}(\mathbf{R}^{n+1})$. If $A(x,t,\xi,\sigma) \in P_{\alpha,m}$, $B(x,t,\xi,\sigma) \in P_{\beta,m}$ then*

$$AB = C + T_{-1}, \tag{46.9}$$

where $C(x,t,\xi,\sigma) = A(x,t,\xi,\sigma)B(x,t,\xi,\sigma)$, and

$$\|T_{-1}u\|_{\frac{s}{m},s} \leq C\|u\|_{\frac{s+1}{m},s+1} \quad \forall u \in C_0^\infty(\mathbf{R}^{n+1}), \ \forall s. \tag{46.10}$$

The following lemma is crucial in this section:

Lemma 46.2. *Let $A(x,t,\xi,\sigma - i\tau) \in P_{\alpha,m}$ be analytic in $z = \sigma - i\tau$ for $\tau > 0$ and continuous for $\tau \geq 0$. Suppose $A(x,t,\xi,\sigma - i\tau)$ satisfies estimates (46.7) with σ replaced by $z = \sigma - i\tau, \tau \geq 0$. Then the ψdo $A(x,t,D_x,D_t)$ maps $\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ into $\overset{\circ}{\Pi}_{\frac{s-\alpha}{m},s-\alpha}(\mathbf{R}^{n+1})$ for any $s \in \mathbf{R}$.*

Proof: For any $u(x,t) \in C_0^\infty(\mathbf{R}_+^{n+1})$ we have:

$$Au = \frac{1}{(2\pi)^{n+1}} \int_{\mathbf{R}^{n+1}} A(x,t,\xi,\sigma)\tilde{u}(\xi,\sigma) e^{-ix\cdot\xi - it\sigma} dxdt. \tag{46.11}$$

Since A and $\tilde{u}(\xi,\sigma)$ are analytic in $z=\sigma-i\tau$ for $\tau>0$ and since $\tilde{u}(\xi,z)$ decays fast when $|\Re z|\to\infty$ we can move the line of integration in z using the Cauchy theorem:

$$(46.12)\quad Au=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{-\infty}^{\infty}A(x,t,\xi,\sigma-i\tau)\tilde{u}(\xi,\sigma-i\tau)e^{-ix\cdot\xi-it(\sigma-i\tau)}d\sigma d\xi,$$

where $\tau>0$ is arbitrary. Therefore

$$(46.13)\qquad |(Au)(x,t)|\leq Ce^{t\tau},\ \forall\tau>0.$$

Fix any $t_0<0$. Taking the limit in (46.13) when $\tau\to+\infty$ we get $(Au)(x,t_0)=0$. Therefore supp $Au\subset\overline{\mathbf{R}^{n+1}_+}$. It follows from Lemma 46.1 that $Au\in\Pi_{\frac{s-\alpha}{m},s-\alpha}(\mathbf{R}^{n+1})$. Therefore $Au\in\overset{\circ}{\Pi}_{\frac{s-\alpha}{m},\frac{s}{m}}$. Since $C_0^\infty(\mathbf{R}^{n+1}_+)$ is dense in $\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ we have that $Au\in\overset{\circ}{\Pi}_{\frac{s-\alpha}{m},s-\alpha}$ for any $u\in\overset{\circ}{\Pi}_{\frac{s}{m},s}$.

We shall denote by $P^+_{\alpha,m}$ the class of symbols in $P_{\alpha,m}$ that are analytic in $z=\sigma-i\tau$ for $\tau>0$ and satisfy estimates (46.7) with σ replaced by $\sigma-i\tau,\tau\geq0$.

Theorem 46.3. *Consider the equation (46.3). Let s be arbitrary. If τ is sufficiently large then for any $g\in\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ there exists a unique $v\in\overset{\circ}{\Pi}_{\frac{s+m}{m},s+m}(\mathbf{R}^{n+1})$ that solves (46.3) in $\mathbf{R}^{n+1}$.*

Proof: Let $R_0(x,t,\xi,\sigma-i\tau)=(i\sigma+\tau+A_0(x,t,\xi))^{-1},\ \tau\geq)$. Note that $R_0\in P^+_{-m,m}$, i.e. R_0 is analytic in $z=\sigma-i\tau$ for $\tau>0$.

It follows from Lemmas 46.1 and 46.2 that the ψdo R_0 maps $\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ to $\overset{\circ}{\Pi}_{\frac{s+m}{m},s+m}(\mathbf{R}^{n+1})$ for any s. Note that $(i\sigma+\tau+A(x,t,\xi))R_0(x,t,\xi,\sigma-i\tau)=1+A_1(x,\xi)(i\sigma+\tau+A_0)^{-1}$. We have

$$(46.14)\qquad |A_1(x,\xi)|\ |R_0|\leq\frac{C(1+|\xi|)^{m-1}}{|\sigma|+\tau+|\xi|^m}\leq C_1(|\sigma|+\tau+|\xi|^m)^{-\frac{1}{m}}.$$

Therefore, $A_1(x,\xi)R_0\in P_{-\frac{1}{m},m}$. Applying (46.9) to the composition of $\frac{\partial}{\partial t}+\tau+A$ and R_0 we get

$$(46.15)\qquad \left(\frac{\partial}{\partial t}+\tau+A\right)R_0=I+T^{(1)}_{-1},$$

where $T_{-1}^{(1)} = A_1(x,D)R_0 + T_{-1}$, ord $T_{-1}^{(1)} \le -1$. Moreover the presence of the parameter τ implies that (c.f. (46.14) and §41)

$$\|T_{-1}^{(1)}u\|_{\frac{s}{m},s} \le \frac{C}{\tau^{\frac{1}{m}}}\|u\|_{\frac{s}{m},s}. \qquad (46.16)$$

Note that $T_1^{(1)}$ acts in $\overset{\circ}{\Pi}_{\frac{1}{m},s}(\mathbf{R}^{n+1})$ since $R_0(x,t,\xi,\sigma - i\tau) \in P^+_{-1,mm}$ and $A_1 \in P^+_{1,m}$. Therefore $T_1^{(1)}$ has a small norm in $\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ when τ is large and $R = R_0(I + T_{-1}^{(1)})^{-1}$ is the right inverse of $\frac{\partial}{\partial t} + \tau + A$. Analoguosly (c.f. \$ 41) one can prove that $\frac{\partial}{\partial t} + \tau + A$ has the left inverse. □

Since $v = e^{-t\tau}u, g = e^{-t\tau}f$, we have:

Theorem 46.4. *Let τ be large. Then for any $f(x,t)$ in $\mathbf{R}^{n+1}$ such that $e^{-t\tau}f \in \overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ there exists a unique solution of (46.1) in $\mathbf{R}^{n+1}$ such that $e^{-t\tau}u \in \overset{\circ}{\Pi}_{\frac{s+m}{m},s+m}(\mathbf{R}^{n+1})$.*

□

Now we shall study the Cauchy problem (46.3) with nonzero initial data(46.4). Analoguosly to the proof of Theorem 13.6 we have:

Lemma 46.5. *Let $s > \frac{m}{2}$. Then $\Pi_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ is embedded into the space $C(\mathbf{R}, H_s(\mathbf{R}^n))$ and*

$$\sup_t \|u(\cdot,t)\|_{s-\frac{m}{2}} \le C\|u\|_{\frac{s}{m},s}. \qquad (46.17)$$

Theorem 46.6. *Let $s > \frac{m}{2}$ and let τ be large. Then for any $u_0(x) \in H_{s-\frac{m}{2}}(\mathbf{R}^n)$ and for any $g \in H_{\frac{s-m}{m},s-m}(\mathbf{R}^{n+1}_+)$ there exists a unique solution $v \in H_{\frac{s}{m},s}(\mathbf{R}^{n+1}_+)$ of (46.3) such that (46.4) holds.*

Proof: Analoguosly to the proof of Example 13.3 for $\forall u_0(x) \in H_{s-\frac{m}{2}}(\mathbf{R}^n)$ there exists $v_0(x,t) \in \Pi_{\frac{s}{m},s}(\mathbf{R}^{n+1})$ such that $v_0(x,0) = u_0(x)$ and

$$\|v_0\|_{\frac{s}{m},s} \le C\|u_0\|_{s-\frac{m}{2}}. \qquad (46.18)$$

Let $w(x,t) = v(x,t) - v_0(x,t)$ for $t \ge 0$. Then

$$w(x,0) = 0 \qquad (46.19)$$

and $w(x,t)$ satisfies

(46.20) $$(\frac{\partial}{\partial t}+\tau+A)w=g_1 \quad \text{when} \quad t>0,$$

where $g_1=g-p(\frac{\partial}{\partial t}+\tau+A)v_0\in\Pi_{\frac{s-m}{m},s-m}(\mathbf{R}_+^{n+1})$. Here p is the restriction operator to the half-space $\mathbf{R}_+^{n+1}$.

Consider first the case $\frac{m}{2}<s\le m$. Since $w\in\Pi_{\frac{s}{m},s}(\mathbf{R}_+^{n+1})$ and $w(x,0)=0$ we have that $w_+=w$ for $t>0$, $w_+=0$ for $t<0$ belongs to $\overset{\circ}{\Pi}_{\frac{s}{m},s}$ $(\mathbf{R}^{n+1})$. Applying Theorem 46.3 we get that such w_+ exists, it satisfies $(\frac{\partial}{\partial t}+\tau+A)w_+=g_+$ and

(46.21) $$\|w_+\|_{\frac{s}{m},s}\le C\|g_+\|_{\frac{s-m}{m},s-m},$$

where $g_+=g_1$ for $t>0$, $g_+=0$ for $t<0$. Choosing $v(x,t)=w(x,t0+v_0(x,t)$ for $t>0$ we prove the Theorem 46.6 for $\frac{m}{2}<s\le m$.

Now consider the case $s>m$. Then $s=mk_0+\gamma$, where $k_0\ge 1$ is an integer, $0<\gamma\le m$. Since $w(x,0)=0$ we get, analogously to (46.21)

(46.22) $$\|w\|_{\frac{\gamma}{m},\gamma}\le C\|g_1\|_{\frac{\gamma-m}{m},\gamma-m}.$$

To show that $w\in\Pi_{\frac{s}{m},s}$ we shall use the same regularity arguments as in §35.

Denote $D_{h_k}w=\frac{w(x+\vec{h}_k,t)-w(x,t)}{h_k}$, where $\vec{h}_k=(0,...,h_k,0,...,0)$. As in §35 one can show that $\|D_{\vec{h}_k}w\|_{\frac{s}{m},s}\le C$ for all small h_k iff $\Lambda w\in\Pi_{\frac{s}{m},s}(\mathbf{R}^{n+1})$, where Λ is the ψdo with symbol $(1+|\xi|^2)^{\frac{1}{2}}$. Applying D_{h_k} to (46.20) and using (46.22) we get that $\Lambda w\in\Pi_{\frac{\gamma}{m},\gamma}(\mathbf{R}_+^{n+1})$. Repeating the same arguments we can prove that

(46.23) $$\Lambda^{s-\gamma}w\in\Pi_{\frac{\gamma}{m},\gamma}(\mathbf{R}_+^{n+1})$$

since $g_1\in\Pi_{\frac{s-m}{m},s-m}(\mathbf{R}_+^{n+1})$, $s=\gamma+mk_0$. It follows from (46.20) and (46.23) that

$$\frac{\partial w}{\partial t}\in\Pi_{\frac{\gamma}{m},\gamma}(\mathbf{R}_+^{n+1})\subset\Pi_{\frac{s-m}{m},s-m}(\mathbf{R}_+^{n+1})$$

since $s-\gamma=k_0m\ge m$. If $k_0=1$ then we get $w\in H_{\frac{s}{m},s}(\mathbf{R}_+^{n+1})$ and the proof is completed. If $k_0>1$ we use that now $w\in\Pi_{\frac{\gamma+m}{m},\gamma+m}(\mathbf{R}_+^{n+1})$. Then the equations (46.20) and (46.23) gives that $\frac{\partial w}{\partial t}\in\Pi_{\frac{\gamma+m}{m},\gamma+m}(\mathbf{R}_+^{n+1})$, $\Lambda^m w\in\Pi_{\frac{\gamma}{m},\gamma}(\mathbf{R}_+^{n+1})$ etc., until we get that $w\in\Pi_{\frac{s}{m},s}(\mathbf{R}_+^{n+1})$.

Remark 46.1 As in the Theorem 46.4 we get that for τ large there exists a unique solution $u(x,t)$ of the Cauchy problem (46.1), (46.2) such that $e^{-\tau t}u \in \Pi_{\frac{s}{m},s}(\mathbf{R}_+^{n+1})$ assuming that $s > \frac{m}{2}$, $u_0(x) \in H_{s-\frac{m}{2}}(\mathbf{R}^n)$, $e^{-\tau t}f \in \Pi_{\frac{s-m}{m},s-m}(\mathbf{R}_+^{n+1})$.

The following estimate holds:

$$\|e^{-\tau t}u\|^+_{\frac{s}{m},s} \le C\|e^{-\tau t}f\|^+_{\frac{s}{m}-1,s-m} + C\|u_0\|_{s-\frac{m}{2}}, \tag{46.24}$$

where $\|\ \|^+_{\frac{s}{m},s}$is the norm in $\Pi_{\frac{s}{m},s}(\mathbf{R}_+^{n+1})$.

Remark 46.2 Consider the Cauchy problem (46.1), (46.2) on a finite time interval $R_T = \mathbf{R}^n \times (0,T)$. Suppose $f \in \Pi_{\frac{s}{m}-1.s-m}(R_T)$. Let f_1 be an extension of $f(x,t)$ such that $e^{-\tau t}f_1 \in \Pi_{\frac{s-m}{m},s-m}(\mathbf{R}_+^{n+1})$ and $\|e^{-\tau t}f_1\|_{\frac{s}{m}-1,s-m} \le C\|f\|_{\frac{s}{m}-1,s-m,R_T}$

By the Remark 46.1 there exists a unique solution u_1 of the Cauchy problem (46.1), (46.2) such that $e^{-\tau t}u_1 \in \Pi_{\frac{s}{m},s}(\mathbf{R}_+^{n+1})$ and the estimate (46.24) holds. Let $u(x,t)$ be the restriction of $u_1(x,t)$ to $R_T = \mathbf{R}^n \times (0,T)$. Note that $e^{-\tau t}$ and $e^{\tau t}$ are C^∞ bounded functions on $[0,T]$. Therefore $u(x,t)$ solves the Cauchy problem (46.1), (46.2) on $\mathbf{R}^n \times (0,T)$ and

$$\|u\|_{\frac{s}{m},s,R_T} \le C\|e^{-\tau t}u_1\|^+_{\frac{s}{m},s} \le C\|u_0\|_{s-\frac{m}{2}} + C\|e^{-\tau t}f_1\|^+_{\frac{s}{m}-1,s-m}$$
$$\le C\|u_0\|_{s-\frac{m}{2}} + C_1\|f\|_{\frac{s}{m}-1,s-m,R_T}.$$

47 The heat kernel.

Consider the Cauchy problem (46.1), (46.2) assuming that the coefficients of A are independent of t and $A_0(x,\xi) \ge C|\xi|^m$. Let $f = 0$ and let $u(x,t)$ be the solution of

$$\frac{\partial u}{\partial t} + A(x,D)u = 0, \quad t > 0,\ x \in \mathbf{R}^n, \tag{47.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}^n. \tag{47.2}$$

It follows from Remark 46.1 that for any $u_0(x) \in H_s(\mathbf{R}^n)$, $s > \frac{m}{2}$ there exists an unique solution $u(x,t)$ of (47.1), (47.2) such that $e^{-\tau t}u(x,t) \in \Pi_{\frac{s}{m},s}(\mathbf{R}_+^{n+1})$ where $\tau \ge \tau_0$, τ_0 is large enough. Let $u_+ = u(x,t)$ for $t > 0$, $u_+ = 0$ for $t < 0$. Since $e^{-\tau t}u(x,t) \in L_2(\mathbf{R}_+^{n+1}) \subset \Pi_{\frac{s}{m},s}(\mathbf{R}_+^{n+1})$ we have that the Fourier transform $\tilde{u}_+(\xi, \sigma - i\tau)$ is analytic in $z = \sigma - i\tau$ for $\tau > \tau_0$. The Parseval's equality holds:

$$(47.3) \quad \int_{\mathbf{R}^n} \int_0^\infty e^{-2t\tau} |u(x,t)|^2 dt dx = \frac{1}{(2\pi)^{n+1}} \int_{\mathbf{R}^n} \int_{-\infty}^\infty |\tilde{u}_+(\xi, \sigma - i\tau)|^2 dx d\sigma$$

for any $\tau \geq \tau_0$. We have

$$(47.4) \qquad \frac{\partial u_+}{\partial t} + A(x,D) u_+ = \delta(t) u_0(x).$$

Performing the Fourier transform in (47.4) we get

$$i(\sigma - i\tau)\hat{u}_+(x, \sigma - i\tau) + A(x,D)\hat{u}_+(x, \sigma - i\tau) = u_0(x),$$

where $\hat{u}_+(x, \sigma - i\tau)$ is the Fourier transform of $u(x,t)$ in t only. It follows from Lemma 41.2 that

$$(47.5) \qquad \hat{u}(x, \sigma - i\tau) = (A + i(\sigma - i\tau))^{-1} u_0(x).$$

Therefore

$$(47.6) \qquad u(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty (A + i(\sigma - i\tau))^{-1} u_0(x) e^{it(\sigma - i\tau)} d\sigma, \quad \tau > \tau_0.$$

Note that the integral (47.6) does not depend on τ by the Cauchy integral theorem and it converges in L_2 norm (c.f. the Parseval's formula (47.3)).

Applying $(A + i(\sigma - i\tau))^{-1}$ to (41.9) from the left we get

$$(47.7) \qquad (A + \lambda I)^{-1} = R^{(N)} + T^{(1)}_{-m-N-1},$$

where $\lambda = i(\sigma - i\tau)$, $R^{(N)} = \sum_{k=0}^N R_k$, $T^{(1)}_{-(m+N+1)} = -(A + \lambda I)^{-1} T_{-N-1}$. Note that ord $T^{(1)}_{-(m+N+1)} \leq -(m + N + 1)$ since ord $(A + \lambda I)^{-1} = -m$. Substitute (47.7) into (47.6). We get

$$u(x,t) = u_1(x,t) + u_2(x,t),$$

where

$$(47.8) \qquad u_1(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty R^{(N)}(x, D, i(\sigma - i\tau) u_0(x) e^{it(\sigma - i\tau)} d\sigma,$$

$$(47.9)\qquad u_2(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(T^{(1)}_{-(m+N+1)}u_0)(x)e^{it(\sigma-i\tau)}d\sigma,$$

and $R^{(N)}(x,\xi,\lambda)$ has the form (41.11).

Integral with respect to σ in (47.8) can be computed easily by the residue theory since

$$(47.10)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{it(\sigma-i\tau)}d\sigma}{(A_0(x,\xi)+i(\sigma-i\tau))^{k+1}}=\frac{t^k}{k!}e^{-tA_0(x,\xi)}.$$

Therefore

$$(47.11)\qquad u_1(x,t)=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\sum_{k=0}^{N}p_k(x,\xi)\frac{t^k}{k!}e^{-tA_0(x,\xi)}\tilde{u}_0(\xi)e^{ix\cdot\xi}d\xi,$$

where $p_0=1,\ \deg p_k\le mk-\frac{k}{2},\ k\ge 1$.

It follows from the proof of Theorem 40.2 that $T^{(1)}_{-(m+N+1)}$ is an integral operator with a continuous kerner, $T^{(1)}_{-(m+N+1)}(x,y,\sigma-i\tau)$ when $-m-N-1<-n$.

Denote by e^{-tA} the solution operator of the Cauchy problem (47.1), (47.2), i.e.

$$(47.12)\qquad u(x,t)=e^{-tA}u_0(x).$$

Let $G(x,y,t)$ be the kernel of e^{-tA}. Note that $G(x,y,t)$ is the solution of the Cauchy problem

$$(47.13)\qquad \left(\frac{\partial}{\partial t}+A\right)G(x,y,t)=0,\quad t>0,\ x\in\mathbf{R}^n,\ y\in R^n,$$
$$G(x,y,0)=\delta(x-y).$$

$G(x,y,t)$ is called the heat kernel.

We have from (47.9), (47.11)

(47.14)
$$G(x,y,t)=\sum_{k=0}^{N}\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}p_k(x,\xi)\frac{t^k}{k!}e^{-tA_0(x,\xi)-i(x-y)\cdot\xi}d\xi+G_N(x,y,t),$$

where

$$(47.15)\qquad G_N(x,y,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}T^{(1)}_{-m-N-1}(x,y,\sigma-i\tau)e^{it(\sigma-i\tau)}d\sigma.$$

Note that $T^{(1)}(x, y, \sigma - i\tau)$ is analytic in $\sigma - i\tau$ for $\tau > \tau_0$ since all other terms in (47.7) are analytic in $z = \sigma - i\tau$. It follows from estimates of the form (41.4) that

$$|T^{(1)}_{-m-N-1}(x, y, \sigma| \leq C|\sigma - i\tau|^{-N_1-2}, \tag{47.16}$$

where $-m-N-1 < -n-N_1-2$. Therefore (47.16) implies that $G_N(x, y, t)$ has N_1 continuous derivatives in t and $G_N(x, y, t) = 0$ for $t < 0$ because of the analyticity of $T^{(1)}_{-m-N-1}$. Therefore

$$|G_N(x, y, t)| \leq C_N t^{N_1}, \quad t > 0. \tag{47.17}$$

Take $x = y$ in (47.14) and make change of variables

$$\eta = \frac{\xi}{t^{\frac{1}{m}}}. \tag{47.18}$$

We get

$$\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-tA_0(x,\xi)} d\xi = \frac{c_0(x)}{t^{\frac{n}{m}}}, \tag{47.19}$$

where $c_0(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-A_0(x,\eta)} d\eta$. Making the change of variables (47.18) for $1 \leq k \leq N$ in (47.14), taking $x = y$ and collecting terms having the same power of t we get

$$G(x, x, t) = \frac{1}{t^{\frac{n}{m}}}(c_0(x) + \sum_{k=1}^{N_2-1} c_k(x) t^{\frac{k}{m}}) + O(t^{\frac{N_2-n}{m}}), \tag{47.20}$$

where coefficients $c_k(x)$ have an explicit form. Note that N_2 in (47.20) is arbitrary since $N_1 > \frac{N_2-n}{m}$ is arbitrary.

Further simplification occurs when $A_0(x, D)$ is a second order elliptic operator, $A_0(x, \xi) = \sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k$. We have

$$c_0(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-\sum_{j,k=1}^n g^{jk}(x)\eta_j\eta_k} d\eta = \frac{1}{(2\sqrt{\pi})^n} \sqrt{g(x)}, \tag{47.21}$$

where $g(x) = \det[g^{jk}(x)]^{-1}$.

To compute (47.21) one should make an orthogonal transformation $\eta = O\zeta$ such that $\sum_{j,k=1}^n g^{jk}(x)\eta_j\eta_k = \sum_{j=1}^n \lambda_j \zeta_j$ and make changes of coordinates $\zeta_j = \frac{1}{\sqrt{\lambda_j}}\zeta_j'$, $1 \leq j \leq n$.