44 Pseudolocal property and microlocal regularity of pseudodifferential operators.

Let A be a linear continuous operator from $D(\mathbf{R}^n)$ to $D'(\mathbf{R}^n)$. The Schwartz kernel of A is the distribution $E \in D'(\mathbf{R}^n \times \mathbf{R}^n)$ such that $(Au, v) = E(u(y)\overline{v(x)})$ for all $u \in C_0^\infty(\mathbf{R}^n)$, $v \in C_0^\infty(\mathbf{R}^n)$. Let Au be a ψdo of the form (43.2). Let

$$E_\varepsilon(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} a(x,y,\xi) e^{i(x-y)\cdot\xi} \chi(\varepsilon\xi) d\xi, \tag{44.1}$$

and let $w \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$. Then $E_\varepsilon(x,y) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and

$$E_\varepsilon(w(x,y)) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} E_\varepsilon(x,y) w(x,y) dx dy$$

converges in $D'(\mathbf{R}^n \times \mathbf{R}^n)$ to $E \in D'(\mathbf{R}^n \times R^n)$ where

$$E(w(x,y)) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} a(x,y,\xi) e^{i(x-y)\cdot\xi} w(x,y) dx dy d\xi. \tag{44.2}$$

The integral in (44.2) is understood as a repeated integral: first integrated in x and y and then integrated in ξ. Since $(Au, v) = E(u(y)\overline{v(x)})$, the distribution E is the Schwartz kernel of A. When $\alpha < -n$ the integral

$$E(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} a(x,y,\xi) e^{i(x-y)\cdot\xi} d\xi \tag{44.3}$$

converges absolutely and defines the Schwartz kernel of A. When $\alpha \geq -n$ we understand (44.3) as a limit in $D'(\mathbf{R}^n \times \mathbf{R}^n)$ of

$$E_\varepsilon(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} a(x,y,\xi) e^{i(x-y)\cdot\xi} \chi(\varepsilon\xi) d\xi$$

when $\varepsilon \to 0$.

Theorem 44.1. *For arbitrary $\delta > 0$ the distribution $(1 - \chi(\frac{x-y}{\delta}))E$ belongs to $C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$.*

Proof: Since $\frac{\partial}{\partial\xi_k} e^{i(x-y)\cdot\xi} = i(x_k - y_k) e^{i(x-y)\cdot\xi}$ we have the following identity:

$$e^{i(x-y)\cdot\xi} = (-\Delta_\xi)^N \frac{e^{i(x-y)\cdot\xi}}{|x-y|^{2N}}, \tag{44.4}$$

where $\Delta_\xi = \sum_{j=1}^n \frac{\partial^2}{\partial\xi_j^2}$. Substituting (44.4) into (44.1) and integrating by parts in ξ we get
(44.5)

$$(1-\chi(\frac{x-y}{\delta}))E_\varepsilon(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}(-\Delta_\xi)^N(a(x,y,\xi)\chi(\varepsilon\xi))\frac{(1-\chi(\frac{x-y}{\delta}))}{|x-y|^{2N}}e^{i(x-y)\cdot\xi}d\xi.$$

Denote

$$E^{(\delta)}(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}(-\Delta_\xi)^N a(x,y,\xi)\frac{(1-\chi(\frac{x-y}{\delta}))}{|x-y|^{2N}}e^{i(x-y)\cdot\xi}d\xi. \tag{44.6}$$

We will study the limit of (44.5) when $\varepsilon \rightarrow 0$. Fix $M > 0$ and assume that $2N > 2M + n + 1 + \alpha$. Note that $|\frac{\partial^k}{\partial\xi^k}\chi(\varepsilon\xi)| \leq C\varepsilon^k$ and $\frac{\partial^k}{\partial\xi^k}\chi(\varepsilon\xi)$ differs from zero only when $\frac{1}{\varepsilon} < |\xi| < \frac{c}{\varepsilon}$ for $|k| \geq 1$. Therefore

$$|(1-\frac{\chi(x-y)}{\delta})E_\varepsilon(x,y) - E^{(\delta)}(x,y)| \leq C\int_{|\xi|>\frac{1}{\varepsilon}}(1+|\xi|)^{\alpha-2N}d\xi$$

$$+C\sum_{|k|=1}^{2N}\int_{\frac{1}{\varepsilon}<|\xi|<\frac{c}{\varepsilon}}\varepsilon^k(1+|\xi|)^{\alpha-2N+k}d\xi \rightarrow 0$$

when $\varepsilon \rightarrow 0$, since $2N > n+1+\alpha$. Since $(1-\frac{\chi(x-y)}{\delta})E_\varepsilon(x,y) \rightarrow (1-\frac{\chi(x-y)}{\delta})E$ in the distribution sense we got that

$$(1-\frac{\chi(x-y)}{\delta})E = E^{(\delta)}, \tag{44.7}$$

where $E^{(\delta})$ is the same as in (44.6). Note that $\frac{(1-\chi(\frac{x-y}{\delta}))}{|x-y|^{2N}} \in C^\infty(\mathbf{R}^n\times\mathbf{R}^n)$ since $\frac{(1-\chi(\frac{x-y}{\delta}))}{|x-y|^{2N}} = 0$ when $|x-y| < \delta$. We shall show that $E^{(\delta)} \in C^\infty(\mathbf{R}^\times\mathbf{R}^n)$. Note that $|(-\Delta_\xi)^N a(x,y,\xi)| \leq C(1+|\xi|)^{\alpha-2N} \leq C(1+|\xi|)^{-(n+1)-2M}$. Therefore differentiating (44.6) in x and y we get that $E^{(\delta)}$ has M continuous derivatives in x and y. Since M is arbitrary we get that $E^{(\delta)} = (1-\frac{\chi(x-y)}{\delta})E \in C^\infty(\mathbf{R}^n\times\mathbf{R}^n)$.

Corollary 44.2 (Pseudolocal property of $\psi do.$)**.** *Let* $\varphi \in C_0^\infty(\mathbf{R}^n), \psi(x) \in C_0^\infty(\mathbf{R}^n)$ *and supp* $\varphi \cap$ *supp* $\psi = \emptyset$. *Let* $A(x,\xi) \in S^\alpha$. *Then* $\psi(x)A\varphi u$ *is an integral operator with* $C^\infty(\mathbf{R}^n\times\mathbf{R}^n)$ *kernel.*

Proof: Since supp $\varphi\cap$ supp $\psi=\emptyset$ there exists $\delta>0$ such that $\psi(x)a(x,y,\xi)\varphi(y)=0$ when $|x-y|<\delta$. Therefore $\psi(x)A\varphi$ is an integral operator with $C^\infty(\mathbf{R}^n\times\mathbf{R}^n)$ kernel. Note that ord $\psi A\varphi=-\infty$.

Lemma 44.3. *Let* $A(x,\xi)\in S^\alpha$ *and let* $u\in H_s(\mathbf{R}^n)$ *for some* $s\in\mathbf{R}$. *We have*

$$\text{sing supp } Au\subset \text{ sing supp } u.$$

Proof: Let $x_0\notin$ sing supp u. Then there exists $\varphi(x)\in C_0^\infty(\mathbf{R}^n)$, $\varphi(x)=1$ near x_0 and supp $\varphi\cap$ sing supp $u=\emptyset$. Let $\psi(x)\in C_0^\infty(\mathbf{R}^n),\psi(x_0)\neq 0$, supp ψ is contained in the interior of the set where $\varphi=1$. Then supp $\psi\cap$ supp $(1-\varphi)=\emptyset$. We have:

$$\psi Au=\psi A\varphi u+\psi A(1-\varphi)u. \tag{44.8}$$

Note that $\varphi u\in C_0^\infty(\mathbf{R}^n)$, since supp $\varphi\cap$ sing supp $u=\emptyset$. Then by Theorem 40.1 $\psi A\varphi u\in C^\infty$. Also $\psi A(1-\varphi)u\in C^\infty$ by the Theorem 44.1. Therefore $\psi Au\in C^\infty$ and $x_0\notin$ sing supp Au since $\psi(x_0)\neq 0$. □

The following theorem gives a refinement of Lemma 44.3.

Theorem 44.4. *Let* $A(x,\xi)\in S^\alpha$ *and let* $u\in H_s(\mathbf{R}^n)$ *for some* $s\in\mathbf{R}^n$. *We have*

$$WF(Au)\subset WF(u).$$

Proof: Let $(x_0,\xi_0)\notin WF(u)$. Then there exists $\varphi(x)\in C_0^\infty(\mathbf{R}^n),\varphi(x)=1$ in a small neighborhood of x_0 and there exists $\alpha(\xi)\in C^\infty(\mathbf{R}^n\setminus\{0\})$, $\alpha(\xi)$ is homogeneous in ξ of degree zero, $\alpha(\xi)=1$ in a small conic neighborhood of ξ_0 such that $\alpha(D)\varphi(x)u\in C^\infty(\mathbf{R}^n)$.

Let $\psi(x)\in C_0^\infty(\mathbf{R}^n),\psi(x_0)\neq 0$, supp ψ is contained in an interior of $\{x:\varphi(x)=1\}$ and let $\beta(\xi)\in C^\infty(\mathbf{R}^n\setminus\{0\})$, $\beta(\xi)$ is homogeneous of degree 0, $\beta(\xi_0)\neq 0$ and supp $\beta\cap\{|\xi|=1\}$ is contained in the interior of the set where $\alpha(\xi)=1$.

To prove Theorem 44.4 we need to show that $\beta(D)\psi(x)Au\in C^\infty(\mathbf{R}^n)$, i.e. $(x_0,\xi_0)\notin WF(Au)$. Since $\beta(\xi)$ is not smooth at $\xi=0$ we replace $\beta(D)$ by $(1-\chi(D))\beta(D)$ where $\chi(\xi)$ is the cutoff function as above. Note that $\chi(D)w\in C^\infty(\mathbf{R}^n)$ when $w\in H_s,\forall s$. Since supp $\psi\cap$ supp $(1-\varphi(x))=\emptyset$ we have that $\psi A(1-\varphi)u\in C^\infty$ because of pseudolocality of $A(x,D)$. Consider now $(1-\chi(D))\beta(D)\psi A\varphi u$. Applying Theorem 40.2 to $(1-\chi(D))\beta(D)$ and ψA we get

$$(1-\chi(D))\beta(D)\psi A\varphi u=\sum_{|k|=0}^{N}C_k(x,D)\varphi u+T_{\alpha-N-1}\varphi u, \tag{44.9}$$

where ord $T_{\alpha-N-1} \leq \alpha - N - 1$, and symbols $C_k(x,\xi)$ contain $\beta(\xi)$ and derivatives $\frac{\partial^k \beta(\xi)}{\partial \xi^k}$. Since $\alpha(\xi) = 1$ on supp $\beta(\xi)$ we have $C(x,\xi) = \sum_{|k|=0}^{N} C_k(x,\xi) = C(x,\xi)\alpha(\xi)$. Therefore $(1-\chi(D))\beta(D)\psi A\varphi u = C\alpha(D)\varphi u + T_{\alpha-N-1}\varphi u$ Since $\alpha(D)\varphi u \in C^\infty$ we get that $C\alpha(D)\varphi u \in C^\infty$ (c.f. Theorem 40.1). Therefore $(1-\chi(D))\beta(D)\psi A\varphi u \in H_{s-\alpha+N+1}(\mathbf{R}^n)$. Since N is arbitrary we get that $\beta(D)\psi A\varphi u \in C^\infty$. □

Theorem 44.5 (Microlocal regularity). *Let $A(x,\xi) \in S^m$ and there exists $A_0(x,\xi)$ homogeneous of degree m in ξ such that $A(x,\xi) - A_0(x,\xi)\chi(\xi) \in S^{m-1}$. Suppose $A(x,\xi)$ is elliptic at (x_0,ξ_0), i.e. $A_0(x_0,\xi_0) \neq 0$. Let $u \in H_s(\mathbf{R}^n)$ be the solution of $Au = f$ and let f be C^∞ at (x_0,ξ_0), i.e. $(x_0,\xi_0) \notin WF(f)$. Then $(x_0,\xi_0) \notin WF(u)$.*

Proof: Let $\varphi(x), \alpha(\xi), \psi(x), \beta(\xi)$ be the same as in the proof of Theorem 44.4. We assume that $A_0(x,\xi) \neq 0$ when $x \in$ supp $\varphi(x)$, $\xi \in$ supp $\alpha(\xi)$ and that $\alpha(D)\varphi(x)f \in C^\infty(\mathbf{R}^n)$.

Let $B(x,\xi)$ be the extension of $A(x,\xi)$ from a neighborhood of supp $\varphi(x)\alpha(\xi)$ to $\mathbf{R}^n \times \mathbf{R}^n$ such that $B(x,\xi) \in S^m$ and $B(x,\xi)$ is elliptic, i.e. $|B(x,\xi)| \geq C(1+|\xi|)^m$ for $|x|^2+|\xi|^2 \geq R^2$. Note that$B(x,\xi)-A(x,\xi) = 0$ on supp $\varphi(x)\alpha(\xi) \cap \{(x,\xi) : |x|^2 + |\xi|^2 \geq R^2\}$. As in Lemma 41.3 one can construct a *ψdo* $R^{(N)}(x,D)$, $R^{(N)}(x,\xi) \in S^{-m}$, such that $R^{(N)}(x,D)B(x,D) = I + T_{-N-1}$, where ord $T_{-N-1} \leq -N-1$. We have $Au = f$. Therefore

$$Bu = f + (B-A)u. \tag{44.10}$$

Applying $R^{(N)}$ to (44.10) we get

$$u = R^{(N)}f + R^{(N)}(B-A))u - T_{-N-1}u. \tag{44.11}$$

Now apply $(1-\chi(D))\beta(D)\psi(x)$ to (44.11). Since $R^{(N)}(x,\xi) \in S^{-m}$ and $\alpha(D)\varphi(x)f \in C^\infty$, the proof of Theorem 44.4 gives that $(1-\chi(D))\beta(D)\psi R^{(N)}f \in C^\infty$. Apply the Theorem 40.2 to the composition C of *ψdo* operators $(1-\chi(D))\beta(D)\psi(x)$, $R^{(N)}(x,D)$ and $B(x,D) - A(x,D)$. We obtain:

$$Cu = \sum_{|k|=0}^{N} C_k(x,D)u + T_{-N-1}^{(1)}u,$$

where ord $T_{-N-1} \leq -N-1$ and $C_k(x,\xi) \in S^{-|k|}$. Note that all $C_k(x,\xi)$ contain either $\psi(x)\beta(\xi)$ or its derivative in ξ and x. From other side $B(x,\xi) -$

$A(x,\xi)=0$ on supp $\varphi(x)\alpha(\xi)$ for $|x|^2+|\xi|^2\geq R$, i.e. $B(x,\xi)-A(x,\xi)=0$ on a neighborhood of supp $\psi(x)\beta(\xi)$. Therefore $C_k(x,\xi)=0$ for $|x|^2+|\xi|^2\geq R^2$ for all $k,\ 0\leq|k|\leq N$. This implies that $(1-\chi(D))\psi(x)u\in H_{s+N+1}(\mathbf{R}^n)$ for arbitrary N. Thus $(1-\chi(D))\psi(x)u\in C^\infty$.

45 Change of variables formula for the pseudodifferential operators.

Let

$$x=s(\hat{x}) \tag{45.1}$$

be a one-to-one diffeomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^n$, $s(\hat{x})=\hat{x}$ for $|\hat{x}|>R$, $J(\hat{x})=\det\frac{Ds(\hat{x})}{D\hat{x}}\neq 0,\ \forall\hat{x}\in\mathbf{R}^n$, where $\frac{Ds(\hat{x})}{D\hat{x}}$ is the Jacobi matrix of $x=s(\hat{x})$. Let $A(x,D)$ be a ψdo operator with symbol $A(x,\xi)\in S^\alpha$. We have

$$A(x,D)u=Bu+T_{-\infty}u, \tag{45.2}$$

where

$$Bu=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}A(x,\xi)\chi(\frac{x-y}{\delta})e^{i(x-y)\cdot\xi}u(y)dyd\xi, \tag{45.3}$$

$$T_{-\infty}u=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}A(x,\xi)(1-\chi(\frac{x-y}{\delta}))e^{i(x-y)\cdot\xi}u(y)dyd\xi. \tag{45.4}$$

It follows from Theorem 44.1 that $T_{-\infty}$ is an integral operator with $C^\infty(\mathbf{R}^n\times\mathbf{R}^n)$ kernel.

Consider $v(x)=A(x,D)u$, where $u(x)\in C_0^\infty(\mathbf{R}^n)$. Make change of variables (45.1). Let $\hat{u}(\hat{x})=u(s(\hat{x})),\ \hat{v}(\hat{x})=v(s(\hat{x}))$ be $u(x)$ and $v(x)$ in new coordinates and let $\hat{v}(\hat{x})=\hat{A}\hat{u}$ be the image of $A(x,D)$ in new coordinates. Then

$$\hat{A}\hat{u}=\hat{B}\hat{u}+\hat{T}_{-\infty}\hat{u}, \tag{45.5}$$

where $\hat{B}$ is the image of B and $\hat{T}_{-\infty}$ is the image of $T_{-\infty}$ in new coordinates. Since $T_{-\infty}$ is an operator with $C^\infty(\mathbf{R}^n\times\mathbf{R}^n)$ kernel, the operator $\hat{T}_{-\infty}$ is also an integral operator with C^∞ kernel.

We shall show that $\hat{B}$ is also a ψdo modulo $T_{\alpha-N}$, where ord $T_{\alpha-N} \leq \alpha - N,\ N$ is arbitrary, and we compute the symbol of $\hat{B}$.

Substituting $x = s(\hat{x}), y = s(\hat{y})$ in (45.3) we get

(45.6)
$$\hat{B}\hat{u} = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} A(s(\hat{x}),\xi)\chi(\frac{s(\hat{x})-s(\hat{y})}{\delta})e^{i(s(\hat{x})-s(\hat{y}))\cdot\xi}u(s(\hat{y}))|J(\hat{y})|d\hat{y}d\xi.$$

Note that

$$s(\hat{x}) - s(\hat{y}) = \int_0^1 \frac{d}{dt}s(\hat{y}+t(\hat{x}-\hat{y})dt = H(\hat{x},\hat{y})(\hat{x}-\hat{y}), \tag{45.7}$$

where $H(\hat{x},\hat{y})$ is C^∞ matrix and $H(\hat{x},\hat{x}) = \frac{Ds(\hat{x})}{D\hat{x}}$. Since det $\frac{Ds(\hat{x})}{D\hat{x}} \neq 0, \forall\hat{x}$, we get that $\det H(\hat{x},\hat{y}) \neq 0$ if $|s(\hat{x})-s(\hat{y})| \leq \delta$ where δ is small. We have

$$(s(\hat{x})-s(\hat{y}))\cdot\xi = H(\hat{x},\hat{y})(x-y)\cdot\xi = (x-y)\cdot H^T(\hat{x},\hat{y})\xi, \tag{45.8}$$

where $H^T(\hat{x},\hat{y})$ is a matrix transpose to $H(\hat{x},\hat{y})$. Substitute (45.8) into (45.6) and make change of variables

$$\eta = H^T(\hat{x},\hat{y})\xi. \tag{45.9}$$

In order to justify the change of variables (45.9) one should introduce the cutoff factor $\chi(\varepsilon\xi)$ into (45.6), make the change of variables and then take the limit when $\varepsilon \to 0$. We get

$$\begin{aligned}\hat{B}\hat{u} = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} & A(s(\hat{x}),(H^T(\hat{x},\hat{y}))^{-1}\eta)\chi(\frac{s(\hat{x})-s(\hat{y})}{\delta})e^{i(\hat{x}-\hat{y})\cdot\eta}\hat{u}(\hat{y}) \\ & \cdot|J(\hat{y})|\ |\det H(\hat{x},\hat{y})|^{-1}d\hat{y}d\eta.\end{aligned} \tag{45.10}$$

Therefore $\hat{B}$ is an ψdo of the form (43.2) with the symbol

(45.11)
$$\hat{b}(\hat{x},\hat{y},\eta) = A(s(\hat{x}),(H^T(\hat{x},\hat{y}))^{-1}\eta)\chi(\frac{s(\hat{x})-s(\hat{y})}{\delta})|J(\hat{y})|\ |\det H(\hat{x},\hat{y})|^{-1}.$$

Using Theorem 43.1 we can represent (45.10) in the form

$$\hat{B}\hat{u} = \sum_{|k|=0}^{N}\hat{B}_k(\hat{x},\hat{D})\hat{u} + \hat{T}_{\alpha-N-1}\hat{u},$$

where

$$\hat{B}_0(\hat{x},\eta)=\hat{b}(\hat{x},\hat{x},\eta)=A(s(\hat{x}),((\frac{Ds(\hat{x})}{D\hat{x}})^T)^{-1}\eta),$$

$$B_k(\hat{x},\eta)=\frac{1}{k!}D_{\hat{y}}^k\frac{\partial^k}{\partial\eta^k}b(\hat{x},\hat{y},\eta)|_{\hat{y}=\hat{x}},$$

ord $\hat{T}_{\alpha-N-1}\leq\alpha-N-1,$ N is arbitrary. □