41 Elliptic operators and parametrices.

Let $A(x,D)$ be elliptic differeial operator of degree m, i.e. $A(x,\xi)=A_0(x,\xi)+A_1(x,\xi)$, where $A_0(x,\xi)=\sum_{|k|=m}a_k(x)\xi^k$, $A_1(x,\xi)=\sum_{|k|=0}^{m-1}a_k(x)\xi^k$, and $A_0(x,\xi)\neq 0,\ \forall x\in\mathbf{R}^n,\ x\neq 0$.

For simplicity we assume that $A_0(x,\xi)>0, \forall x\in\mathbf{R}^n,\ \xi\neq 0$ and $A(x,\xi)=A(\infty,\xi)$ for $|x|\geq R$.

Denote by $R_0(x,\xi,\lambda)$ the symbol $(A_0(x,\xi)+\lambda)^{-1}$ where $\lambda\in C_\delta$,

$$C_\delta=\{\lambda\in\mathbf{C}\setminus\{0\},\quad -\pi+\delta<\arg\lambda<\pi-\delta,\quad 0<\delta<\frac{\pi}{2}\}. \tag{41.1}$$

Note that $R_0(x,\xi,\lambda)\in S^{-m}$ and satisfies estimates

$$|R_0(x,\xi,\lambda)|\geq(|\xi|^m+|\lambda|)\geq C_1(|\xi|+|\lambda|^{\frac{1}{m}})^m, \tag{41.2}$$
$$\left|\frac{\partial^{k+p}R_0(x,\xi,\lambda)}{\partial\xi^k\partial x^p}\right|\leq C_{pk}|R_0(x,\xi,\lambda)|\ (|\xi|+|\lambda|^{\frac{1}{m}})^{-|k|},\ \forall p,\forall k,\ \lambda\in C_\delta.$$

Theorem 41.1. *For any $N\geq 0$, there exists a ψdo operator $R^{(N)}(x,D,\lambda)$, $R^{(N)}(x,\xi,\lambda)\in S^{-m}$ such that*

$$(A(x,D)-\lambda I)R^{(N)}(x,D,\lambda)=I+T_{-N-1}, \tag{41.3}$$

where I is the identity operator, ord $T_{-N-1}\leq -N-1$ and

$$|||T_{-N-1}|||_{(s)}\leq C_N|\lambda|^{-\frac{N+1}{m}},\quad \lambda\imath C_\delta, \tag{41.4}$$

where $|||T_{-N-1}|||_{(s)}$ is the norm of the operator T_{-N-1} acting from $H_s(\mathbf{R}^n)$ to $H_s(\mathbf{R}^n)$.

Operator $R^{(N)}(x,D,\lambda)$ is called the parametrix to $A(x,D)-\lambda$.

Proof: Using Theorem 40.2 we have

$$(A(x,D)+\lambda I)R_0(x,D,\lambda)=I+C_{-1}(x,D,\lambda),$$

where

$$C_{-1}(x,\xi,\lambda)=\sum_{|k|=1}^{m}\frac{1}{k!}\frac{\partial A(x,\xi)}{\partial\xi^k}D_x^kR_0(x,\xi,\lambda)+A_1(x,\xi)R_0(x,\xi,\lambda). \tag{41.5}$$

Note that there is no remainder in (41.5) since $A(x,\xi)$ is a polynomial.

Let

$$R_{-1}(x,\xi,\lambda) = -R_0(x,\xi,\lambda)C_{-1}(x,\xi,\lambda).$$

Then by the Theorem 40.2 we have

$$(A(x,D) - \lambda I)(R_0 + R_1(x,D,\lambda)) = I + C_{-2}(x,D,\lambda), \tag{41.6}$$

where $C_{-2}(x,\xi,\lambda) \in S^{-2}$ and

$$C_{-2}(x,\xi,\lambda) = A_1(x,\xi)R_{-1}(x,\xi,\lambda) + \sum_{|k|=1}^{m} \frac{1}{k!}\frac{\partial^k A(x,\xi)}{\partial \xi^k} D_x^k R_{-1}(x,\xi,\lambda). \tag{41.7}$$

Analoguously defining $R_{-2}(x,\xi,\lambda) = -R_0(x,\xi,\lambda)C_{-2}(x,\xi,\lambda)$ etc. we construct $R_{-p}(x,\xi,\lambda) = -R_0(x,\xi,\lambda)C_{-p}(x,\xi,\lambda)$ where $C_{-p}(x,\xi,\lambda) \in S^{-p}$, $1 \le p \le N$, and

$$C_{-p}(x,\xi) = A_1(x,\xi)R_{-p+1}(x,\xi,\lambda) + \sum_{|k|=1}^{m} \frac{1}{k!}\frac{\partial^k A(x,\xi)}{\partial \xi^k} D_x^k R_{-p+1}(x,\xi,\lambda). \tag{41.8}$$

Therefore we get:

$$(A + \lambda I)(R_0 + R_{-1} + ... + R_{-N}) = I + T_{-N-1}, \tag{41.9}$$

where ord $T_{-N-1} \le -N-1$ and T_{-N-1} satisfies (41.4). Note that the sum

$$R^{(N)}(x,\xi,\lambda) = R_0(x,\xi,\lambda) + ... + R_{-N}(x,\xi,\lambda) \tag{41.10}$$

can be rewritten in the following form:

$$R^{(N)}(x,\xi,\lambda) = (A_0(x,\xi) + \lambda)^{-1} + \sum_{k=1}^{N}(A_0(x,\xi)+\lambda)^{k+1}P_k(x,\xi), \tag{41.11}$$

where $P_k(x,\xi)$ are polynomials in ξ, deg $P_x(x,\xi) \le km - \frac{k}{2}$.

Lemma 41.2. *Let $A(x,D)$ be the same as in Theorem 41.1. Let s be arbitrary. Suppose λ is sufficiently large, and (41.1) holds. Then for any $f \in H_{s-m}(\mathbf{R}^n)$ there exists a unique solution $u \in H_s(\mathbf{R}^n)$ of the equation*

$$(A(x,D) + \lambda I)u = f.$$

Proof: Consider (41.3) with $N = 0$. It follows from (41.4) that T_{-1} is an operator with a small norm norm. Therefore $I + T_{-1}$ is invertible in $H_{s-m}(\mathbf{R}^n)$ and $R = R_0(x, D, \lambda)(I + T_{-1})^{-1}$ is the right inverse to $A + \lambda I$, i.e. $(A + \lambda I)Rf = f, \forall f \in H_{s-m}(\mathbf{R}^n)$. Also $u = Rf \in H_s(\mathbf{R}^n)$ since ord $R = -m$.

Applying Theorem 40.2 we have $R_0(x, D, \lambda)(A(x, D) + \lambda) = I + T_{-1}^{(1)}$, where $T_{-1}^{(1)}$ satisfies (41.4) with $N = 0$ and $(I + T_{-1}^{(1)})^{-1}$ exists when $\lambda \in C_\delta$ is sufficiently large. Therefore $R^{(1)} = (I + T_{-1}^{(1)})^{-1}R_0$ is the left inverse of $A(x, D) + \lambda I$. Since $A(x, D) + \lambda I$ has the left and the right inverses it is invertible and $R = R^{(1)} = (A(x, D) + \lambda I)^{-1}$. □

In the case when $A(x, \xi) \in S^\alpha$ is not a polynomial in ξ we shall call $A(x, \xi)$ elliptic if there exists $R > 0$ such that

$$|A(x, \xi)| \geq C(1 + |\xi|)^\alpha \quad \text{when} \quad |x|^2 + |\xi|^2 \geq R^2.$$

Lemma 41.3 (Elliptic regularity)**.** *Let $A(x, \xi) \in S^\alpha$ be elliptic and let $u \in H_s(\mathbf{R}^n)$ be a solution of $A(x, D)u = f$ for some $s \in \mathbf{R}$. Suppose $f \in H_{s-\alpha+r}(\mathbf{R}^n), r > 0$. Then $u \in H_{s+r}(\mathbf{R}^n)$. In particular, if $f \in H_N(\mathbf{R}^n), \forall N$, then $u \in H_N(\mathbf{R}^n), \forall N$.*

Proof: Let $R(x, \xi) = A^{-1}(x, \xi)(1 - \chi(\frac{x}{R})\chi(\frac{\xi}{R}))$, where $\chi(x) \in C_0^\infty(\mathbf{R}^n), \chi(x) = 1$ when $|x| \leq 1$. It follows from Theorem 40.2 that

$$R(x, D)A(x, D)u = u - \chi(\frac{x}{R})\chi(\frac{D}{R})u + T_{-1}u, \tag{41.12}$$

where ord $T_{-1} \leq -1$. Therefore

$$u = Rf + \chi(\frac{x}{R})\chi(\frac{D}{R})u - T_{-1}u.$$

Note that $Rf \in H_{s+r}$ since ord $R = -m$. Also $T_{-1}u \in H_{s+1}(\mathbf{R}^n)$, $\chi(\frac{x}{R})\chi(\frac{D}{R})u \in H_N(\mathbf{R}^n), \forall N$. Therefore $u \in H_{s+r_0}$, where $r_0 = \min(1, r)$. If $r > 1$ we can repeat the same arguments with $u \in H_{s+1}(\mathbf{R}^n)$ instead of $u \in H_s(\mathbf{R}^n)$. □

42 Compactness and Fredholm property.

Theorem 42.1. *Let ord $T' \leq -\delta$ and T' has the property (40.26), i.e.*

$$\|(1 + |x|^2)^M T'u\|_{s+\delta} \leq C\|u\|_s, \ \forall s. \tag{42.1}$$

Then T' is compact in $H_s(\mathbf{R}^n), \forall s$.

Proof: We have the identity:

$$T' = (1 + |x|^2)^{-M} \Lambda^{-\delta} \Lambda^{\delta} (1 + |x|^2)^M T'.$$

Denote $T_1 = \Lambda^\delta (1 + |x|^2)^M T'$, where Λ^δ is a ψdo with symbol $\Lambda^\delta(\xi) = (1 + |\xi|^2)^{\frac{\delta}{2}}$. It follows from (42.1) that T_1 is bounded in $H_s(\mathbf{R}^n), \forall s$. Therefore it is enough to prove that $T_2 = (1 + |x|^2)^{-M} \Lambda^{-\delta}$ is compact in $H_1(\mathbf{R}^n)$ since the product of a compact and a bounded operators is compact.

Let $\chi(\xi) \in C_0^\infty(\mathbf{R}^n),\ \chi(\xi) = 1$ for $|\xi| \leq 1, |\chi(\xi)| \leq 1$. Let $T_\varepsilon = T_2 \chi(\varepsilon D)$. We have $T_2 - T_\varepsilon = (1 + |x|^2)^{-M} \Lambda^{-\delta} (1 - \chi(\varepsilon D))$. Therefore

$$\|(T_2 - T_\varepsilon) u\|_s^2 \leq C \|\Lambda^{-\delta} (1 - \chi(\varepsilon D)) u\|_s^2$$
$$= C \int_{\mathbf{R}^n} \frac{|1 - \chi(\varepsilon\xi)|^2 |\tilde{u}(\xi)|^2 (1 + |\xi|)^{2s}}{(1 + |\xi|^2)^\delta} d\xi \leq C \varepsilon^{2\delta} \|u\|_s^2, \tag{42.2}$$

i.e. $|||T_2 - T_\varepsilon|||_{(s)} \to 0$ when $\varepsilon \to 0$. It is enough to prove that T_ε is compact, $\forall \varepsilon$, since the limit in the operator norm of a sequence of compact operators is compact.

Let $v(x) = T_\varepsilon u$. Then

$$\tilde{v}(\xi) = \int_{\mathbf{R}^n} \tilde{\psi}(\xi - \eta) (1 + |\eta|^2)^{-\frac{\delta}{2}} \chi(\varepsilon\eta) \tilde{u}(\eta) d\eta, \tag{42.3}$$

where $\psi(x) = (1 + |x|^2)^{-M}, M > \frac{n}{2}$. We have

$$(1 + |\xi|^2)^{\frac{s}{2}} \tilde{v}(\xi) = \int_{\mathbf{R}^n} K(\xi, \eta) (1 + |\eta|^2)^{\frac{s}{2}} \tilde{u}(\eta) d\eta,$$

where

$$K(\xi, \eta) = (1 + |\xi|^2)^{\frac{s}{2}} \tilde{\psi}(\xi - \eta) (1 + |\eta|^2)^{-\frac{s+\delta}{2}} \chi(\varepsilon\eta). \tag{42.4}$$

Note that

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |K(\xi, \eta)|^2 d\xi d\eta < +\infty,$$

i.e. the integral operator $Kw = \int_{\mathbf{R}^n} K(\xi, \eta) w(\eta) d\eta$ is the Hilbert-Schmidt operator. Since the Hilbert-Schmidt operator is compact in $L^2(\mathbf{R}^n)$ we get that operator (42.3) is compact $\tilde{H}_s(\mathbf{R}^n) = F H_s(\mathbf{R}^n)$ (c.f. the proof of Theorem 23.2). $\square$

Theorem 42.2. *Let Ω be a smooth bounded domain in $\mathbf{R}^n$, and let T be a bounded operator from $H_s(\Omega)$ to $H_{s+\varepsilon}(\Omega), \varepsilon > 0$. Then T is compact operator in $H_s(\Omega)$.*

Proof: Let l be an extension operator from Ω to $\mathbf{R}^n$ such that

$$\|lu\|_{s,\mathbf{R}^n} \leq C\|u\|_{s,\Omega} \quad \text{for} \quad \forall u \in H_s(\Omega)$$

and $lu = 0$ for $|x| > N$ (c.f. Theorem 33.2).

Denote by $\psi_0(x)$ a $C_0^\infty(\mathbf{R}^n)$ function such that $\psi_0(x) = 1$ when $|x| \leq N$. Let p be the restriction operator to the domain Ω. We have, for any $u \in H_s(\Omega)$:

$$Tu = pl(Tu) = p\psi_0 l(Tu) = p\psi_0\Lambda^{-\varepsilon}\Lambda^\varepsilon l(Tu).$$

Operator $\Lambda^\varepsilon l(Tu)$ is bounded from $H_s(\Omega)$ to $H_s(\mathbf{R}^n)$ and operator p is bounded from $H_s(\mathbf{R}^n)$ to $H_s(\Omega)$. It follows from the proof of Theorem 42.1 that $\psi_0(x)\Lambda^{-\varepsilon}$ is compact in $H_s(\mathbf{R}^n)$. Therefore T is compact in $H_s(\Omega)$ as a product of bounded opeators and a compact operator. □

Remark 42.1 Analogously to the proof of Theorem 42.2 one can show that the embedding of $H_{s+\varepsilon}(\Omega)$ into $H_s(\Omega)$ is compact: we have as above $u = p\psi_0 lu = p\psi_0\Lambda^{-\varepsilon}\Lambda^\varepsilon lu$ for any $u \in H_{s+\varepsilon}(\Omega)$. Operator $\Lambda^\varepsilon lu$ is bounded from $H_{s+\varepsilon}(\Omega)$ to $H_s(\mathbf{R}^n)$ therefore $Iu = p(\psi_0\Lambda^{-\varepsilon})(\Lambda^\varepsilon lu)$ is compact operator from $H_{s+\varepsilon}(\Omega)$ to $H_s(\Omega)$ as the product of bounded operators p and $\Lambda^\varepsilon lu$ and the compact operator $\psi_0\Lambda^{-\varepsilon}$. □

Note that the operator A, bounded from $H_s(\mathbf{R}^n)$ to $H_{s-\alpha}(\mathbf{R}^n)$, is called Fredholm if

a) $\ker A = \{u \in H_s(\mathbf{R}^n), Au = 0\}$ is finite-dimensional.

b) $\Im A = \{v \in H_{s-\alpha}(\mathbf{R}^n) : v = Au$ for some $u \in H_s(\mathbf{R}^n)\}$ is closed in $H_{s-\alpha}(\mathbf{R}^n)$.

c) coker A is finite-dimensional where coker A is the orthogonal complement to $\Im A$ in $H_{s-\alpha}(\mathbf{R}^n)$.

One can prove that A is Fredholm iff there exists a bounded operatoe R from $H_{s-\alpha}(\mathbf{R}^n)$ to $H_s(\mathbf{R}^n)$ such that

$$AR = I + T_1, \quad RA = I + T_2, \tag{42.5}$$

where T_1 is a compact operator in $H_{s-\alpha}(\mathbf{R}^n)$, T_2 is a compact operator in $H_s(\mathbf{R}^n)$. Operator R is called regularizer to A.

Theorem 42.3. *Let $A(x,D) \in S^\alpha$ be an elliptic operator. Then $A(x,D)$ is a Fredholm operator from $H_s(\mathbf{R}^n)$ to $H_{s-m}(\mathbf{R}^n)$, $\forall s$.*

Proof Let $\chi(\xi)$ be as in Theorem 42.1 and let $R(x,\xi) = A^{-1}(x,\xi)(1 - \chi(\frac{x}{R})\chi(\frac{\xi}{R})$. Applying Theorem 40.2 we have:

$$A(x,D)R(x,D) = \left(I - \chi(\frac{x}{R})\chi(\frac{D}{R})\right) + T_{-1}^{(1)}), \tag{42.6}$$

where ord $T_{-1}^{(1)} \leq -1$. Moreover, by Remark 40.2 ord $(1+|x|^2)^M T_{-1}^{(1)} \leq -1$. By Theorem 42.1 the operator $T_1^{(1)} - \chi(\frac{x}{R})\chi(\frac{D}{R})$ is compact in $H_{s-\alpha}(\mathbf{R}^n)$. Therefore $R(x,D)$ is the right regularizer of $A(x,D)$. Analogously it follows from (41.12) that $R(x,D)$ is a left regularizer of $A(x,D)$ since

$$-\chi(\frac{x}{R})\chi(\frac{D}{R}) + T_1$$

is also compact in $H_s(\mathbf{R}^n)$.

43 The adjoint of pseudodifferential operator.

We shall rewrite the formula for the ψdo (40.3) in the form

$$Au = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} A(x,\xi)e^{i(x-y)\cdot\xi}u(y)dyd\xi, \quad \forall u \in C_0^\infty(\mathbf{R}^n), \tag{43.1}$$

where the integral in (43.1) is understood as a repeated integral : first the integration in y and then the integration in ξ. It is convenient to consider instead of (43.1) a more general form of a pseudodifferential operator:

$$Au = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} a(x,y,\xi)e^{i(x-y)\cdot\xi}u(y)dyd\xi, \tag{43.2}$$

where the integral is understood as a repeated integral and $a(x,y,\xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n)$ with the following properties:

$$\left|\frac{\partial^{p+k+r}a(x,y,\xi)}{\partial x^p \partial y^k \partial \xi^r}\right| \leq C_{pkr}(1+|\xi|)^{\alpha-|r|}, \quad \forall p,k,r, \tag{43.3}$$

(43.4)
$$a(x,y,\xi) = a(\infty,y,\xi) \text{ when } |x| \geq R, \; a(x,y,\xi) = a(x,\infty,\xi) \text{ when } |y| \geq R.$$

We shall represent $a(x,y,\xi)$ in the following form

$$a(x,y,\xi)=a(x,\infty,\xi)+a'(x,y,\xi), \tag{43.5}$$

where $a'(x,y,\xi)=0$ when $|y|\geq R$.

The following theorem shows that an operator of the form (43.2) can be represented in the form (40.2) up to an operator of an arbitrary low order.

Theorem 43.1. *Let A be an operator of the form (43.2). Then for any N*

$$Au=\sum_{|k|=0}^{N}A_k(x,D)u+T_{\alpha-N-1}u, \tag{43.6}$$

where $A_k(x,D)$ are operators of the form (40.2)

$$A_0(x,\xi)=a(x,\infty,\xi)+a'(x,x,\xi)=a(x,x,\xi), \tag{43.7}$$

$$A_k(x,\xi)=\frac{1}{k!}D_y^k\frac{\partial^k}{\partial\xi^k}a'(x,y,\xi)|_{y=x},\quad ord\ T_{\alpha-N-1}\leq\alpha-N-1,\ |k|\geq 1. \tag{43.8}$$

Proof: We have for any $u\in C_0^\infty(\mathbf{R}^n)$

$$Au=a(x,\infty,D)u+A'u, \tag{43.9}$$

where

$$A'u=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}a'(x,y,\xi)u(y)e^{i(x-y)\cdot\xi}dyd\xi. \tag{43.10}$$

Compute the integral in y using the convolution formula:

$$\int_{\mathbf{R}^n}a'(x,y,\xi)u(y)e^{-iy\cdot\xi}dy=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\tilde{a}'(x,\xi-\eta,\xi)\tilde{u}(\eta)d\eta, \tag{43.11}$$

where $\tilde{a}'(x,\eta,\xi)=F_ya'(x,y,\xi)$. We get

$$A'u=\frac{1}{(2\pi)^{2n}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\tilde{a}'(x,\xi-\eta,\xi)\tilde{u}(\eta)e^{ix\cdot\xi}d\eta d\xi. \tag{43.12}$$

Making the change of variables $\xi-\eta=\zeta$ we obtain

$$A'u=\frac{1}{(2\pi)^{2n}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\tilde{a}'(x,\zeta,\eta+\zeta)\tilde{u}(\eta)e^{ix\cdot(\eta+\zeta)}d\eta d\zeta. \tag{43.13}$$

Expand $\tilde{a}'(x,\zeta,\eta+\zeta)$ by the Taylor formula:

$$\tilde{a}'(x,\zeta,\eta+\zeta) = \sum_{|k|=0}^{N} \frac{1}{k!}\frac{\partial^k \tilde{a}'(x,\zeta,\eta)}{\partial \eta^k}\zeta^k + R_N(x,\zeta,\eta). \tag{43.14}$$

Taking into account that

$$\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \frac{\partial^k \tilde{a}'x,\zeta,\eta)}{\partial \eta^k}\zeta^k e^{ix\cdot\zeta}d\zeta = \frac{\partial^k}{\partial \eta^k} D_y^k a'(x,y,\eta)|_{y=x}, \tag{43.15}$$

we get (43.6) with $T_{\alpha-N-1}u$ having the form:

$$T_{\alpha-N-1}u = \frac{1}{(2\pi)^{2n}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} R_N(x,\zeta,\eta)e^{ix\cdot(\eta+\zeta)}\tilde{u}(\eta)d\eta d\zeta. \tag{43.16}$$

It remains to show that ord $T_{\alpha-N-1} \le \alpha - N - 1$. Note that

$$\tilde{a}'(x,\zeta,\eta) = \tilde{a}'(\infty,\zeta,\eta) + \tilde{a}''(x,\zeta,\eta), \tag{43.17}$$

where $\tilde{a}''(x,\zeta,\eta) = 0$ for $|x| \ge R$. Then

$$R_N(x,\zeta,\eta) = R_{N1}(\infty,\zeta,\eta) + R_{N2}(x,\zeta,\eta), \tag{43.18}$$

where R_{N1} corresponds to $\tilde{a}'(\infty,\zeta,\eta)$ and R_{N2} corresponds to $\tilde{a}''(x,\zeta,\eta)$. The Fourier transform in x of $T_{\alpha-N-1}u$ has the form:

$$(F_x(T_{\alpha-N-1}u))(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} R_{N1}(\infty,\xi-\eta,\eta)\tilde{u}(\eta)d\eta + \frac{1}{(2\pi)^{2n}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \tilde{R}_{N2}(\xi-\eta-\zeta,\zeta,\eta)\tilde{u}(\eta)d\eta d\zeta \tag{43.19}$$

It follows from (40.13), (40.14) and (43.14) that

$$|R_{N1}(\infty,\xi-\eta,\eta)| \le C_M(1+|\xi-\eta|)^{-M}(1+|\eta|)^{\alpha-N-1}(1+|\xi-\eta|)^{N+1+|\alpha|}, \ \ \forall M, \tag{43.20}$$

$$|\tilde{R}_{N2}(\xi-\eta-\zeta,\zeta,\eta)| \le C_{M_1,M}(1+|\xi-\eta-\zeta|)^{-M_1}(1+|\zeta|)^{-M}(1+|\eta|)^{\alpha-N-1}(1+|\zeta|)^{N+1+|\alpha|}, \ \forall M_1, \forall M. \tag{43.21}$$

Estimating the term in (43.19) containing R_{N1} as in (40.19), (40.20) and the term in (43.19) containing R_{N2} as in (40.21), (40.22) we get

$$\|T_{\alpha-N-1}u\|_s \le C\|u\|_{s+\alpha-N-1}, \quad \forall s. \tag{43.22}$$

$\square$

Let A^* be the adjoint opetator to the ψdo $A(x,D)$, i.e.

$$(Au, v) = (u, A^*v), \quad \forall u \in C_0^\infty(\mathbf{R}^n),\ v \in C_0^\infty(\mathbf{R}^n),$$

where (u,v) is the L^2 inner product.

Writing $A(x,D)u$ in the form (43.1) we get

$$(43.23) \qquad (Au, v) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} A(x,\xi)u(y)e^{i(x-y)\cdot\xi}dyd\xi)\overline{v(x)}dx.$$

We can rewrite (43.23) in the form

$$(43.24) \qquad (Au, v) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} u(y)(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} A(x,\xi)e^{i(x-y)\cdot\xi}\overline{v(x)}dxd\xi)dy.$$

Therefore

$$(43.25) \qquad (A^*v)(y) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \overline{A(x,\xi)}v(x)e^{i(y-x)\cdot\xi}dxd\xi.$$

Changing the notations from y to x and vice versa we get

$$(43.26) \qquad (A^*v)(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \overline{A(y,\xi)}v(y)e^{i(x-y)\cdot\xi}dyd\xi,$$

i.e. A^*v has the form (43.2). Applying Theorem 43.1 to (43.26) we get

Theorem 43.2. *Let $A(x,D)$ be a ψdo with symbol $A(x,\xi) \in S^\alpha$. Then the adjoint operator A^* is also a ψdo operator and*

$$(43.27) \qquad (A^*v|(x) = \sum_{|k|=0}^{N} A_k(x,D)v + T_{\alpha-N-1}v,$$

where

$$(43.28) \qquad A_0(x,\xi) = \overline{A(x,\xi)},$$

$$(43.29) \quad A_k(x,\xi) = \frac{1}{k!}D_x^k\frac{\partial^k}{\partial\xi^k}\overline{A(x,\xi)},\ 1 \le |k| \le N,\ \mathit{ord}\ T_{\alpha-N-1} \le \alpha - N - 1.$$

□

Let $A(x,\xi) \in S^\alpha$. Denote by $A_w(x,D)$ the following operator of the form (43.2):

$$A_w(x,D)u = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} A(\frac{x+y}{2},\xi)e^{i(x-y)\cdot\xi}u(y)dyd\xi. \tag{43.30}$$

Operators of the form (43.30) are called Weyl's pseudodifferential operators

Applying Theorem 43.1 to (43.30) we obtain

$$A_w(x,D) = A(x,D) + A_1(x,D) + T_{\alpha-N-1}, \tag{43.31}$$

where ord $T_{\alpha-N-1} \leq \alpha - N - 1$ and

$$A_1(x,\xi) = \sum_{|k|=1}^{N} \frac{1}{2^k k!} D_x^k \frac{\partial^k}{\partial\xi^k} A(x,\xi).$$

Therefore $A_w(x,D)$ and $A(x,D)$ differ by terms of order $\alpha - 1$.

Consider the case when $A(x,\xi)$ is real valued. Then the ψdo $A(x,D)$ is not self-adjoint if the terms (43.29) are not zero. However $A_w(x,D)$ is self-adjoint when $A(x,\xi)$ is real since

$$(A_w(x,D)u,v) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} A(\frac{x+y}{2},\xi)u(y)e^{-(x-y)\cdot\xi}dyd\xi\overline{v(x)}dx$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} A(\frac{x+y}{2})u(y)e^{-y\cdot\xi}\overline{v(x)e^{-ix\cdot\xi}}dydxd\xi = (u, A_w(x,D)v).$$

This show the usefulness of the Weyl's operators.