63 Parametrix for the hyperbolic Cauchy problem.

Let $H(x,t,D_x,D_t)$ be the second order strictly hyperbolic operator, i.e.

$$H(x,t,D_x,D_t) = g^{00}(x,t)\frac{\partial^2}{\partial t^2} + 2\sum_{j=1}^{n} g^{j0}(x,t)\frac{\partial^2}{\partial t\partial x_j} \tag{63.1}$$

$$- \sum_{j,k=1}^{n} g^{jk}(x,t)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial}{\partial x_j} + b_0(x,t)\frac{\partial}{\partial t} + c(x,t).$$

We assume that $g^{00}(x,t) = 1$ and that H is strictly hyperbolic, i.e.

$$H_0(x,t,\xi,\sigma) = (\sigma - \lambda_1(x,t,\xi))(\sigma - \lambda_2(x,t,\xi)), \tag{63.2}$$

$$\lambda_1(x,t,\sigma) < \lambda_2(x,t,\sigma) \quad \text{for all} \quad (x,t,\xi),\ \xi \neq 0. \tag{63.3}$$

Here H_0 is the principal part of (63.1). Consider the Cauchy problem

$$Hu(x,t) = 0, \quad t > 0,\ x \in \mathbf{R}^n, \tag{63.4}$$

$$u(x,0) = g_1(x), \quad \frac{\partial u}{\partial t}(x,0) = g_2(x), \quad x \in \mathbf{R}^n. \tag{63.5}$$

We shall construct a parametrix for (63.4), (63.5). Denote by $\varphi_j(x,t,\eta), j = 1,2$, the solution of the eiconal equation

$$\frac{\partial \varphi_j(x,t)}{\partial t} - \lambda_j(x,t,\frac{\partial \varphi_j}{\partial x}) = 0, \quad t > 0, \tag{63.6}$$

$$\varphi_j(x,0,\eta) = x \cdot \eta, \quad j = 1,2. \tag{63.7}$$

The solution of (63.6), (63.7) exists for $|t| < \delta$, when δ is small. To find $\varphi_j(x,t,\eta)$ we solve the system of bicharacteristics

$$\frac{dx_j}{dt} = -\frac{\partial \lambda_j(x_j(t),t,p_j(t))}{\partial p}, \quad x_j(0) = y, \tag{63.8}$$

$$\frac{dp_j}{dt} = \frac{\partial \lambda_j(x_j(t),t,p_j(t))}{\partial x}, \quad p_j(0) = \eta,\ \eta \neq 0.$$

Note that the solution

$$x = x_j(t,y,\eta), \quad p = p_j(t,y,\eta) \tag{63.9}$$

of the system (63.8) exists for all $t \in \mathbf{R}$ (c.f. Remark 50.1).If $\varphi_j(x,t,\eta)$ is a solution of (63.6) then

$$\frac{d}{dt}\varphi_j(x_j(t),t,\eta) = \frac{\partial \varphi_j(x_j(t),t,\eta)}{\partial x}\cdot\frac{dx_j}{dt} + \frac{\partial \varphi_j(x_j(t),t,\eta)}{\partial t}$$

$$= \lambda_j(x_j(t),t,\frac{\partial \varphi_j(x_j(t))}{\partial x_j}) - \frac{\partial \lambda_j(x_j(t),t,p_j(t))}{\partial p}\cdot\frac{\partial \varphi_j(x_j(t),t,\eta)}{\partial x} = 0.$$

We used that $p_j(t) = \frac{\partial \varphi_j(x_j(t),t,\eta)}{\partial x}$ and $\lambda_j = \frac{\partial \lambda_j}{\partial p}\cdot p$. Therefore $\frac{d}{dt}\varphi_j(x_j(t),t,\eta) = 0$ and

$$\varphi_j(x_j(t),t,\eta) = \varphi_j(x_j(0),0,\eta) = \sum_{k=1}^{n} y_k\eta_k. \tag{63.10}$$

Since $x_j(0,y,\eta) = y$ we can solve $x = x_j(t,y,\eta)$ in y and get

$$y = y^{(j)}(x,t,\eta), \quad |t| \leq \delta. \tag{63.11}$$

Therefore

$$\varphi_j(x,t,\eta) = \sum_{k=1}^{n} y_k^{(j)}(x,t,\eta)\eta_k, \quad j = 1,2, \tag{63.12}$$

is the solution of (63.6), (63.7).

We shall look for the solution of (63.4) in the form

$$u_N(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\sum_{j=1}^{2}\sum_{k=0}^{N} a_{jk}(x,t,\eta)e^{i\varphi_j(x,t,\eta)}(1-\chi(\eta))d\eta, \tag{63.13}$$

where $a_{jk}(x,t,\eta) \in C^\infty$ when $\eta \neq 0$, $\deg_\eta a_{jk} = -k$. Substituting (63.13) into (63.4) we get the following equations

$$H_0(x,t,\varphi_{jx},\varphi_{jt}) = 0, \quad j = 1,2, \tag{63.14}$$

$$\frac{\partial H_0(x,t,\varphi_{jx},\varphi_{jt})}{\partial \sigma}\frac{\partial a_{j0}}{\partial t} + \sum_{k=1}^{n}\frac{\partial H_0(x,t,\varphi_{jx},\varphi_{jt})}{\partial p_k}\frac{\partial a_{j0}(x,t,\eta)}{\partial x_k} \tag{63.15}$$

$$+(H_0(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t})\varphi_j)a_{j0} + i\left(\sum_{k=1}^{n} b_k(x,t)\varphi_{jx_k} + b_0(x,t)\varphi_{jt}\right)a_{j0}(x,t,\xi) = 0,$$

$$L_j a_{jk} = -H(x,t,D_x,D_t)a_{j,k-1}, \quad k \geq 1, \ j = 1,2, \tag{63.16}$$

where L_j is the left hand side of (62.15). Since we want to have $u_N(x,0) = g_1(x)$, $\frac{\partial u_N(x,0)}{\partial t} = g_2(x)$ we shall impose the following initial conditions on $a_{jk},\ j=1,2,\ 0 \le k \le N$:

$$\sum_{j=1}^{2}\sum_{k=0}^{N} a_{jk}(x,0,\eta) = \tilde{g}_1(\eta), \tag{63.17}$$

$$\begin{aligned} i\varphi_{1t}(x,0,\eta)\sum_{k=0}^{N} a_{1k}(x,0,\eta) + i\varphi_{2t}(x,0,\eta)\sum_{k=0}^{N} a_{2k}(x,0,\eta) \\ + \sum_{j=1}^{2}\sum_{k=0}^{N} \frac{\partial a_{jk}(x,0,\eta)}{\partial t} = \tilde{g}_2(\eta). \end{aligned} \tag{63.18}$$

Since $\varphi_{jt}(x,0,\eta) = \lambda_j(x,0,\eta),\ j=1,2,$ we can determine $a_{10}(x,0,\eta)$ and $a_{20}(x,0,\eta)$ as the unique solution of the 2×2 algebraic system

$$\begin{aligned} a_{10}(x,0,\eta) + a_{20}(x,0,\eta) = \tilde{g}_1(\eta), \\ i\lambda_1(x,0,\eta)a_{10} + i\lambda_2(x,0,\eta)a_{20} = \tilde{g}_2(\eta). \end{aligned} \tag{63.19}$$

Note that $\det\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} = \lambda_2 - \lambda_1 \ne 0, \quad \forall x,t,\eta \ne 0.$ We determine $a_{jk}(x,0,\eta)$ from the equations

$$\begin{aligned} a_{1k}(x,0,\eta) + a_{20}(x,0,\eta) = 0, \\ i\lambda_1(x,0,\eta)a_{1k} + i\lambda_2(x,0,\eta)a_{2k} \\ -\frac{\partial a_{1,k-1}(x,0,\eta)}{\partial t} - \frac{\partial a_{2,k-1}(x,0,\eta)}{\partial t}, \\ 1 \le k \le N. \end{aligned} \tag{63.20}$$

Denote $w_N(x,t,\eta) = \sum_{j=1}^{2}\sum_{k=0}^{N} a_{jk}(x,t,\eta)e^{i\varphi_j(x,t,\eta)}(1-\chi(\eta))$. We have

$$\begin{aligned} Hw_N(x,t,\eta) = \sum_{j=1}^{2} t_{0j}(x,t,\eta)\tilde{g}_j(\eta), \\ w_N(x,0,\eta) = \tilde{g}_1(\eta) + \sum_{j=1}^{2} t_{1j}(x,\eta)\tilde{g}_j(\eta), \\ \frac{\partial w_N(x,0,\eta)}{\partial t} = \tilde{g}_2(\eta) + \sum_{j=1}^{2} t_{2j}(x,\eta)\tilde{g}_j(\eta), \end{aligned} \tag{63.21}$$

where $t_{pj} = O(\frac{1}{(|\eta|+1)^N}), \quad j = 1, 2, \quad 0 \le p \le 2.$

Denote by $w^{(N+1)}(x, t, \eta)$ the solution of the Cauchy problem (c.f Theorem 48.3)

(63.22) $$Hw^{(N+1)}(x, \eta) = -\sum_{j=1}^{2} t_{0j}(x, t, \eta)\tilde{g}_j(\eta),$$

$$\frac{\partial^k w^{(N+1)}(x, 0, \eta)}{\partial t^k} = -\sum_{j=1}^{2} t_{1+k,j}(x, \eta)\tilde{g}_j(\eta), \quad k = 0, 1, \quad j = 1, 2.$$

It follows from Theorem 48.3 with $s > \frac{n}{2} + 1$ that

(63.23) $$w^{(N+1)}(x, t, \eta) = O(\frac{1}{(1 + |\eta|)^{N-\frac{n}{2}-2}}).$$

Therefore

(63.24) $$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} (w_N(x, t, \eta) + w^{(N+1)}(x, t, \eta))d\eta$$

is the solution of (62.4), (62.5). □

Now we shall find the wave front set of $u(x, t)$ knowing $WF(g_j), \ j = 1, 2$. Note that (63.13) is a sum of FIO with phase functions $\varphi_j(x, t, \eta), \ j = 1, 2$. Denote by $\varphi_j \circ WF(g)$ the image of $WF(g)$ under the canonical transformation with the generating function $\varphi_j(x, t, \eta)$. We get, using Theorem 62.1, that

$$WF(u(\cdot, t)) \subset \cup_{j=1}^{2}(\varphi_j \circ (WF(g_1) \cup WF(g_2))),$$

where $WF(u(\cdot, t))$ is the wave front of $u(x, t)$ for fixed t. Using the Remark 50.2 we can also describe the $WF(u)$ in $\mathbf{R}^{n+1} \times (\mathbf{R}^{n+1} \setminus \{0\})$.

Compare this result with Theorem 50.2. The canonical transformation defined by generating function $\varphi_j(x, t, \eta)$ for t fixed is

(63.25) $$\xi = \varphi_{jx}(x, t, \eta), \quad y = \varphi_{j\eta}(x, t, \eta).$$

Let $x = x_j(t, y, \eta), \ p = p_j(t, y, \eta)$ be the solution of (63.8). Since $p_j(t, t, \eta) = \varphi_{jx}(x_j, y, \eta)$ we get

$$\xi = p_j(t, y, \eta) \quad \text{when } x = x_j(t, y, \eta).$$

Also differentiating (63.6) in η_k we get

$$\varphi_{jt\eta_k}(x,t,\eta) - \sum_{i=1}^{n} \frac{\partial \lambda_j(x,t,\varphi_{jx})}{\partial p_i} \varphi_{jx_i\eta_k} = 0. \tag{63.26}$$

Cosider the identity

$$\varphi_{j\eta_k}(x,t,\eta) = y_k^{(j)}(x,t,\eta).$$

Substituting $x = x_j(t)$ and differentiating in t we get

(63.27)

$$\varphi_{j\eta_k t}(x_j(t),t,\eta) + \sum_{i=1}^{n} \varphi_{j\eta_k x_i}(x_j(t),t,\eta)\Big(-\frac{\partial \lambda_j(x_j(t),t,p_j(t))}{\partial p_i}\Big) = \frac{d}{dt} y_k^{(j)}(x_j(t),t,\eta).$$

Comparing (63.26), (63.27) and taking into account that $p_j(t) = \varphi_{jx_i}(x_j(t),t,\eta)$ we get

$$\frac{d}{dt} y_k^{(j)}(x_j(t),t,\eta) = 0, \quad \text{i.e. } y_k^{(j)}(x_j(t),t,\eta) = y_k.$$

Therefore the canonical transformation (63.25) coincides with the map $(y,\eta) \to (x_j(t,y,\eta), p_j(t,y,\eta))$ where $x = x_j(t,y,\eta),\ p = p_j(t,y,\eta)$ is the solution of (63.8).

Therefore we got another proof of the Theorem 50.2.

64 Global Fourier Integral Operator.

Let $x = x(y,\eta), \xi = \xi(y,\eta)$ is a global canonical transformation of $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ onto $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$. An example of such canonical transformation is the map

$$(y,\eta) \to (x(t), \xi(t)), \tag{64.1}$$

where $(x(t),\xi(t)),\ t \in \mathbf{R}$ is the solution of the system (63.8), with $p(t)$ replaced by $\xi(t)$. The generating function $\varphi(x,t,\eta)$ for the canonical transformation (64.1) exists when $|t| < \delta$ and may not exist for large t. Therefore the FIO of the form (60.4) may not exist globally.

We shall define the Fourier integral operator corresponding to the global canonical transformation. Consider a Fourier integral operator of the form

$$\Phi u = \int_{\mathbf{R}^n} \int_{\mathbf{R}^r} a(x,y,\theta) e^{i\psi(x,y,\theta)} u(y)\, dy\, d\theta,$$

where $\psi(x, y, \theta) \in C^\infty$ for all (x, y, θ), $\theta \neq 0$ and $\psi(x, y, \theta)$ is homogeneous in θ of degree one. Suppose that the set $C_0 = \{x, y, \theta) : \psi_{\theta_j} = 0, 1 \leq j \leq r\}$ is smooth and has dimension $2n$. Consider the subset C of $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ of the form (x, ψ_x, y, ψ_y), where $(x, y, \theta) \in C_0$.

A manifold $\Lambda \subset \mathbf{R}^N \times (\mathbf{R}^N \setminus \{0\})$ is called Lagrangian if $\dim \Lambda = N$ and the symplectic form

$$\sum_{j=1}^N dx_j \wedge d\xi_j = 0 \quad \text{on } \Lambda$$

Consider a Lagrangian manifold Λ in $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$

$$dx \wedge d\xi + dy \wedge d\eta = 0 \quad \text{on} \ \ \Lambda, \tag{64.2}$$

We say that the points $(x, \xi, y, \eta) \in \Lambda$ form a canonical relation between (x, ξ) and (y, η).

We shall show that C is a canonical relation. We have $\xi dx + \eta dy = \psi_x dx + \psi_y dy = d\psi - \psi_\theta d\theta = d\psi$ since $\psi_\theta = 0$ on C. Also $\psi = \theta \cdot \psi_\theta = 0$ since $\psi(x, y, \theta)$ is homogeneous of degree 1. Therefore $\xi dx + \eta dy = d\psi = 0$ when $(x, y, \theta) \in C_0$. Since $d^2 = 0$ we have that $\xi dx + \eta dy = 0$ implies that $d\xi \wedge dx + d\eta \wedge dy = 0$ on C.

□

In the case of the FIO (60.4) we have

$$\psi(x, y, \eta) = S(x, \eta) - y \cdot \eta, \quad \psi_x = S_x(x, \eta), \quad \psi_y = -\eta, \quad \psi_\eta = S_\eta(x, \eta) - y = 0. \tag{64.3}$$

Therefore the canonical relation has the form $(x, S_x, y, -\eta)$ where $y - S_\eta = 0$. Since $S(x, t)$ is the generating function of the canonical transformation $\alpha : \ (y, \eta) \rightarrow (x, \xi)$ (c.f. (50.4)), we have $d\xi \wedge dx - d\eta \wedge dy = 0$ when $y = S_\eta(x, \eta)$, $\xi = S_x(x, \eta)$. Note that the graph of the canonical transformation $(x, \xi, y, \eta) = (x, S_x(x, \eta)), S_\eta(x, \eta), \eta)$ differs from the canonical relation (64.3) corresponding to (60.4) by the change of the sign of η. We shall consider in this section only the FIO whose canonical relations correspond to the graphs of canonical transformations.

The following lemma holds:

Lemma 64.1. *Let $\alpha : (y, \eta) \rightarrow (x, \xi)$ be an arbitrary canonical transformation. Then for any (y_0, η_0) there exists a conic neighborhood in $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ with the following properties:*

There is a splitting $x = (x^{(1)}, x_{(1)})$, where $x^{(1)} \in \mathbf{R}^{n_1}, x_{(1)} \in \mathbf{R}^{n-n_1}$, and analogously the splitting $y = (y^{(1)}, y_{(1)})$, $\xi = (\xi^{(1)}, \xi_{(1)})$, $\eta = (\eta_{(1)}, \eta_{(1)})$ such that $(x^{(1)}, y_{(1)}, \eta^{(1)}, \xi_{(1)})$ are coordinates of the graph of α near $(y_0, \eta_0, x_0, \xi_0)$. There exists a generating function $L(x^{(1)}, \eta^{(1)}, y_{(1)}, \xi_{(1)})$ such that

$$\xi^{(1)} = L_{x^{(1)}}, \quad \eta_{(1)} = -L_{y_{(1)}}, \quad x_{(1)} = -L_{\xi_{(1)}}, \quad y^{(1)} = L_{\eta(1)}. \tag{64.4}$$

Proof: Suppose we know that $(x^{(1)}, \eta^{(1)}, y_{(1)}, \xi_{(1)})$ are coordinates of α near $(y_0, \eta_0, x_0, \xi_0)$. We shall find the generating function L and prove (64.4). Since $x = x(y, \eta)$, $\xi = \xi(y, \eta)$ are homogeneous functions of η the fact that α is canonical is equivalent to

$$\xi^{(1)} \cdot dx^{(1)} + \xi_{(1)} \cdot dx_{(1)} - \eta^{(1)} \cdot dy^{(1)} - \eta_{(1)} \cdot dy_{(1)} = 0. \tag{64.5}$$

We can rewrite (64.5) in the form

$$\begin{aligned} \xi^{(1)} \cdot dx^{(1)} + d(x_{(x)} \cdot \xi_{(1)}) - x_{(1)} \cdot d\xi_{(1)} - d(y^{(1)} \cdot \eta^{(1)}) \\ + y^{(1)} \cdot d\eta^{(1)} - \eta_{(1)} \cdot dy_{(1)} = 0 \end{aligned} \tag{64.6}$$

Denote

$$L(x^{(1)}, \eta^{(1)}, y_{(1)}, \xi_{(1)}) = y^{(1)} \cdot \eta^{(1)} - x_{(1)} \cdot \xi_{(1)}, \tag{64.7}$$

where we expressed $x_{(1)}$ and $y^{(1)}$ in the right hand side of (64.7) as functions of $(x^{(1)}, \eta^{(1)}, y_{(1)}, \xi_{(1)})$. Then (64.6) has the form

$$dL(x^{(1)}, \eta^{(1)}, y_{(1)}, \xi_{(1)})) = \xi^{(1)} \cdot dx^{(1)} - x_{(1)} d\xi_{(1)} + y^{(1)} d\eta^{(1)} - \xi_{(1)} \cdot dy_{(1)}. \tag{64.8}$$

Then (64.4) follows from (64.8). □

We can find a cover $\{U_j\}$ of $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ by conic neighborhoods U_j such that the generating function $L^{(j)}(x^{(j)}, \eta^{(j)}, y_{(j)}, \xi_{(j)})$ exists in $\overline{U_j}$. Let $\{\varphi_j(y, \eta)\}$ be a partition of unity corresponding to $\{U_j\}$. By Φ_j we denote the following FIO

(64.9)

$$\Phi_j a \varphi_j u = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} a(L^{(j)}_{\eta^{(j)}}, y_{(j)}, \eta^{(j)}, -L_{y_j}) \varphi_j(y, \eta^{(j)}, -L^{(j)}_{y_{(j)}})$$
$$\cdot e^{ix_{(j)} \cdot \xi_{(j)} + iL^{(j)}(x^{(j)}, \eta^{(j)}, y_{(j)}, \xi_{(j)}) - iy^{(j)} \cdot \eta^{(j)}} \left| L^{(j)}_{x^{(j)} \eta^{(j)} y_{(j)} \xi_{(j)}} \right|^{\frac{1}{2}} u(y) dy d\eta^{(j)} d\xi_{(j)},$$

where $a(y,\eta)\in S^m$, $|L^{(j)}_{x^{(j)}\eta^{(j)}y_{(j)}\xi_{(j)}}| = \det L^{(j)}_{x^{(j)}\eta^{(j)}y_{(j)}\xi_{(j)}}$.

We shall prove that

$$\Phi_j\varphi_j\varphi_k au = e^{i\frac{\pi}{4}\sigma_{jk}}\Phi_k\varphi_j\varphi_k au + T_{m-1}u, \tag{64.10}$$

where ord $T_{m-1}\le m-1$, $\sigma_{jk} = \text{sgn } H$, H is the Jacobi matrix $\frac{D(x^{(k)},\eta^{(k)},y_{(k)},\xi_{(k)})}{D(x^{(j)},\eta^{(j)},y_{(j)},\xi_{(j)})}$. Using the identity

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{-i(x-z)\cdot\zeta} f(z)dzd\zeta$$

we have

(64.11)

$$\Phi_j\varphi_j\varphi_k au = \lim_{\varepsilon\to 0}\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\chi(\varepsilon(|\xi_{(j)}|+|\eta^{(j)}|+|\hat{\xi}_{(k)}|+|\hat{\eta}^{(k)}|))$$
$$\cdot\exp i\{(x_{(k)}-z_{(k)})\cdot\hat{\xi}_{(k)} + \psi_j(x^{(k)},z_{(k)},y_{(k)},z^{(k)},\eta^{(j)},\xi_{(j)}) + (z^{(k)}-y^{(k)})\cdot\hat{\eta}^{(k)}\}$$
$$\cdot|L_{x^{(j)}\eta^{(j)}y_{(j)}\xi_{(j)}}|^{\frac{1}{2}} a\varphi_j\varphi_k u(y)dydzd\xi_{(j)}d\eta^{(j)}d\hat{\xi}_{(k)}d\hat{\eta}^{(k)},$$

where

(64.12)

$$\psi_j(x,y,\eta^{(j)},\xi_{(j)}) = -x_{(j)}\cdot\xi_{(j)} + L^{(j)}(x^{(j)},\eta^{(j)},y_{(j)},\xi_{(j)}) - y^{(j)}\cdot\eta^{(j)},\quad z=(z^{(k)},z_{(k)}).$$

Applying the stationary phase lemma 62.1 to the integral (64.11) in $(z^{(k)},z_{(k)},\eta^{(j)},\xi_{(j)})$ we get the following equations for the critical point

$$-\hat{\xi}_{(k)} + \psi_{jz_{(k)}} = 0,\quad \hat{\eta}^{(k)} + \psi_{jz^{(k)}} = 0, \tag{64.13}$$
$$\psi_{j\eta^{(j)}} = L^{(j)}_{\eta^{(j)}} - y^{(j)} = 0,\quad \psi_{j\xi_{(j)}} = -x_{(j)} + L^{(j)}_{\xi_{(j)}} = 0.$$

Since $\psi_{j\eta^{(j)}} = \psi_{j\xi_{(j)}} = 0$ and ψ_j is homogeneous in $(\xi_{(j)},\eta^{(j)})$ we have that $\psi_j = 0$ at the critical point. Therefore after applying Lemma 62.1 the phase function in (64.11) becomes

$$-z_{(k)}(x^{(k)},y_{(k)},\hat{\xi}_{(k)},\hat{\eta}^{(k)})\cdot\hat{\xi}_{(k)} + z^{(k)}(x^{(k)},y_{(k)},\hat{\xi}_{(k)},\hat{\eta}^{(k)})\cdot\hat{\eta}^{(k)}. \tag{64.14}$$

Changing the notations in (64.14): $\hat{\eta}^{(k)}$ to $\eta^{(k)}$ and $\hat{\xi}_{(k)}$ to $\xi_{(k)}$ we get that (64.14) coincide with (64.7).

It follows from (64.13) that the Hessian H is equal to the Jacobi matrix $\frac{D(x^{(k)},y_{(k)},\hat{\xi}_{(k)},\hat{\eta}^{(k)})}{D(x^{(j)},y_{(j)},\xi_{(j)},\eta^{(j)})}$.

For the Jacobians we have

$$\left|\frac{D(x^{(k)}, y_{(k)}, \xi_{(k)}, \eta^{(k)})}{D(x^{(j)}, y_{(j)}, \xi_{(j)}, \eta^{(j)})}\right| = \left|\frac{D(x^{(k)}, y_{(k)}, \xi_{(k)}, \eta^{(k)})}{D(y, \eta)}\right| \left|\frac{D(y, \eta)}{D(x^{(j)}, y_{(j)}, \xi_{(j)}, \eta^{(j)})}\right|$$

$$= \left|\frac{D(x^{(k)}, \xi_{(k)})}{D(y^{(k)}, \eta_{(k)})}\right| \left|\frac{D(y^{(j)}, \eta_{(j)})}{D(x^{(j)}, \xi_{(j)})}\right|$$

Note that

$$\left|\frac{D(y^{(j)}, \eta_{(j)})}{D(x^{(j)}, \xi_{(j)})}\right| = \left|L^{(j)}_{x^{(j)}\eta^{(j)}y_{(j)}\xi_{(j)}}\right|$$

and

$$\left|\frac{D(x^{(k)}, \xi_{(k)})}{D(y^{(k)}, \eta_{(k)})}\right| = \left|L^{(k)}_{x^{(k)}\eta^{(k)}y_{(k)}\xi_{(k)}}\right|^{-1}.$$

Therefore

$$\frac{\left|L^{(j)}_{x^{(j)}\eta^{(j)}y_{(j)}\xi_{(j)}}\right|^{\frac{1}{2}}}{|H|^{\frac{1}{2}}} = \left|L^{(k)}_{x^{(k)}\eta^{(k)}y_{(k)}\xi_{(k)}}\right|^{\frac{1}{2}}. \tag{64.15}$$

The integers σ_{jk} are defined on $U_j \cap U_k$ and have the properties:

$$e^{i\frac{\pi}{4}\sigma_{jk}} = e^{i\frac{\pi}{4}\sigma_{kj}} \quad \text{on} \quad U_j \cap U_k,$$

$$e^{i\frac{\pi}{4}(\sigma_{jk}+\sigma_{ki}+\sigma_{ij})} = 1 \quad \text{on} \quad U_j \cap U_k \cap U_i.$$

Note that σ_{jk} are even because H is $2n \times 2n$ matrix. The bundle with the transition functions $e^{i\frac{\pi}{4}\sigma_{jk}}$ is called the Maslov bundle.

Since $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ is simply connected we can choose integers σ_j in U_j such that $e^{i\frac{\pi}{4}\sigma_{jk}} = e^{i\frac{\pi}{4}\sigma_j - i\frac{\pi}{4}\sigma_k}$ in $U_j \cap U_k$.

We define a global FIO as

$$\Phi u = \sum_j e^{i\frac{\pi}{4}\sigma_j}\Phi_j a\varphi_j u. \tag{64.16}$$

Using again the stationary phase lemma we can obtain the following theorem:

Theorem 64.2. *Let α_1, α_2 be canonical transformation of $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ onto $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ and let Φ_1 and Φ_2 be corresponding global FIO with the principal symbols $a_1(y,\eta)$, $a_2(y,\eta)$ respectively, $a_j \in S^{m_j}, j = 1, 2$. Let $\alpha_3 = \alpha_2 \circ \alpha_1$ be the composition of α_1 and α_2. Then $\Phi_3 = \Phi_2\Phi_1 + T_{m_1+m_2-1}$*

is a global FIO corresponding to the canonical transformation α_3, *ord* $T \leq m_1 + m_2 - 1$. *The principal symbol* $a_3(y, \eta)$ *of* Φ_3 *is the composition of* $\alpha_1 \circ a_1$ *and* a_2 *where* $\alpha_1 \circ a_1$ *is the image of* $a_1(y, \eta)$ *under the canonical transformation* α_1.

Note that the pseudodifferential operators are obviously particular cases of FIO when the canonical transformation is the identity and operator Φ^* (c.f. (60.6)) corresponds to the canonical transformation α^{-1} where α is the canonical transformation corresponding to Φ. Lemmas 61.2 - 61.5 and Theorem 61.7 are particular cases of Theorem 64.2. □

Denote by Φ_0 the global FIO (c.f. (64.16)) with $a(y, \eta) = 1$ corresponding to the canonical transformation α. Analogously to the proof of Lemma 61.4 and Theorem 64.2 we have

$$\Phi_0\Phi_0^* = I + C_N^{(i)} + T_{-N}^{(1)}, \quad \Phi_0^*\Phi_0 = I + C_N^{(2)} + T_{-N}^{(2)}, \tag{64.17}$$

where ord $T_{-N}^{(k)} \leq -N$, $C_N^{(k)}(x, \xi) \in S^{-1}$, $k = 1, 2$, N is arbitrary.

Let $A(x, \xi) \in S^m$. Then analogously to Theorems 61.2 and 64.2 we get

$$\Phi_0^* A(x, D)\Phi_0 = B_N(x, D) + T_{m-N}, \tag{64.18}$$

where $B_N(x, \xi) \in S^m$, ord $T \leq m - N$ and the principal symbol of $B(x, D)$ is the image of the principal symbol of $A(x, \xi)$ under the canonical transformation α. □

Consider the canonical transfomation defined by the Hamiltonian system (c.f. (50.11), (50.12))

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\partial \lambda(x, t, \xi)}{\partial \xi}, \quad x(0) = y, \\ \frac{d\xi}{dt} &= \frac{\partial \lambda(x, t, \xi)}{\partial x}, \quad \xi(0) = \eta, \\ \frac{d\xi_0}{dt} &= \frac{\partial \lambda(x, t, \xi)}{\partial t}, \quad \xi_0(0) = \eta_0. \end{aligned} \tag{64.19}$$

The canonical transformation α defined by (64.19) maps (y, t, η, η_0) to $(x(t), t, \xi(t), \xi_0(t))$. The last equation in (64.19) is equivalent to

$$\xi_0(t) - \lambda(x(t), t, \xi(t)) = \eta_0, \quad \forall t. \tag{64.20}$$

Therefore applying (64.18) with $A(x, t, \xi, \xi_0) = \xi_0 - \lambda(x, t, \xi)$ we get

$$\Phi_0^*(D_t - \lambda(x, t, D_x))\Phi_0 = D_t + T, \tag{64.21}$$

where ord $T \leq 0$ (c.f. Example 61.1).