Part VIII

Fourier Integral Operators.

60 The definition and the boundness of the Fourier integral operators.

Let $S(x,\eta)$ be the same as in (50.4), i.e. $S(x,\eta)\in C^\infty(\mathbf{R}^n\times(\mathbf{R}^n\setminus\{0\}),\ \deg S_\eta(x,\eta)=1$ and

$$\det S_{x\eta}\neq 0,\quad \text{where } S_{x\eta}=\left[\frac{\partial^2 S}{\partial x_i\partial\eta_j}\right]_{i,j=1}^n. \tag{60.1}$$

Denote $\xi=S_x(x,\eta),\ y=S_\eta(x,\eta)$. It follows from (60.1) that (c.f. (50.4))

$$y=S_\eta(x,\eta),\qquad \xi=S_x(x,\eta) \tag{60.2}$$

define a diffeomorphism of $(y,\eta)\rightarrow(x,\xi)$ in a neighborhood of every point $(x,\eta)\in\mathbf{R}^n\times(\mathbf{R}^n\setminus\{0\})$. We assume for simplicity that $S(x,\eta)=x\cdot\eta$ for $|x|>R$ and that(60.2) is a diffeomorphism of $\mathbf{R}^n\times(\mathbf{R}^n\setminus\{0\})\rightarrow\mathbf{R}^n\times(\mathbf{R}^n\setminus\{0\})$.

Let $a(x,\eta)\in S^m$,

$$a(x,\eta)=\sum_{k=0}^N a_k(x,\eta)(1-\chi(\eta)))+S^{m-N-1},\quad \forall N, \tag{60.3}$$

where $a_k(x,\eta)$ are homogeneous in η of degree $m-k$.

An operator of the form

$$\Phi u=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}a(x,\eta)e^{iS(x,\eta)}\tilde{u}(\eta)d\eta,\quad u\in C_0^\infty(\mathbf{R}^n), \tag{60.4}$$

is called the Fourier integral operator with the phase function $S(x,\eta)$ and the symbol $a(x,\eta)$. Denote by Φ^* the formally adjoint operator to Φ (c.f. (43.26)):

$$(\Phi u,v)=(u,\Phi^*v),\quad \forall u,v\in C_0^\infty(\mathbf{R}^n),$$

where (u,v) is the L_s-inner product. We have

$$\Phi^*v=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\overline{a(y,\xi)}e^{ix\cdot\xi-iS(y,\xi)}v(y)dyd\xi,\quad v\in C_0^\infty(\mathbf{R}^n). \tag{60.5}$$

As in §43 we treat the integral in (60.5) as a repeated integral, i.e. first integrate in y and then integrate in ξ. We shall call (60.5) also a Fourier integral operator. Operators (60.4) and (60.5) are particular cases of a more general class of Fourier integral operators of the form

$$\Psi u = \int_{\mathbf{R}^n}\int_{\mathbf{R}^r} a(x,y,\xi)e^{i\psi(x,y,\xi)}u(y)dyd\xi, \tag{60.6}$$

when $r \geq n$.

Theorem 60.1. *Operators Φ and Φ^* are bounded from $H_s(\mathbf{R}^n)$ to $H_{s-m}(\mathbf{R}^n)$ for any s.*

Proof: Consider the product $\Phi\Lambda^{2s}\Phi^*u$ where $\Lambda^{2s}(\xi) = (1+|\xi|^2)^s$. We have

$$\Lambda^{2s}\Phi^*u = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(1+|\xi|^2)^s\overline{a(y,\xi)}e^{ix\cdot\xi - iS(y,\xi)}u(y)dyd\xi$$

and

$$\Phi\Lambda^{2s}\Phi^*u = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} a(x,\xi)\overline{a(y,\xi)}(1+|\xi|^2)^s e^{iS(x,\xi)-iS(y,\xi)}u(y)dyd\xi. \tag{60.7}$$

As in (45.7) we get

$$S(x,\xi) - S(y,\xi) = \sum(x,y,\xi)\cdot(x-y), \tag{60.8}$$

where

$$\sum(x,y,\xi) = \int_0^1 S_x(y+t(x-y))dt. \tag{60.9}$$

Denote

$$K_1u = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} a(x,\xi)\overline{a(y,\xi)}(1+|\xi|^2)^s\chi(\frac{x-y}{\delta})e^{iS(x,\xi)-iS(y,\xi)}u(y)dyd\xi, \tag{60.10}$$

where $\chi(x) \in C_0^\infty(\mathbf{R}^n)$, $\chi(x) = 1$ for $|x| \leq 1$, $\chi(x) = 0$ for $|x| > 2$, δ is small. Let

$$K_2u = \Phi\Lambda^{2s}\Phi^*u - K_1u,$$

i.e. K_2 has the form (60.10) with $\chi(\frac{x-y}{\delta})$ replaced by $1-\chi(\frac{x-y}{\delta})$. Since $S_\xi(x,\xi)-S_\xi(y,\xi)\neq 0$ when $x-y\neq 0$ we have

$$S_\xi(x,\xi)-S_\xi(y,\xi)|\geq C|\xi|$$

when $|x-y|\geq\delta>0$. We have for $\forall N$

$$e^{iS(x,\xi)-iS(y,\xi)}=\left[\frac{S_\xi(x,\xi)-S_\xi(y,\xi)}{|S_\xi(x,\xi)-S_\xi(y,\xi)|^2}\cdot\left(-i\frac{\partial}{\partial\xi}\right)\right]^N e^{iS(x,\xi)-iS(y,\xi)}. \tag{60.11}$$

Substituting the identity (60.11) in K_2u and integrating by parts in ξ N times we get (c.f. (44.5), (44.6)) that K_2u is an integral operator with $C^\infty(\mathbf{R}^n\times\mathbf{R}^n)$ kernel.

In (60.10) make the change of variables

$$\eta=\Sigma(x,y,\xi). \tag{60.12}$$

Since $\Sigma(x,x,\xi)=S_x(x,\xi)$ the inverse map $\xi=\Sigma^{-1}(x,y,\eta)$ exists when $|x-y|<2\delta$. To justify the change of variables we introduce the cutoff function $\chi(\varepsilon\xi)$, make the change of variables and then take $\varepsilon\to 0$. We get

$$K_1u=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}a(x,\Sigma^{-1}(x,y,\eta))\overline{a(y,\Sigma^{-1}(x,y,\eta)}(1+|\Sigma^{-1}(x,y,\eta)|^2)^s$$

$$\cdot e^{i(x-y)\cdot\eta}v(y)\left|\frac{D\Sigma^{-1}}{D\eta}\right|dyd\eta,$$

where $\left|\frac{D\Sigma^{-1}}{D\eta}\right|=\left|\frac{D\eta}{D\xi}\right|$ and $\frac{D\eta}{D\xi}$ is the Jacobian of the map (60.12). Note that $\left|\frac{D\eta}{D\xi}\right|=|S_{x\xi}(x,\xi)|$ when $x=y$, where $|S_{x\xi}(x,\xi)|$ is the determinant of $S_{x\xi}(x,\xi)$. Note that K_1 is a pseudodifferential operator of the form (43.2) and has order $2m+2s$. The principal symbol of K_1 is

$$|a(x,\Sigma^{-1}(x,x,\eta))|^2(1+|\Sigma^{-1}(x,x,\eta)|^2)^s\ |S_{x\xi}(x,\Sigma^{-1}(x,x,\eta)|^{-1}. \tag{60.13}$$

Since ord $K_2=-\infty$ we get

$$\|\Phi\Lambda^{2s}\Phi^*u\|_0^2=\|K_1u+K_2u\|_0^2\leq C\|u\|_{m+s}^2. \tag{60.14}$$

Therefore

$$\|\Phi^*u\|_s^2=(\Lambda^s\Phi^*u,\Lambda^s\Phi^*u)=((K_1+K_2)u,u)\leq C\|u\|_{m+s}^2, \tag{60.15}$$

i.e. ord $\Phi^* = m$. It follows from (60.15) that

$$|(\Phi u, v)| = |(u, \Phi^* v)| \le C\|u\|_s \|\Phi^* v\|_{-s} \le C\|u\|_s \|v\|_{m-s}$$

for $\forall v \in H_{m-s}(\mathbf{R}^n)$. Therefore (c.f. §13)

$$\|\Phi u\|_{s-m} \le C\|u\|_s. \tag{60.16}$$

Since s is arbitrary Theorem 60.1 is proven. □

61 Operations with Fourier integral operators.

We shall need the stationary phase lemma (c.f. Lemma 19.4)

Lemma 61.1. *Let $\psi(x, \xi, y, \eta)$ be a C^∞ function for $\xi \ne 0, \eta \ne 0$, homogeneous in (ξ, η) of degree one. Suppose $y_0 = y(x, \xi)$, $\eta_0 = \eta(x, \xi)$, $\xi \ne 0$, is the unique solutions of the equations*

$$\psi_y(x, \xi, y, \eta) = 0, \quad \psi_\eta(x, \xi, y, \eta) = 0. \tag{61.1}$$

Suppose $|H| \ne 0$, where $H = \begin{bmatrix} \psi_{yy} & \psi_{y\eta} \\ \psi_{\eta y} & \psi_{\eta\eta} \end{bmatrix}$, $|H| = \det H$. Suppose $C(x, \xi, y, \eta) \in C^\infty$ for $\xi \ne 0$, $\eta \ne 0$, homogeneous in (ξ, η), and the support of $C(x, \xi, y, \eta)$ is contained in a neighborhood of (y_0, η_0). Then

$$\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} C(x, \xi, y, \eta) e^{i\psi(x,\xi,y,\eta)} dy d\eta \tag{61.2}$$
$$= \frac{C(x, \xi, y_0(x, \xi), \eta_0(x, \xi)) e^{i\psi(x,\xi,y_0,\eta_0) + i\frac{\pi}{4} sgn\ H}}{\sqrt{|H(x, \xi, y_0, \eta_0)|}} \cdot \left(1 + \sum_{k=1}^{N} C_k(x, \xi) + O(\frac{1}{|\xi|^{N+1}}) \right),$$

where $sgn\ H(x, \xi, y_0, \eta_0)$ is the signature of the matrix H, $ord_\xi\ C_k(x, \xi) = -k$.

Note that (61.2) is understood as an oscillatory integral, i.e. we take cutoff $\chi(\varepsilon\eta)$ and consider the limit in D' when $\varepsilon \to 0$.

The proof of Lemma 61.1 follows from the proof of Lemma 19.4 if we make change of variables $\eta = |\xi|\eta'$ and apply (19.16) with $\lambda = |\xi|$ and n replaced by $2n$.

Note that if the symbol $C(x,\xi,y,\eta)$ is such that there is no solution to (61.1) on supp C and

$$|\psi_y| + |\psi_\eta|(|\xi| + |\eta|) \geq C(|\xi| + |\eta|) \quad \text{on supp } C, \tag{61.3}$$

then using an identity of the form (60.11) and integrating by parts in (y,η) we get that the integral (61.2) is $O(\frac{1}{|\xi|^N})$, $\forall N$.

Lemma 61.2. *Let $A(x,\xi) \in S^\alpha$ and let Φ be an operator of the form (60.4). Then*

$$A(x,D)\Phi u = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} C_N(x,\eta) e^{iS(x,\eta)} \tilde{u}(\eta) d\eta + T_{\alpha+m-N} u, \tag{61.4}$$

where $C_N(x,\eta) \in S^{\alpha+m}$, ord $T_{\alpha+m-N} \leq \alpha + m - N$ and the principal part C_0 of $C_N(x,\eta)$ has the form

$$C_0(x,\eta) = A_0(x, S_x(x,\eta)) a(x,\eta), \tag{61.5}$$

where A_0 is the principal part of $A(x,\xi)$.

Proof: We have

$$A(x,D)\Phi u = \frac{1}{(2\pi)^{2n}} \int_{\mathbf{R}^n} A(x,\xi) a(y,\eta) e^{i(x-y)\cdot\xi + iS(y,\eta)} \tilde{u}(\eta) d\eta dy d\xi.$$

Compute the integral in (y,ξ) using Lemma 61.1. We have

$$\psi = (x-y)\cdot\xi + S(y,\eta), \quad \psi_y = -\xi + S_y(y,\eta) = 0, \quad \psi_\xi = x - y = 0,$$

$$\psi_{\xi y} = -I, \quad \psi_{yy} = S_{yy}, \quad \psi_{\xi\xi} = 0.$$

Therefore at the critical point $\psi = S(x,\eta)$, $H = \begin{bmatrix} S_{yy} & -I \\ -I & 0 \end{bmatrix}$, $\det H = 1$ and the signature of H is zero. Therefore Lemma 61.1 implies (61.4), (61.5).

Analogously we can prove

Lemma 61.3. *Let $B(x,\xi) \in S^\alpha$ and Φ is the same as in (60.4). Then*

$$\Phi B u = \frac{1}{(2\pi)^{2n}} \int_{\mathbf{R}^n} C_N^{(1)}(x,\eta) e^{iS(x,\eta)} \tilde{u}(\eta) d\eta + T_{\alpha+m-N} u, \tag{61.6}$$

where $C_N^{(1)} \in S^{m+\alpha}$, ord $T_{\alpha+m-N} \leq \alpha + m - N$ and the principal symbol $C_0^{(1)}$ of $C_N^{(1)}(x,\eta)$ is

$$C_0^{(1)}(x,\eta) = B_0(S_\eta(x,\eta), \eta) a(x,\eta). \tag{61.7}$$

Proof: We have

$$\Phi Bu = \frac{1}{(2\pi)^{2n}} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{iS(x,\xi)-iy\cdot\xi} a(x,\xi) B(y,\eta) e^{iy\cdot\eta} \tilde{u}(\eta) d\xi dy d\eta.$$

Compute the integral in (y,ξ) using Lemma 61.1. We have

$$\psi = S(x,\xi) - y\cdot\xi + y\cdot\eta, \quad \psi_y = -\xi + \eta = 0,$$

$$\psi_\xi = S_\xi(x,\xi) - y = 0, \quad \psi_{y\xi} = -I, \quad \psi_{yy} = 0, \quad \psi_{\xi\xi} = S_{\xi\xi}(x,\xi).$$

Therefore at the critical point

$$\psi = S(x,\eta), \quad H = \begin{bmatrix} S_{yy} & -I \\ -I & 0 \end{bmatrix}, \quad |H| = 1, \quad \text{sgn } H = 0,$$

and we get from (61.2) that (61.6) and (61.7) hold.

Lemma 61.4. *Let Φ and Φ^* have the form (60.4) and (60.5), respectively. Then for arbitrary N*

$$\Phi^*\Phi u = C_N u + T_{2m-N}, \tag{61.8}$$

where C_N is a ψdo, $C_N(x,\xi) \in S^{2m}$.

Proof: We have

$$(\Phi^*\Phi)u(y) = \frac{1}{(2\pi)^{2n}} \int_{\mathbf{R}^n} \overline{a(x,\xi)} a(x,\eta) e^{iy\xi - iS(x,\xi) + iS(x,\eta)} \tilde{u}(\eta) d\eta dx d\xi.$$

Apply Lemma 61.1 to the integral in (x,ξ). Let $\psi = y\cdot\xi - S(x,\xi) + S(x,\eta)$. Then

$$\psi_x = -S_x(x,\xi) + S_x(x,\eta) = 0, \quad \psi_\xi = y - S_\xi(x,\xi) = 0, \quad \psi_{x\xi} = -S_{x\xi}(x,\xi),$$

$$\psi_{xx} = -S_{xx}(x,\xi) + S_{xx}(x,\eta), \quad \psi_{\xi\xi} = -S_{\xi\xi}(x,\xi).$$

Since we assume that the map $\xi = S_x(x,\eta)$ is one-to-one for each x we get that $\psi_x = 0$ implies that $\xi = \eta$ at the critical point. Therefore $S(x,\eta) = S(x,\xi)$, $\psi_{xx} = 0$, $H = \begin{bmatrix} 0 & -S_{x\xi}(x,\xi) \\ -S_{x\xi}(x,\xi) & -S_{\xi\xi}(x,\xi) \end{bmatrix}$. Therefore $|H| = |S_{x\xi}(x,\xi)|^2$ and $\text{sgn } H = 0$. We have

$$(\Phi^*\Phi u)(y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} C_N(y,\eta) e^{iy\eta} \tilde{u}(\eta) d\eta + T_{2m-N} u,$$

where the principal symbol of $C_N(y,\eta)$ is

$$|a_0(x_0,\eta)|^2|S_{x\eta}(x_0,\eta)|^{-1}, \tag{61.9}$$

where $x_0(y,\eta)$ is such that $S_\eta(x_0,\eta)=y$.

Denote by Φ_0 the Fourier integral operator of the form

$$\Phi_0 u=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}|S_{x\eta}(x,\eta)|^{\frac{1}{2}}(1-\chi(\eta))e^{iS(x,\eta)}\tilde{u}(\eta)d\eta. \tag{61.10}$$

Then

$$\Phi_0^* u=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}|S_{x\eta}(x,\xi)|^{\frac{1}{2}}(t-\chi(\xi))e^{ix\cdot\xi-iS(y,\xi)}u(y)dy. \tag{61.11}$$

The following lemma follows from (60.13) with $s=0$, and (61.9):

Lemma 61.5. *Let $\Phi_0,\ \Phi_0^*$ be as in (61.10), (61.11). Then*

$$\begin{aligned}\Phi_0^*\Phi_0&=I+T_{-1}^{(1)},\\ \Phi_0\Phi_0^*&=I+T_{-1}^{(2)},\end{aligned} \tag{61.12}$$

where ord $T_{-1}^{(p)}\le -1,\ p=1,2.$

□

We shall call the Fourier integral operator (60.4) an elliptic Fourier integral operator (FIO) if the principal part $a_0(x,\eta)$ of $a(x,\eta)\in S^m$ is elliptic, i.e. $a_0(x,\eta)\neq 0,\ \forall x\in\mathbf{R}^n,\ \forall\eta\neq 0.$

Lemma 61.6. *Elliptic FIO is Fredholm in $H_s(\mathbf{R}^n)$ for any $s\in\mathbf{R}$.*

Proof: It follows from (61.8), (61.9) that $\Phi^*\Phi=C+T_{-1}$, where C is elliptic ψdo. Therefore there exists an opeartor R_1, ord $R_1\le -2m$ such that $R_1C=I+T_{-1}^{(1)}$ where ord $T_{-1}^{(1)}\le -1$ (c.f. §42). Therefore $R\Phi^*$ is the left regularizer for Φ. Analogously it follows from (60.7), with $s=0$, and (60.13) that

$$\Phi\Phi^*=C_1+T_{-1}^{(2)},$$

where C_1 is elliptic ψdo, ord $T_{-1}^{(2)}\le -1$. Therefore Φ has a right regularizer too, and subsequently Φ is Fredholm (c.f. §42).

The following theorem is called the Egorov's theorem.

Theorem 61.7. *Let* $A_0(x,\xi)$ *be the principal part of* $A(x,\xi)\in S^m$*. Then*

$$\Phi_0^* A(x,D)\Phi_0 = B_N(x,D) + T_{m-N}, \quad \forall N, \tag{61.13}$$

where $B_N(x,\xi)\in S^m$ *with the principal part*

$$B_0(y,\eta) = A_0(x,\xi), \tag{61.14}$$

where

$$y = S_\eta(x,\eta), \quad \xi = S_x(x,\eta), \tag{61.15}$$

i.e. (y,η) *and* (x,ξ) *are related by the canonical transformation (61.15) (c.f. (50.4)).*

Proof: It follows from Lemmas 61.2 and 61.4 that B_N is a *ψdo.* Since $a_0(x,\eta) = |S_{x\eta}(x,\eta)|^{\frac{1}{2}}$ we get from (61.5) that the principal symbol of $A\Phi$ is

$$A_0(x, S_x(x,\eta)|S_{x\eta}(x,\eta)|^{\frac{1}{2}}.$$

Analogously to (61.9) we have

$$B_0(y,\eta) = A_0(x_0(y,\eta), S_x(x_0(y,\eta),\eta)), \tag{61.16}$$

where $y = S_\eta(x_0,\eta)$. Therefore

$$B_0(S_\eta(x,\eta),\eta) = A_0(x, S_x(x,\eta)). \tag{61.17}$$

The equality (61.17) is equivalent to (61.14). □

Remark 61.1 Theorem 61.7 follows also from Lemmas 61.2, 61.3. If $A\Phi u = \Phi B u + T_{-N}$ then compairing (61.5) and (61.7) we get that $A_0(x, S_x(x,\eta)) = B_0(S_\eta(x,\eta),\eta)$. Replacing Φ by Φ_0 and multiplying $A\Phi_0$ by Φ_0^* we get $\Phi_0^* A\Phi = \Phi_0^*\Phi_0 B + T_{-N} = B + T_{-N}^{(1)}$.

Example 61.1. Consider the symbol $\xi_n - \lambda(x,\xi')$ where $\xi'\in\mathbf{R}^{n-1}$, $\lambda(x,\xi')\in C^\infty$ when $\xi'\neq 0$, $\deg_{\xi'}\lambda(x,\xi') = 1$ and $\lambda(x,\xi')$ is real-valued. We shall find a phase $S(x,\eta)$ such that

$$\Phi_0^* A\Phi_0 = B + T_{-1}, \tag{61.18}$$

where $A(x,D) = D_n - \lambda(x,D')(1-\chi(D'))$, B is a *ψdo* with the symbol $B(y,\eta) = \eta_n$ and Φ_0 has the form (61.10). Let $\varphi(x,\eta)$ be the solution of the eiconal equation

$$\begin{gathered} \varphi_{x_n} - \lambda(x,\varphi_{x'}) = 0, \\ \varphi(x,\eta)|_{x_n=0} = x'\cdot\xi'. \end{gathered} \tag{61.19}$$

Denote $S(x,\eta) = \varphi(x,\eta') + x_n\eta_n$. We have

$$\xi' = S_{x'} = \varphi_{x'}(x,\eta'), \tag{61.20}$$
$$\xi_n = S_{x_n} = \eta_n + \varphi_{x_n}(x,\xi') = \eta_n + \lambda(x,\varphi_{x'}(x,\eta')),$$
$$y' = S_{\eta'} = \varphi_{\eta'}(x,\eta'), \quad y_n = S_{\eta_n} = x_n.$$

Note that $y' = \varphi_{\eta'}(x,\eta') = x'$ when $x_n = 0$. Therefore $S(x,\eta)$ defines a canonical transformation

$$(y,\eta) \to (x,\xi)$$

equal to identity when $x_n = 0$ It is a diffeomorphism when $|x_n|$ is small. Note that the solution of (61.19) also exists when $|x_n|$ is small. We assume in this section that the solution of (61.19) exists for all $x_n \in \mathbf{R}$ and that the canonical transformation $(y,\eta) \to (x,\xi)$ defined by (61.19) is a diffeomorphism on $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$. We shall consider the general case in §64. Note that (61.20) implies that $\eta_n = \xi_n - \lambda(x,\xi')$ and therefore (61.18) holds.

62 The wave front set of Fourier integral operators.

Consider FIO of thr form (60.4). Denote by $\alpha \circ \Sigma$ the image of the set $\Sigma \subset \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ under the canonical transformation $\alpha : (y,\eta) \to (x,\xi)$ given by (60.2).

Theorem 62.1. *Suppose $u \in H_s(\mathbf{R}^n)$ for some $s \in \mathbf{R}$. The wave front set $WF(\Phi u)$ of Φu is contained in $\alpha \circ WF(u)$, where $WF(u)$ is the wave front set of u.*

Proof: We have

$$\Phi u = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{iS(x,\eta) - iy\cdot\eta} a(x,\eta) u(y) dy d\eta.$$

We represent Φ as the sum $\Phi = \Phi_1 + \Phi_2$, where

$$\Phi_1 u = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{iS(x,\eta) - iy\cdot\eta} \chi\Big(\frac{S_\eta(x,\eta) - y}{\delta}\Big) a(x,\eta) u(y) dy d\eta$$

and $\Phi_2 = \Phi - \Phi_1$. The following identity holds (c.f. (60.11)):

$$\left[\frac{(S_\eta(x,\eta) - y)}{|S_\eta - y|^2} \cdot \left(-i\frac{\partial}{\partial\eta}\right)\right]^N e^{iS(x,\eta) - iy\cdot\eta} = e^{iS(x,\eta) - iy\cdot\eta}, \quad \forall N. \tag{62.1}$$

Substituting (62.1) in $\Phi_2 u$ and integrating by parts in η we get that $\Phi_2 u \in C^\infty$.

Suppose $(x_0, \xi_0) \notin \alpha \circ WF(u)$. Then there exists $\varphi_0(x) \in C_0^\infty(\mathbf{R}^n)$, $\varphi_0(x_0) \neq 0$ and $\alpha_0(\xi) \in C^\infty$ for $\xi \neq 0$, $\deg_\xi \alpha_0(\xi) = 0$, $\alpha_0(\xi_0) \neq 0$ such that $\operatorname{supp} \varphi_0(x)\alpha_0(\xi) \cap \alpha \circ WF(u) = \emptyset$. Consider the Fourier transform of $\varphi_0 \Phi_1 u$. We get

$$I(\xi) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \alpha_0(\xi)\varphi_0(x) e^{-ix\cdot\xi + iS(x,\eta) - iy\cdot\eta} a(x,\eta)\chi(\frac{S_\eta - y}{\delta}) u(y)\,dy\,d\eta\,dx. \tag{62.2}$$

Let $I = I_1 + I_2$, where

$$I_1 = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \alpha_0(\xi)\varphi_0(x)\chi(\frac{\xi - S_x(x,\eta)}{\delta\sqrt{|\xi|^2 + |\eta|^2}})\chi(\frac{S_\eta - y}{\delta}) a(x,\eta) \cdot e^{-ix\cdot\xi + iS(x,\eta) - iy\cdot\eta} u(y)\,dy\,d\eta\,dx, \tag{62.3}$$

and $I_2 = I - I_1$.

Using the identity

$$\left[\frac{(\xi - S_x(x,\eta))}{|\xi - S_x(x,\eta)|^2} \cdot \left(-i\frac{\partial}{\partial x}\right)\right]^N e^{iS(x,\eta) - ix\cdot\xi} = e^{iS(x,\eta) - ix\cdot\xi}, \quad \forall N, \tag{62.4}$$

and integrating by parts in I_2 we get that

$$|I_2(\xi)| \leq C_N(1 + |\xi|)^{-N}, \quad \forall N. \tag{62.5}$$

Let (y_0, η_0) be such that $(x_0, \xi_0) = \alpha \circ (y_0, \eta_0)$. By the assumption $(y_0, \eta_0) \notin WF(u)$. Let $\psi_0(y) \in C_0^\infty(\mathbf{R}^n)$, $\psi_0(y) = 1$ in a neighborhood of y_0 and let $\beta_0(\eta) \in C^\infty(\mathbf{R}^n \setminus \{0\})$, $\deg_\eta \beta_0(\eta) = 0$, $\beta_0(\eta) = 1$ in a conic neighborhod of η_0. We assume that $\operatorname{supp} \psi_0(y)\beta_0(\eta) \cap WF(u) = \emptyset$. We take $\delta > 0$ so small that $\varphi_0(y)\beta_0(\eta) = 1$ on $\operatorname{supp} \alpha_0(\xi)\varphi_0(x)\chi(\frac{\xi - S_x}{\delta(|\xi|^2)})\chi(\frac{y - S_\eta}{\delta})$. Then we get

$$I_1 = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \alpha_0(\xi)\varphi_0(x)\chi(\frac{\xi - S_x(x,\eta)}{\delta\sqrt{|\xi|^2 + |\eta|^2}})\chi(\frac{y - S_\eta}{\delta}) a(x,\eta) \cdot e^{-ix\cdot\xi + iS(x,\eta) - iy\cdot\eta} \beta_0(\eta)\psi_0(y) u(y)\,dy\,d\eta\,dx. \tag{62.6}$$

Since $\operatorname{supp} \beta_0(\eta)\psi_0(y) \cap WF(u) = \emptyset$ we get that (c.f. §14)

$$\left|\int_{\mathbf{R}^n} e^{-iy\cdot\eta}\chi(\frac{y - S_\eta}{\delta})\beta_0(\eta)\psi_0(y)u(y)\,dy\right| \leq C_N(1 + |\eta|)^{-N}, \ \forall N. \tag{62.7}$$

Therefore

$$|I_1(\xi)| \leq C_N(1 + |\xi|)^{-N}, \quad \forall N,$$

i.e. $(x_0, \xi_0) \notin WF(\Phi u)$.