57 The parametrix for the Dirichlet-to-Newmann operator.

Consider the Dirichlet problem in Ω

$$A(x,D)u = 0, \quad x \in \Omega, \tag{57.1}$$

$$u|_{\partial\Omega} = g(x'), \tag{57.2}$$

where A is the Laplace-Beltrami operator (56.1).

Denote by Λ the operator (the Dirichlet-to-Neumann operator) that transforms $g(x')$ to the Neumann data on $\partial\Omega$ (c.f. (56.28)):

$$\Lambda f = \sum_{j,k=1}^{n} g^{jk}(x)\frac{\partial u}{\partial x_j}\nu_k(x)\left(\sum_{p,r=1}^{n} g^{pr}(x)\nu_p\nu_r\right)^{-\frac{1}{2}}\Bigg|_{\partial\Omega}. \tag{57.3}$$

We shall show that Λ is a pseudodifferential operator on $\partial\Omega$ and we will find its symbol.

Let $U_j \cap \partial\Omega \neq \emptyset$. In the semigeodesics coordinates (y', y_n) (c.f. (56.11)) we have

$$A_j(y, D_y)u = 0, \tag{57.4}$$

$$\frac{\partial u}{\partial y_n}\bigg|_{y_n=0} = f. \tag{57.5}$$

Since $\lambda = 0$ in (57.1) we shall define

$$\tilde{R}_j^+ = \sum_{p=0}^{N} \tilde{R}_{jp}^+(y, D), \tag{57.6}$$

where $\tilde{R}_{j0}^+(y,\xi) = (\xi_n - i\sigma_j(y,\xi')(1-\chi(\xi')))^{-1}$, $\tilde{A}_{j0}^- = (\xi_n + i\sigma_j(1-\chi(\xi'))$, $\tilde{R}_{jp}^+, \tilde{A}_{jp}^-(y,\xi)$ are defined as in (55.18) with $A_\pm$ replaced by $\tilde{A}_{j0}^\pm$. Note that $\tilde{R}_{jp}^+(y,\xi) = O(\frac{1}{\xi_n^2})$ when $p \geq 1$. Therefore

$$p'\tilde{R}_j^+(g(y'))\delta(y_n)) = g(y'), \tag{57.7}$$

$$A_j(y, D_y)\tilde{R}_j^+(g(x')\delta(y_n)) = T_{-N}(g(y')\delta(y_n)), \tag{57.8}$$

where T_{-N} satisfies an estimate of the form (55.22) with $m_+ = 1$.

Denote

$$\Lambda_j g = \lim_{y_n\to+0} \frac{\partial}{\partial y_n} \tilde{R}_j^+(g(y')\delta(y_n)). \tag{57.9}$$

Then Λ_j is a pseudodifferential operator in $\mathbf{R}^{n-1}$ with symbol

$$\Lambda_j(y',\xi') = \Pi'\left(i\xi_n \sum_{p=0}^{N} \tilde{R}_{jp}^+(y',0,\xi)\right), \tag{57.10}$$

where $\Pi' D_+(y',\xi) = \frac{1}{2\pi}\int_{\gamma_+} D_+(y',\xi',\xi_n)d\xi_n$, γ_+ is the contour containing all poles of $D_+(y',0,\xi',\xi_n)$ (c.f. (55.24)). It follows from (57.10) that

$$\Lambda_j(y',\xi') = \sum_{p=0}^{N} \Lambda_{jp}(y',\xi'), \tag{57.11}$$

where ord $\Lambda_{jp} = 1-p$. We shall compute several terms in (57.11). Note that $A_j(y,D_y)$ has the form

$$A_j(y,D_y) = -\frac{\partial^2}{\partial y_n^2} - \sum_{p,r=1}^{n-1} g_j^{pr}(y)\frac{\partial^2}{\partial y_p \partial y_r} + A_1(y,D_y), \tag{57.12}$$

where

$$A_1 = -\frac{g_{jy_n}}{2g_j}\frac{\partial u}{\partial y_n} - \sum_{p,r=1}^{n-1} \frac{1}{\sqrt{g_j}}\frac{\partial}{\partial y_p}(\sqrt{g_j} g_j^{pr})\frac{\partial}{\partial y_r}. \tag{57.13}$$

We have

$$\Lambda_{j0}(y',\xi') = \Pi' \frac{i\xi_n}{\xi_n - i\tilde{\sigma}_j} = -i\tilde{\sigma}_j = -i\sqrt{\sum_{p,r=1}^{n-1} g_j^{pr}(y',0)\xi_p\xi_r}(1-\chi(\xi')), \tag{57.14}$$

$$\Lambda_{jp}(y',\xi') = \Pi' i\xi_n \tilde{A}_+^{-1}\Pi^+\tilde{A}_-^{-1}T_p, \quad p\geq 1, \tag{57.15}$$

where T_p is given by (55.16). In particular

$$T_1 = A_1(y,\xi)R_{j0}^+ + i\sum_{k=1}^{n} \frac{\partial A_0}{\partial \xi_k}\frac{\partial R_{j0}^+}{\partial y_k}. \tag{57.16}$$

Note that $\tilde{A}_-^{-1}T_p$ has poles at $\xi_n = \pm i\tilde{\sigma}_j$, and therefore

$$\Pi^+\tilde{A}_+^{-1}T_p = \sum_{k=1}^{p+1}\alpha_{pk}(y',\xi')(\xi_n - i\tilde{\sigma}_j)^{-k}. \tag{57.17}$$

Note that $\Pi' i\xi_n(\xi_n - i\tilde{\sigma}_j)^{-r} = 0$ when $r \geq 3$. Therefore

$$\Lambda_{jp}(y',\xi') = \Pi'\frac{i\xi_n\alpha_{p1}(y',\xi')}{(\xi_n - i\tilde{\sigma}_j)^2} = -\alpha_{p1}(y',\xi'), \tag{57.18}$$

i.e. to compute $\Lambda_{jp}(y',\xi')$ one needs to find the residue $\alpha_{p1}(y',\xi')$ of $\tilde{A}_-^{-1}T_p$ at $\xi_n = i\tilde{\sigma}_j$. In particular, when $p = 1$ we get

$$\Lambda_{j1}(y',\xi') = \mathrm{res}_{\xi_n = i\tilde{\sigma}_j}\left[\frac{A_1(y,\xi',\xi_n)}{\xi_n^2+\sigma_j^2} - \frac{\sum_{k=1}^n \frac{\partial A_0(y',0,\xi',\xi_n)}{\partial\xi_k}\sigma_{jy_k}}{(\xi_n+i\sigma_j)(\xi_n - i\sigma_j)2}\right]$$

$$= \frac{A_1(y',\xi',i\sigma_j)}{-2i\sigma_j} + \frac{\sum_{k=1}^n \frac{\partial A_0(y',0,\xi',-i\sigma_j)}{\partial\xi_k}\sigma_{jy_k}}{4\sigma_j^2}.$$

Now we shall construct the Dirichlet-to-Neumann operator (D-to-N) operator in Ω. Let U_j, φ_j, ψ_j will be the same as in (55.31), and let R_j^+ be the same as in (57.6). Denote by Rf the following operator

$$Rf = \sum_j{}' \psi_j(x)\tilde{R}_j^+\varphi_j(s_j^{-1}(y',0))f(s_j^{-1}(y',0)). \tag{57.19}$$

Here $\sum_j'$ is the summation over j such that $U_j \cap \partial\Omega \neq \emptyset$. We have

$$A(x,D)Rf = T_{-N}f, \tag{57.20}$$

$$Rf|_{\partial\Omega} = f + T_{-N}^{(1)}f, \tag{57.21}$$

where $\mathrm{ord}\ T_{-N} \leq -N$, $\mathrm{ord}\ T_{-N}^{(1)} \leq -N$.

Denote by v the solution of Dirichlet problem

$$A(x,D)v = -T_{-N}f, \tag{57.22}$$

$$v|_{\partial\Omega} = -T_N^{(1)}f. \tag{57.23}$$

Then (c.f. §54)

$$\|v\|_{s,\Omega} \leq C\|T_{-N}f\|_{s-2,\Omega} + C[T^{(1)}f]_{s-\frac{1}{2},\partial\Omega} \leq C[f]_{s-N,\partial\Omega}. \tag{57.24}$$

Therefore $Rf + v$ satisfies

$$A(Rf + v) = 0 \quad \text{in} \quad \Omega \tag{57.25}$$

$$(Rf + v)|_{\partial\Omega} = f. \tag{57.26}$$

Substituting $u = Rf + v$ into (57.3) we get

$$\Lambda f = \sum_j{}' p'\psi_j \Lambda_j p'\varphi_j f + \Lambda_{-N} f, \tag{57.27}$$

where ord $\Lambda_{-N} \leq -N$ and Λ_j are the same as in (57.9). □

We can use the parametrix of the D-to-N operator to solve the inverse problem of determining the metric tensor on the boundary.

Theorem 57.1. *Suppose we know Λ on $\partial\Omega$ for all $f \in C^\infty(\partial\Omega)$. Then we can determine the metric on $\partial\Omega$ induced by (56.1) and all its normal derivatives at $\partial\Omega$.*

Proof: It follows from (57.14) that we can determine $\sum_{p,r=1}^{n-1} g_j^{pr}(y', 0)\xi_p\xi_r$, i.e. we know the metric on $\partial\Omega$. Therefore we know all deivatives of $g_j^{pr}(y', 0)$ in y'. It remains to determine derivatives $\frac{\partial^k g^{pr}(y',0)}{\partial y_n^k}$, $1 \leq p, r, \leq n - 1$, for all $k \geq 1$. It follows from (57.16) that the only term in T_1 that contains a derivative $\frac{\partial}{\partial y_n}$ is $T_{11} = -\frac{1}{2g_j}\frac{\partial g_j}{\partial y_n} - 2\xi_n(\xi_n - i\tilde{\sigma}_j)^{-2}\tilde{\sigma}_{jy_n}$. Note that $\sigma_{jy_n} = \frac{1}{2\sigma_j}(\sum_{p,r=1}^{n-1} g_{jy_n}^{pr}\xi_p\xi_r)$. Computing the residue of $\tilde{A}_-^{-1}T_{11}$ we can recover $\frac{g_{jy_n}}{g_j}$ and σ_{jy_n}. Analogously, from $\Lambda_{jp}, p \geq r$, we can recover $\frac{\partial^p \sigma_j}{\partial y_n^p}$ and therefore we can find all $\frac{\partial^p g_j^{pr}(y',0)}{\partial y_n^p}$, $p \geq 1$.

58 Spectral theory of elliptic operators.

Let $A(x, D)$ be an elliptic oparator of order m, $A_0(x, \xi) > 0$, $\forall x \in \overline{\Omega}, \forall \xi \neq 0$. Consider the boundary value problem in Ω

$$(A(x, D) + \lambda)u(x) = f(x), \quad x \in \Omega, \tag{58.1}$$

$$B_j(x, D)u|_{\partial\Omega} = g_j(x'), \quad 1 \leq j \leq m_+, \tag{58.2}$$

where ord $B_j = m_j$, $1 \leq j \leq m_+$. We assume that $B_j(x, D)$, $1 \leq j \leq m_+$, satisfy the condition (53.12) for all $x \in \partial\Omega$. Then Theorem 54.3 implies that the left hand sides of (58.1), (58.2) define a Fredholm operator $\mathcal{A}+\lambda$ from $H_s(\Omega)$ to $\mathcal{H}_{(s)} = H_{s-m}(\Omega)\times\Pi_{j=1}^{m-1}H_{s-m_j-\frac{1}{2}}(\partial\Omega)$ assuming that $s\geq m_0$ where $m_0=\max_{1\leq j\leq m_+}(m_j+\frac{1}{2})$.

Let $C_\delta=\{\lambda\in\mathbf{C}\setminus\{0\},\ -\pi+\delta<\arg\lambda<\pi-\delta,\ 0<\delta<\frac{\pi}{2}\}$ (c.f. (41.1)). As in §41 we can prove a stronger result:

Theorem 58.1. *Assume that the condition (53.12) is satisfied for all $x\in\partial\Omega$. If $\lambda\in C_\delta,\ |\lambda|\geq\lambda_0$, where λ_0 is large then $\mathcal{A}+\lambda$ is an invertible operator from $H_s(\Omega)$ to $\mathcal{H}_{(s)},\ s>m_0$.*

Proof: Consider the boundary value problem (53.2), (53.3), replacing $\hat{B}_{j0}$ by B_{j0} and $\hat{A}_0$ by $\mathcal{A}_0+\lambda$. It fillows from the explicit formulas (53.9), (53.13) , where $A_0+\lambda=A_-A_+$ is the factorization of $A_0+\lambda$, that the inverse operator $R_0(\lambda)$ exists and it is analytic in λ for $\lambda\in C_\delta$.

As in §41 we have that operators T and T_1 in (54.4) and (54.8) satisfy the estimates

$$|||TR_0|||\leq\frac{C}{|\lambda|^{\frac{1}{m}}},\quad |||T_1R^{(0)}|||\leq\frac{C}{|\lambda|^{\frac{1}{m}}}, \tag{58.3}$$

i.e. have a small norm when $\lambda\in C_\delta,|\lambda|\geq\lambda_0$. Therefore if R and $R^{(1)}$ are given by (54.11), (54.12) then $(\mathcal{A}+\lambda)R=I+T,\ R^{(1)}(\mathcal{A}+\lambda)=I+T^{(1)}$, where $T,\ T^{(1)}$ have a small norm and are analytic when $\lambda\in C_\delta,\ |\lambda|\geq\lambda_0$. Therefore $\mathcal{A}+\lambda$ has an inverse. □

Consider now the case of homogeneous boundary conditions (58.2), i.e when $g_j=0, 1\leq j\leq m_+$.

Denote by D_B the subspace of $H_s(\Omega)$ consisting of $u(x)\in H_s(\Omega)$ such that $B_ju|_{\partial\Omega}=0,\ 1\leq j\leq m_+$. Denote by A_B the operator A on the domain D_B. Since $A_B+\lambda$ are Fredholm operators depending analyticly on λ and invertible (c.f. Theorem 58.1) when $\lambda\in C_\delta,\ |\lambda|\geq\lambda_0$, there exists a sequence of eigenvalues $\lambda_j,\ \Re\lambda_j\to-\infty$ and a sequence $\{\varphi_j\}$ of eigenvector $\varphi_j\in D_B$ such that

$$(A+\lambda_j)\varphi_j=0,\quad 1\leq j\leq+\infty.$$

For the simplicity of notation we will write eigenvalues λ_j taking in account their multiplicity. Since A_B is not assumed to be self-adjoint for each λ_j there is a finite-dimentional subspace of root vectors corresponding to φ_j:

$$\varphi_{jr},\quad 1\leq r\leq k_j,\quad \varphi_{jk_j}=\varphi_j,\ \text{such that}$$

$$(A_B+\lambda_j)\varphi_{j1}=\varphi_{j2},...,(A_B+\lambda_j)\varphi_{j,k_j-1}=\varphi_{jk_j},$$
$$(A_B+\lambda_j)\varphi_{jk}=0,$$
i.e we have a $k_j\times k_j$ Jordan block.

Denote by P_j the projector on the span of $(\varphi_{j1},...,\varphi_{jk_j})$, i.e. $P_j^2=P_j$, $P_j\varphi_{jk}=\varphi_{jk}$, $1\le k\le k_j$. We have $P_ju_0=\sum_{k=1}^{n_j}(u_0,\psi_{jk})\varphi_{jk}(x)$ where ψ_{jr} belong to the dual space $H_s^*(\Omega)$, $(\varphi_{jk},\psi_{ir})=0$ when $(j,k)\neq(i,r)$, $(\varphi_{jk},\psi_{jk})=1$. Therefore $(A_B+\lambda)^{-1}$ is a meromorphic operator-function, $\lambda\in\mathbf{C}$, and the singular part of $(A_B+\lambda)^{-1}u_0$ corresponding to φ_j has the form

$$\sum_{k=1}^{k_j}(\lambda-\lambda_j)^{-k}P_{jk}u_0, \tag{58.4}$$

where

$$P_{jk}u_0=\sum_{r=1}^{k_j-k+1}(u_0,\psi_{j,r+k-1})\varphi_{jr},\quad P_{j1}u_0=P_ju_0. \tag{58.5}$$

Note that there is only a finite number of poles in any strip $-N<\Re\lambda<\lambda_0,\ \forall N$.

Consider the initial-boundary value problem

$$\frac{\partial u(x,t)}{\partial t}+A(x,D)u=0,\quad t>0,\ x\in\Omega, \tag{58.6}$$
$$B_j(x,D)u(x,t)|_{\partial\Omega\times(0,+\infty)}=0,$$
$$u(x,0)=u_0(x),\quad x\in\Omega.$$

Performing the Fourier-Laplace transform as in §47 we get (c.f. (47.6)):

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(A_B+i(\sigma-i\tau))^{-1}u_0(x)e^{it(\sigma-i\tau)}d\sigma, \tag{58.7}$$

where $\tau\ge\lambda_0$ is arbitrary. Since $(A_B+i(\sigma-i\tau))^{-1}$ is a meromorphic operator-function of $\lambda=i(\sigma-i\tau)$ and since $\|(A_B+i(\sigma-i\tau)^{-1}\|\le\frac{C}{|\sigma-i\tau|}$ when $|\sigma-i\tau|\ge\lambda_0,\ |\sigma|\to\infty$, we get by the Cauchy integral theorem

$$u(x,t)=\sum_{\Re\lambda_j>-N}\sum_{k=1}^{k_j}\frac{t^{k-1}}{(k-1)!}e^{t\lambda_j}P_{kj}u_0 \tag{58.8}$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}(A+i(\sigma-i\lambda_0+iN))^{-1}u_0(x)e^{it(\sigma-i\lambda_0+iN)}d\sigma$$

We assume that $(A+i(\sigma-i\lambda_0+iN))^{-1}$ has no poles on the line $\sigma-i\lambda_0+iN,\ -\infty<\sigma<+\infty$.

Since the integral in the right hand side of (58.8) is $O(e^{-tN})$ we get, taking the limit when $N\to\infty$:

$$u(x,t)=\sum_{j=1}^{\infty}\sum_{k=1}^{k_j}\frac{t^{k-1}}{(k-1)!}e^{t\lambda_j}P_{kj}u_0(x).$$

As in §46 one can prove that $u(x,t)\to u_0(x)$ when $t\to 0$ in the norm of $H_{s-\frac{m}{2}}(\Omega)$ assuming that $s>m_0, s>m_+=\frac{m}{2}$, and $u_0(x)\in D_B$.

Let $G_B(x,y,t)$ be the heat kernel corresponding to the initial-boundary value problem (58.6), i.e.

$$\text{(58.9)}\qquad \begin{aligned} \frac{\partial G_B(x,y,t)}{\partial t}+A(x,D)G_B(x,y,t)=0,\quad t>0,\ x\in\Omega,\ y\in\Omega,\\ B_j(x,D)G_B(x,y,t)|_{\partial\Omega\times(0,+\infty)}=0,\quad 1\le j\le m_+,\\ G_B(x,y,0)=\delta(x-y).\end{aligned}$$

We shall find the trace $\int_\Omega G_B(x,x,t)dx$. It follows from (58.5) that the trace of $P_j=P_{j1}$ is equal to k_j and the traces of $P_{jk},\ 2\le k\le k_j$, are zero. Therefore

$$\text{(58.10)}\qquad \int_\Omega G(x,x,t)dx=\sum_{j=1}^{\infty}k_je^{\lambda_j t}.$$

Consider a particular case when A_B is self-adjoint with respect to some inner product $(u,v)_\sigma=\int_\Omega u(x)\overline{u(x)}\sigma(x)dx$. We have

$$(Au,v)_\sigma=(u,Av)_\sigma$$

for all $u\in D_B,\ v\in D_B$. Suppose A_B is invertible. Consider $s=m$. Then A_B^{-1} is compact in $L_2(\sigma,\Omega)$ and is self-adjoint:

$$\text{(58.11)}\qquad (A_B^{-1}f,g)_\sigma=(f,A_B^{-1}g)_\sigma,\ \forall f,g\in L_2(\sigma,\Omega).$$

Therefore there exists an orthonormal basis in $L_2(\sigma,\Omega)$ consisting of the eigenfunctions $\varphi_j(x)$

$$A_n^{-1}\varphi_j=\mu_j\varphi_j,\quad \mu_j\to 0,$$

μ_j are real valued. Therefore $(A_B + \lambda_j)\varphi_j = 0$, where $\lambda_j = -\frac{1}{\mu_j}$. Compairing with (58.10) we have that $k_j = 1$ (i.e. the Jordan blocks are 1×1), and the trace formula (58.10) has the form

$$\int_\Omega G_B(x,x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t}, \tag{58.12}$$

Consider, for example, the Laplace-Beltrami operator (56.1). Then combining (58.12) and (56.26) we get

$$\sum_{j=1}^{\infty} e^{\lambda_j t} = \frac{1}{(\sqrt{2\pi})^n} \frac{1}{t^{\frac{n}{2}}} \text{Vol } (\Omega) - \frac{1}{2^{n+1}\pi^{\frac{n-1}{2}}} \frac{1}{t^{\frac{n-1}{2}}} \text{Vol } (\partial\Omega) + O(\frac{1}{t^{\frac{n}{2}-1}}). \tag{58.13}$$

Remark 58.1 The parametrix construction of §55 can be carried out for the heat kernel $G_B(x,y,t)$. Note that the principal term will be equal to (c.f. §47)

$$\frac{1}{(2\pi)^n} t^{-\frac{n}{m}} \int_\Omega \int_{\mathbf{R}^n} e^{-A_0(y,\eta)} dy d\eta. \tag{58.14}$$

One can show (c.f. §56) that contribution of the boundary will be $O(\frac{1}{t^{\frac{n-1}{m}}})$. Therefore we get

$$\sum_{j=1}^{\infty} k_j e^{t\lambda_j} = \sum_{k=0}^{N} \frac{1}{t^{\frac{n}{m}}} c_k t^{\frac{k}{m}} + O(t^{\frac{N-n+1}{m}}), \tag{58.15}$$

where the principal term $\frac{c_0}{t^{\frac{n}{m}}}$ has the form (58.14).

Remark 58.2 Consider the elliptic operator on a compact manifold M, $\partial M = \emptyset$, of the form

$$(A + \lambda I)u(x) = f(x), \quad x \in M, \tag{58.16}$$

where A is a pseudodifferential operator on M (c.f. §52) with the principal symbol $A_0(x, \xi_x) > 0$, $A_0(x, \xi_x)$ is a homogeneous in ξ_x of degree $m > 0$. As in Theorem 58.1 we can prove that $A + \lambda I$ is an invertible operator from $H_s(M)$ to $H_{s-m}(M)$ when $\lambda \in C_\delta, |\lambda| \geq \lambda_0$. Consider the parabolic equation

$$\frac{\partial u(x,t)}{\partial t} + Au(x,t) = 0, \quad t > 0, \; x \in M, \tag{58.17}$$

$$u(x,0) = u_0(x), \quad x \in M. \tag{58.18}$$

We have

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(A + i(\sigma - i\tau))^{-1}u_0(x)e^{it(\sigma-i\tau)}d\sigma, \tag{58.19}$$

where $\tau \geq \lambda_0$. Analogously to (58.8), (58.9) we get that $u(x,t)$ has the expansion (58.8), where $P_{jr}u_0$ are finite rank projectors in $H_{s-\frac{m}{2}}(M)$ and $u(x,t) \in C(\mathbf{R}, H_{s-\frac{m}{2}}(M))$, i.e. $u(x,t)$ depends continuously on t with values in $H_{s-\frac{m}{2}}(M)$. In particular, $\|u(x,t) - u_0\|_{s-\frac{m}{2},M} \to 0$ when $t \to +0$.

We define the heat kernel of the Cauchy problem (58.17), (58.18) as the kernel of the operator $u_0(x) \to u(x,t)$, i.e.

$$u(x,t) = \int_M G(x,y,t)u_0(y)dy.$$

Note that the heat kernel in the case of a manifold is $G(x,y,t)dy$, i.e. it is a differential of the order n in y.

The parametrix R_j of $(A + \lambda I)^{-1}$ in local system of coordinates in U_j is constructed as in §41. The global parametrix $R(\lambda)$ is given by the formula of the form (54.11). Substituting the parametrix $R(\lambda)$ in (58.19) we get, as in §56

$$\sum_{j=1}^{\infty} k_j e^{\lambda_j t} = \frac{1}{t^{\frac{n}{m}}}\left(\sum_{k=0}^{N} c_k t^{\frac{k}{m}} + O(t^{\frac{N+1}{m}})\right), \tag{58.20}$$

where

$$c_0 = \frac{1}{(2\pi)^n}\int_{T_0^*(M)} e^{-A_0(x,\xi)}dxd\xi,$$

$dx \wedge d\xi = dx_1 \wedge ... \wedge dx_n \wedge d\xi_1 \wedge ... \wedge d\xi_n$ is the differential form on $T_0^*(M)$.