Part I
Pseudodifferential operators.

40 Boundness and composition of pseudodifferential operators.

Denote by S^α the class of $C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ functions $A(x,\xi)$ such that

$$\left|\frac{\partial^{k+p}A(x,\xi)}{\partial x^p \partial \xi^k}\right| \leq C_{pk}(1+|\xi|)^{\alpha-|k|}, \quad \forall\ k,p, \tag{40.1}$$

and $A(x,\xi)$ is independent of x for $|x| > R$. Let $A(\infty,\xi) = A(x,\xi)$ for $|x| > R$ and $A'(x,\xi) = A(x,\xi) - A(\infty,\xi)$, i.e. $A'(x,\xi)$ has a compact support in x.

For $A(x,\xi) \in S^\alpha$ we define an operator

$$Au = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} A(x,\xi)\tilde{u}(\xi)e^{ix\cdot\xi}d\xi, \quad \forall\ C_0^\infty(\mathbf{R}^n), \tag{40.2}$$

where $\tilde{u}(\xi)$ is the Fourier transform of $u(x)$. If $A(x,\xi) = \sum_{|k|=0}^m a_k(x)\xi^k$ is a polynomial in ξ then

$$Au = \sum_{|k|=0}^m a_k(x)(-i\frac{\partial}{\partial x})^k u \tag{40.3}$$

is a differental operator $A(x,D)u,\ D = -i\frac{\partial}{\partial x}$. We shall call (40.2) the pseudodifferential operator and $A(x,\xi)$ its symbol. We shall often denote by $A(x,D)$ the pseudodifferential operator (ψdo) with symbol $A(x,\xi)$.

Theorem 40.1 *Pseudodifferential operator $A(x,D)$ with symbol $A(x,\xi) \in S^\alpha$ is bounded from $H_s(\mathbf{R}^n)$ to $H_{s-\alpha}(\mathbf{R}^n)$ for all $s \in \mathbf{R}$:*

$$\|A(x,D)u\|_{s-\alpha} \leq C\|u\|_s, \quad \forall u \in H_s(\mathbf{R}^n). \tag{40.4}$$

Proof: It is clear that

$$\|A(\infty,D)u\|_{s-\alpha}^2 = \int_{\mathbf{R}^n} |A(\infty,\xi)\tilde{u}(\xi)|^2(1+|\xi|)^{2(s-\alpha)}d\xi \leq C_0^2\|u\|_s^2,$$

where $C_0 = \sup_\xi \frac{|A(\infty,\xi)|}{(1+|\xi|)^\alpha}$, $\forall u \in S$.

Let

$$\tilde{A}'(\eta,\xi) = \int_{\mathbf{R}^n} A'(x,\xi)e^{-ix\cdot\eta}dx. \tag{40.5}$$

Taking the Fourier transform of $v(x) = A'(x,d)u$ we get

$$\tilde{v}(\eta) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \tilde{A}'(\eta-\xi,\xi)\tilde{u}(\xi)d\xi. \tag{40.6}$$

Since $A'(x,\xi) \in C_0^\infty$ in x we get by the integration by parts in (40.5):

$$|\tilde{A}(\eta,\xi)| \le C_N(1+|\eta|)^{-N}(1+|\xi|)^\alpha, \quad \forall N.$$

Therefore

$$|\tilde{v}(\eta)| \le C_N \int_{\mathbf{R}^n} \frac{(1+|\xi|)^\alpha|\tilde{u}(\xi)|}{(1+|\xi-\eta|)^N}d\xi.$$

It follows from (13.9) that

$$(1+|\xi-\eta|)^{-|t|} \le C_t\frac{(1+|\xi|)^t}{(1+|\eta|)^t}, \quad \forall t \in \mathbf{R}. \tag{40.7}$$

Taking $N \ge n+1+|s-\alpha|$ and using (40.7), $t = s-\alpha$, we get:

$$(1+|\eta|)^{s-\alpha}|\tilde{v}(\eta)| \le C\int_{\mathbf{R}^n} \frac{(1+|\xi|)^s|\tilde{u}(\xi)|}{(1+|\xi-\eta|)^{n+1}}d\xi.$$

Now as in (13.10), (13.11) we get

$$\|v\|_{s-\alpha} \le C\|u\|_s, \quad \forall u \in C_0^\infty(\mathbf{R}^n).$$

Since $C_0^\infty(\mathbf{R}^n)$ is dense in $H_s(\mathbf{R}^n)$ (see Theorem 13.2), we can take a closure in $H_s(\mathbf{R}^n)$ and get (40.4).

Definition 40.1 *We say that operator T_α is an operator of order $\le \alpha$ if*

$$\|T_\alpha u\|_s \le C\|u\|_{s+\alpha} \ \ \textit{for any} \ \ s. \tag{40.8}$$

It follows from Theorem 40.1 that $A(x,D)$ with symbol $A(x,\xi) \in S^\alpha$ has order α.

Theorem 40.2 *Let $A(x,\xi)\in S^{\alpha},\ B(x,\xi)\in S^{\beta}$. Then for any N the composition $A(x,D)B(x,D)$ has the form*

$$A(x,D)B(x,D)=C_N(x,D)+T_{\alpha+\beta-N-1}, \tag{40.9}$$

where $C_N(x,\xi)\in S^{\alpha+\beta}$, ord $T_{\alpha+\beta-N-1}\le\alpha+\beta-N-1$ and

$$C_N(x,\xi)=\sum_{|k|=0}^{N}\frac{1}{k!}\frac{\partial^k A(x,\xi)}{\partial\xi^k}D_x^k B(x,\xi),\ \ D_x=-i\frac{\partial}{\partial x}. \tag{40.10}$$

Proof: We have that $A(x,D)B(\infty,D)$ is a ψdo operator with symbol $A(x,\xi)B(\infty,\xi)$. Denote $v(x)=B'(x,D)u=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}B'(x,\eta)\tilde{u}(\eta)e^{ix\cdot\eta}d\eta$. As in (40.6) we have:

$$A(x,D)v=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}A(x,\xi)\tilde{v}(\xi)e^{ix\cdot\xi}d\xi$$

$$=\frac{1}{(2\pi)^{2n}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}A(x,\xi)\tilde{B}'(\xi-\eta,\eta)e^{ix\cdot\xi}\tilde{u}(\eta)d\xi d\eta,\quad\forall u\in C_0^{\infty}(\mathbf{R}^n).$$

Make change of variables $\xi-\eta=\zeta$. Then

$$A(x,D)v=\frac{1}{(2\pi)^{2n}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}A(x,\eta+\zeta)\tilde{B}'(\zeta,\eta)e^{ix\cdot(\eta+\zeta)}\tilde{u}(\eta)d\eta d\zeta.$$

Note that $\tilde{B}'(\zeta,\xi)$ is rapidly decreasing in ζ and $\tilde{u}(\eta)$ is rapidly decreasing in η. Expand $A(x,\eta+\zeta)$ by the Taylor formula in ζ:

$$A(x,\eta+\zeta)=\sum_{|k|=0}^{N}\frac{1}{k!}\frac{\partial^k A(x,\eta)}{\partial\eta^k}\zeta^k+R_N(x,\eta,\zeta), \tag{40.11}$$

where $R_N(x,\eta,\zeta)$ is the remainder. Note that

$$\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\zeta^k\tilde{B}'(\zeta,\eta)e^{ix\cdot\zeta}d\zeta=D_x^kB'(x,\eta).$$

Therefore

$$A(x,D)v=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\sum_{|k|=0}^{N}\frac{1}{k!}\frac{\partial^k A(x,\eta)}{\partial\eta^k}D_{x'}^kB'(x,\eta)\tilde{u}(\eta)e^{ix\cdot\eta}d\eta+T_{\alpha+\beta-N-1}u,$$

where $T_{\alpha+\beta-N-1}u = \frac{1}{(2\pi)^{2n}} \int_{\mathbf{R}^n} R_N(x,\eta,\zeta)\tilde{B}'(\zeta,\eta)e^{ix\cdot(\eta+\zeta)}\tilde{u}(\eta)d\eta d\zeta$. It remains to prove that $T_{\alpha+\beta-N-1}$ has order $\alpha+\beta-N-1$. We have

$$R_N(x,\eta,\zeta) = R_N(\infty,\eta,\zeta) + R'_N(x,\eta,\zeta), \tag{40.12}$$

where $R_N(\infty,\eta,\zeta)$ is the remainder for $A(\infty,\xi)$ and $R'_N(x,\eta,\zeta)$ is the remainder for $A'(x,\xi)$.

Let $\tilde{R}'_N(\xi,\eta,\zeta)$ be the Fourier transform of $R'_N(x,\eta,\zeta)$ in x. We shall prove the following estimates for $R_N(\infty,\eta,\zeta)$ and $\tilde{R}'_N(\xi,\eta,\zeta)$:

$$|R_N(\infty,\eta,\zeta)| \leq C(1+|\eta|)^{\alpha-N-1}|\zeta|^{N+|\alpha|+1}, \tag{40.13}$$

$$|\tilde{R}'_N(\xi,\eta,\zeta)| \leq C_M(1+|\xi|)^{-M}(1+|\eta|)^{\alpha-N-1}|\zeta|^{N+|\alpha|+1}, \quad \forall M. \tag{40.14}$$

To prove (40.13) consider two cases: $|\zeta| \leq \frac{1}{2}(1+|\eta|)$ and $|\zeta| > \frac{1}{2}(1+|\eta|)$. When $|\zeta| \leq \frac{1}{2}(1+|\eta|)$ we use the Lagrange form of the remainder in the Taylor's formula:

$$R_N(\infty,\eta,\zeta) = \sum_{|k|=N+1} \frac{1}{k!}\frac{\partial^k A(\infty,\eta+\theta\zeta)}{\partial\eta^k}\zeta^k,$$

where $|\theta| < 1$. Then

$$\left|\frac{\partial^k A(\infty,\eta+\theta\zeta)}{\partial\eta^k}\right| \leq C(1+|\eta+\theta\zeta|)^{\alpha-N-1} \leq C_1(1+|\eta|)^{\alpha-N-1}, \tag{40.15}$$

since $1+|\eta+\theta\zeta| \geq 1+|\eta|-|\zeta| \geq \frac{1}{2}(1+|\eta|)$ and $(1+|\eta+\theta\zeta|) \leq 1+|\eta|+|\zeta| \leq \frac{3}{2}(1+|\eta|)$. Therefore (40.15) holds.

If $|\zeta| > \frac{1}{2}(1+|\eta|)$ we get from (40.11):

$$|R_N(\infty,\eta,\zeta)| \leq |A(\infty,\eta+\zeta)| + \sum_{|k|=0}^{N} \frac{1}{k!}\left|\frac{\partial^k A(\infty,\eta)}{\partial\eta^k}\right| |\zeta|^{|k|}$$

$$\leq C(1+|\eta+\zeta|)^{\alpha} + C\sum_{|k|=0}^{N}(1+|\eta|)^{\alpha-|k|}|\zeta|^{|k|}.$$

Note that

$$(1+|\eta|)^{\alpha-|k|}|\zeta|^k \leq (1+|\eta|)^{\alpha-N-1}(1+|\eta|)^{N-|k|+1}|\zeta|^{|k|} \leq C(1+|\eta|)^{\alpha-N-1}|\zeta|^{N+1}$$

since $(1+|\eta|)^{N-|k|+1} \leq C|\zeta|^{N-|k|+1}$. Using (40.7) with $t=\alpha$ we get

$$(1+|\eta+\zeta|)^\alpha \leq C(1+|\eta|)^\alpha|\zeta|^{|\alpha|} \leq C(1+|\eta|)^{\alpha-N-1}|\zeta|^{|\alpha|+N+1}$$

since $|\zeta| \geq \frac{1}{2}(1+|\eta|)$. Therefore (40.13) holds.

The proof of (40.14) is similar. One should replace the estimate $|\frac{\partial^k A(\infty,\eta)}{\partial \eta^k}| \leq C(1+|\eta|)^{\alpha-|k|}$ by the estimate $|\frac{\partial^k \tilde{A}(\xi,\eta)}{\partial \eta^k}| \leq C_M(1+|\xi|)^{-M}(1+|\eta|)^{\alpha-|k|}, \ \forall M$.

Let

$$T_{\alpha+\beta-N-1} = T^{(1)}_{\alpha+\beta-N-1} + T^{(2)}_{\alpha+\beta-N-1},$$

where

$$(40.16)\quad T^{(1)}_{\alpha+\beta-N-1}u = \frac{1}{(2\pi)^{2n}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} R_N(\infty,\eta,\zeta)\tilde{B}'(\zeta,\eta)e^{ix\cdot(\eta+\zeta)}\tilde{u}(\eta)d\eta d\zeta,$$

$$(40.17)\quad T^{(2)}_{\alpha+\beta-N-1}u = \frac{1}{(2\pi)^{2n}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} R'_N(x,\eta,\zeta)\tilde{B}'(\zeta,\eta)e^{ix\cdot(\eta+\zeta)}\tilde{u}(\eta)d\eta d\zeta.$$

Since $(F_x T^{(1)}_{\alpha+\beta-N-1}u)(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} R_N(\infty,\eta,\xi-\eta)\tilde{B}'(\xi-\eta,\eta)\tilde{u}(\eta)d\eta$ and

$$(40.18)\qquad |\tilde{B}'(\xi-\eta,\eta)| \leq C_M(1+|\xi-\eta|)^{-M}(1+|\eta|)^\beta$$

we get, using (40.13):

$$|(1+|\xi|)^s|(F_x T^{(1)}_{\alpha+\beta-N-1}u)(\xi)| \leq C_M\int_{\mathbf{R}^n}\frac{(1+|\xi|)^s|\xi-\eta|^{N+|\alpha|+1}(1+|\eta|)^{\alpha+\beta-N-1}|\tilde{u}(\eta)|}{(1+|\xi-\eta|)^M}d\eta.$$

(40.19)

Taking $M > N+1+|\alpha|+|s|+n+1$ and using (40.7) with $t=s$ we get

$$(40.20)\quad |(1+|\xi|)^s|(F_x T^{(1)}_{\alpha+\beta-N-1}u)(\xi)| \leq C\int_{\mathbf{R}^n}\frac{(1+|\eta|)^{s+\alpha+\beta-N-1}|\tilde{u}(\eta)|}{(1+|\xi-\eta|)^{n+1}}d\eta.$$

Applying arguments in (13.10), (13.11) to (40.20) we obtain:

$$\|T^{(1)}_{\alpha+\beta-N-1}u\|_s \leq C\|u\|_{s+\alpha+\beta-N-1}.$$

Analogously, we have

$$(F_x T^{(2)}_{\alpha+\beta-N-1}u)(\xi) = \frac{1}{(2\pi)^{2n}}\int_{\mathbf{R}^n}\tilde{R}'_N(\xi-\eta-\zeta,\eta,\zeta)\tilde{B}'(\zeta,\eta)\tilde{u}d\eta d\zeta.$$

For any M_1 we have (c.f. (40.7))

$$(1+|\xi-\eta-\zeta|)^{-M_1} \leq C_{M_1}(1+|\xi-\eta|)^{-M_1}(1+|\zeta|)^{M_1}. \tag{40.21}$$

Taking $M \geq M_1 + N + 1 + |\alpha|$ in (40.18) and using (40.21) we get:

$$|(1+|\xi|)^s|(F_x T^{(2)}_{\alpha+\beta-N-1}u)(\xi)| \leq C\int_{\mathbf{R}^n} \frac{(1+|\xi|)^s(1+|\eta|)^{\alpha+\beta-N-1}|\tilde{u}(\eta)|}{(1+|\xi-\eta|)^{M_1}}d\eta. \tag{40.22}$$

Treating (40.22) as in (13.10), (13.11) we get

$$\|T^{(2)}_{\alpha+\beta-N-1}u\|_s \leq C\|u\|_{\alpha+\beta-N-1+s}.$$

Therefore $T_{\alpha+\beta-N-1}$ is an operator of order $\leq \alpha+\beta-N-1$.

We shall often call the operator of a negative order a smoothing operator.

Corollary 40.3 *Let $A(x,\xi) \in S^\alpha$, $B(x,\xi) \in S^\beta$. Then the commutator $[A,B] = A(x,D)B(x,D) - B(x,D)A(x,D)$ has the form $\sum_{k=1}^N C_k(x,D) + T_{\alpha+\beta-N-1}$, where ord $T_{\alpha+\beta-N-1} \leq \alpha+\beta-N-1$, $C_k(x,\xi) \in S^{\alpha+\beta-k}$,*

$$C_1(x,\xi) = -i\sum_{j=1}^n \left(\frac{\partial A(x,\xi)}{\partial \xi_j}\frac{\partial B(x,\xi)}{\partial x_k} - \frac{\partial A(x,\xi)}{\partial x_j}\frac{\partial B(x,\xi)}{\partial \xi_j}\right),$$

i.e.

$$iC_1(x,\xi) = \{A(x,\xi), B(x,\xi)\} \tag{40.23}$$

is the Poisson bracket of $A(x,\xi)$ and $B(x,\xi)$.

The proof of the corollary follows immediatly from the application of Theorem 40.2 to $A(x,D)B(x,D)$ and $B(x,D)A(x,D)$.

Remark 40.1. Let $|||A(x,D)|||_{(s)}$ be the norm of an operator $A(x,D)$ acting from $H_{s-\alpha}(\mathbf{R}^n)$ to $H_s(\mathbf{R}^n)$. It follows from the proof of Theorem 40.1 that

$$|||A(x,D)|||_{(s)} \leq \sup_\xi \frac{|A(\infty,\xi)|}{(1+|\xi|)^\alpha} + C\sup_{\xi,\eta}\frac{(1+|\eta|)^N|\tilde{A}'(\eta,\xi)|}{(1+|\xi|)^\alpha}, \tag{40.24}$$

where $N \geq n+1+|s-\alpha|$. Note that

$$(1+|\eta|)^N|\tilde{A}'(\eta,\xi)| \leq \sum_{|k|=0}^N \int_{\mathbf{R}^n}|D_x^k A'(x,\xi)|dx. \tag{40.25}$$

The estimate (10.25) follows from the equality

$$\eta^k \tilde{A}'(\eta,\xi) = \int_{\mathbf{R}^n} (D_x^k A'(x,\xi)) e^{-ix\cdot\eta} d\eta.$$

Remark 40.2 We will use later the following property of the remainder $T\alpha+\beta-N-1$ in (40.9): $(1+|x|^2)^M T_{\alpha+\beta-N-1}$ is an operator of order $\leq \alpha+\beta-N-1$ for any M, i.e.

$$\|(1+|x|^2)^M T_{\alpha+\beta-N-1} u\|_s \leq C_M \|u\|_{s+\alpha+\beta-N-1}, \quad \forall s. \tag{40.26}$$

Proof: It follows from (40.17) that $T^{(2)}_{\alpha+\beta-N-1} = 0$ when $|x| > R$ since $R'_N(x,\eta,\zeta) = 0$ when $|x| > R$. Consider now $T^{(1)}_{\alpha+\beta-N-1}$ (see (40.16)). Integrating by parts in ζ in (40.16) we get

$$(1+|x|^2)^M T^{(1)}_{\alpha+\beta-N-1} u = \frac{1}{(2\pi)^{2n}} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} (1-\Delta_\zeta)^M (R'_N(\infty,\eta,\zeta)\tilde{B}'(\zeta,\eta))\cdot$$

$$\cdot e^{ix\cdot(\xi+\eta)} \tilde{u}(\eta) d\eta d\zeta.$$

We have

$$|(1-\Delta_\zeta)^M (R'_N(\infty,\eta,\zeta)\tilde{B}'(\zeta,\eta)| \leq C_{M_1} \frac{(1+|\eta|)^{\alpha+\beta-N-1}}{(1+|\zeta)^{M_1}}, \quad \forall M_1.$$

Therefore as in (40.19), (40.20) we get

$$\|(1+|x|^2)^M T^{(1)}_{\alpha+\beta-N-1} u\|_s \leq C\|u\|_{\alpha+\beta-N-1+s}.$$