Problem 1: Using the limit definition of the derivative, prove that the function

\[ f(x) = \begin{cases} 
  x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0,
\end{cases} \]

is differentiable at \( x = 0 \).

Solution: Consider the quotient in the limit definition of the derivative:

\[ \frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin \left( \frac{1}{x} \right)}{x} = x \sin \left( \frac{1}{x} \right) \]

This is a continuous function at 0. So if we take the limit as \( x \to 0 \) we get that the limit exists and is equal to 0. That is, \( f'(0) = 0 \).

Problem 2: Suppose that \( f \) is defined on an open interval \( I \) containing \( a \). Prove that \( f'(a) \) exists if and only if there is a function \( \epsilon : I \to \mathbb{R} \) such that

\[ f(x) - f(a) = (x - a)[f'(a) + \epsilon(x)] \quad \text{and} \quad \lim_{x \to a} \epsilon(x) = 0. \]

Solution: Without loss of generality, \( a = 0 \) and \( f(a) = f(0) = 0 \) (why?).

\( \Rightarrow \): We assume \( f'(a) \) is differentiable. So, define the function

\[ \epsilon(x) := \frac{f(x) - xf'(0)}{x}. \]

Then \( \epsilon \) is a function defined on \( I \) and satisfies the first equality mentioned in the problem. All that is left to show is that it has the correct limit relationship.

\[ \lim_{x \to 0} \epsilon(x) = \lim_{x \to 0} \frac{f(x)}{x} - f'(0) = f'(0) - f'(0) = 0. \]

\( \Leftarrow \): Now we assume that \( \epsilon(x) \) exists, \( \lim_{x \to 0} \epsilon(x) = 0 \) and

\[ f(x) = x(m + \epsilon(x)) \]

for some \( m \in \mathbb{R} \). Consider the limit definition of the derivative,

\[ \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} m + \epsilon(x) = m \]

because of the limit property of \( \epsilon \). So \( f \) is differentiable at 0 and \( f'(0) = m \).

Problem 3: Prove that

\[ |\cos x - \cos y| \leq |x - y| \]
for all \( x, y \in \mathbb{R} \).

**Solution:** Fix \( x, y \in \mathbb{R} \). Without loss of generality let \( x < y \). By the mean value theorem we know that there exists a \( c \in (x, y) \) such that
\[
\frac{\cos x - \cos y}{x - y} = -\sin c.
\]
So
\[
| \cos x - \cos y | = | -\sin c | | x - y | \leq | x - y |.
\]

**Problem 4:** Suppose that \( f: \mathbb{R} \rightarrow \mathbb{R} \) satisfies \( | f(x) - f(y) | \leq (x - y)^2 \) for all \( x, y \in \mathbb{R} \). Prove that \( f \) is a constant function.

**Solution:**
\[
0 \leq \left| \frac{f(x) - f(y)}{x - y} \right| \leq | x - y |
\]
So by the squeeze theorem and taking limits as \( y \rightarrow x \), we know that \( f'(x) \) exists for all \( x \in \mathbb{R} \) and \( f'(x) = 0 \). So \( f \) must be constant (why is a function with 0 derivative constant?).

**Problem 5:** Suppose that \( f \) and \( g \) are differentiable on \( \mathbb{R} \) and that \( f(0) = g(0) \) and \( f'(x) \leq g'(x) \) for all \( x \geq 0 \). Prove that \( f(x) \leq g(x) \) for all \( x \geq 0 \).

**Solution:** Let \( h = g - f \). Then \( h(0) = 0 \) and \( h'(x) \geq 0 \) for all \( x \geq 0 \). By the mean value theorem,
\[
h(x) = h(x) - h(0) = (x - 0)h'(c) \geq 0
\]
for \( x \geq 0 \). So \( f(x) \leq g(x) \).