Problem 1:
(a) Suppose that $f$ is uniformly continuous on a bounded set $S$. Prove that $f$ is bounded on $S$.

(b) Use (a) to prove that $f(x) = 1/x^2$ is not uniformly continuous on $(0, 1)$.

Solution:
(a) Suppose $f$ is unbounded. Then for all $n \in \mathbb{N}$ there exists $a_n \in S$ such that $|f(a_n)| \geq N$. Since $S$ is bounded, by the Bolzano-Weierstrass theorem there exists a subsequence $(a_n_k)$ that is a Cauchy sequence. Since $f$ is uniformly continuous, $f(a_n_k)$ will be a Cauchy sequence as well. Therefore it will be bounded. However, $|f(a_n_k)| \geq n_k$ so it is not bounded which is a contradiction.

(b) The function $1/x^2$ is unbounded on $(0, 1)$ so it cannot be uniformly continuous by (a).

Problem 2: Show that if $f: X \to \mathbb{R}$ is Lipschitz then it is uniformly continuous on $X$.

Solution: Fix $\epsilon > 0$. Choose $\delta = \epsilon/M$. If $|x - y| < \delta$, then

$$|f(x) - f(y)| \leq M|x - y| < M\delta = \epsilon$$

Since $\delta$ did not depend on $x$ or $y$ we see that $f$ is uniformly continuous.

Problem 3: Suppose that $f$ is continuous on $[0, \infty)$. Prove that if $f$ is uniformly continuous on $[k, \infty)$ for some $k \geq 0$ then $f$ is uniformly continuous on $[0, \infty)$.

Solution: Fix $\epsilon > 0$. The function $f$ is continuous on $[0, k+1]$ and therefore also uniformly continuous on $[0, k+1]$. So we can choose $\delta_1$ such that for all $x, y \in [0, k+1]$ with $|x - y| < \delta_1$ we have that $|f(x) - f(y)| < \epsilon$. Similarly we get a $\delta_2$ such that for all $x, y \in [k, \infty)$ with $|x - y| < \delta_2$ we have that $|f(x) - f(y)| < \epsilon$. Now let $\delta = \min\{\delta_1, \delta_2, 1\}$. Then if $|x - y| < \delta$ then $x$ and $y$ are either in $[0, k+1]$ or $[k, \infty)$ but not both. So we can apply the above to see that $|f(x) - f(y)| < \epsilon$.

Problem 4: Let $f(x) = \sqrt{x}$. Prove that $f$ is uniformly continuous on $[0, \infty)$.

Solution: By Problem 3 it suffices to show that $f$ is uniformly continuous on $[1, \infty)$.

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \left|\frac{x - y}{\sqrt{x} + \sqrt{y}}\right| \leq \frac{|x - y|}{2}.$$

So $f$ is Lipschitz on $[1, \infty)$ and by Problem 2 it is uniformly continuous.

Problem 6: Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that $f$ is differentiable at $x = 0$. 

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Solution: Fix $\epsilon > 0$. Let $\delta = \epsilon$. Then if $|x| < \delta$, and $x \in \mathbb{Q}$ then

$$\frac{|f(x)|}{|x|} = 0 < \epsilon.$$ 

If $x \notin \mathbb{Q}$ then

$$\frac{|f(x)|}{|x|} = |x| < \epsilon.$$ 

So $f$ is differentiable at 0.