Math 131A Homework 5 Solutions

Problem 1: Let \((a_n)\) and \((b_n)\) be sequences and suppose that there exists \(N \in \mathbb{N}\) such that \(a_n \leq b_n\) for all \(n \geq N\). Prove that \(\lim \sup a_n \leq \lim \sup b_n\).

Solution: Let \(\lim \sup a_n = A\). Then there exists a subsequence \((a_{n_k})\) such that \(\lim_{k \to \infty} a_{n_k} = A\). By our assumption, \(a_{n_k} \leq b_{n_k}\). So
\[
A = \lim_{k \to \infty} a_{n_k} \leq \lim \inf b_{n_k} \leq \lim \sup b_n.
\]

Problem 3: Prove that \((a_n)\) is bounded if and only if \(\lim \sup |a_n|\) is finite.

Solution: \(\Rightarrow\): Suppose \((a_n)\) is bounded by \(M\).
\[
\lim \sup |a_n| = \lim_{N \to \infty} \sup_{n > N} \{|a_n|\} \leq \lim_{N \to \infty} M = M.
\]

\(\Leftarrow\): Suppose \(\lim \sup |a_n|\) is finite. Then \(\lim_{N \to \infty} \sup_{n > N} \{|a_n|\} = M\). Then there exists \(N\) large such that \(\sup_{n \geq N} |a_n| < M + 1\). There are only finitely many terms less than \(N\) so they are bounded as well, say by \(M'\). So \(\sup_{n \in \mathbb{N}} |a_n| \leq \sup_{n \in \mathbb{N}} |a_n| \leq \max(M + 1, M')\). This shows that \((a_n)\) is bounded.

Problem 4: Determine which of the following series is convergent.

(a) \(\sum_{n=2}^{\infty} \frac{1}{\log n}\)

(b) \(\sum_{n=1}^{\infty} \frac{n!}{n^n}\)

Solution:

(a) First, we claim that \(\log n < n\). This is true for \(n = 2\). Assume this is true for \(n\), then

\[
\log(n + 1) \leq \log(en) = \log e + \log n \leq 1 + n
\]

by the fact that \(e > 2\) and the induction hypothesis. So by induction we have that \(\log n < n\). So, \(\frac{1}{\log n} > \frac{1}{n}\). We then compare our series to the harmonic series to see that it diverges.

(b) We will compare our series to \(\sum_{n=1}^{\infty} \frac{2}{n^2}\) to see that it converges.

\[
\frac{n!}{n^n} = \frac{n(n-1)\cdots(2)(1)}{(n)(n-1)\cdots(n)(n)} \leq (1)(1)\cdots \frac{2}{n^2}.
\]

By the \(p\)-test, \(\sum_{n=1}^{\infty} \frac{2}{n^2}\) converges. So, by the comparison test, our series converges.

Problem 5: Prove that if \(\sum |a_n|\) converges and \((b_n)\) is a bounded sequence then \(\sum a_nb_n\) converges.

Solution: Let \(B\) be the bound for \((b_n)\). Fix \(\epsilon > 0\). Since \(\sum |a_n|\) converges, we have that there exists \(N \in \mathbb{N}\) such that for all \(M_1, M_2 > N\),
\[
\sum_{n=M_1}^{M_2} |a_n| < \frac{\epsilon}{B}.
\]
So,

\[ \left| \sum_{n=M_1}^{M_2} a_n b_n \right| \leq \sum_{n=M_1}^{M_2} |a_n| |b_n| \leq B \sum_{n=M_1}^{M_2} |a_n| < B \frac{\epsilon}{B} = \epsilon. \]