Problem 1: Suppose that \((a_n)\) is a convergent subsequence and for each \(n \geq 1\) let \(b_n = a_{n+1}\). Prove that \((b_n)\) is also convergent and that 
\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n.
\]

Solution: \((a_n)\) is a convergent subsequence. Let \(L\) be its limit. Fix \(\epsilon > 0\), since \(a_n\) converges, there exists an \(N \in \mathbb{N}\) such that for all \(n \geq N - 1\),
\[
|a_n - L| < \epsilon.
\]
So for \(n \geq N\),
\[
|b_n - L| = |a_{n+1} - L| < \epsilon
\]
since \(n + 1 \geq N\). So \((b_n)\) also converges to \(L\).

Problem 2: Let \((a_n)\) be the sequence recursively defined by \(a_1 = \sqrt{2}\) and
\[
a_{n+1} = \sqrt{2a_n}.
\]

(a) Prove that \(a_n \leq 2\) for all \(n \geq 1\).

(b) Prove that \(a_{n+1} \geq a_n\) for all \(n \geq 1\).

(c) Compute \(\lim_{n \to \infty} a_n\).

Solution:

(a) Prove this by induction. For \(n = 1\) we know that \(\sqrt{2} \leq 2\). \(a_{n+1} = \sqrt{2a_n} \leq \sqrt{2 \cdot 2}\) by the induction hypothesis. So \(a_{n+1} \leq 2\). So by the principle of induction the claim is shown.

(b) \(a_{n+1} = \sqrt{2a_n} \geq \sqrt{a_n \cdot a_n}\) by (a). So \(a_{n+1} \geq a_n\).

(c) By problem 1,
\[
L = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{2a_n} = \sqrt{2 \lim_{n \to \infty} a_n} = 2\sqrt{L}
\]
So \(L = 0\) or \(2\). \(L\) cannot be \(0\) since by (b) \(a_n \geq a_1 = \sqrt{2}\). So \(L = 2\).