Problem 1: Let $k \in \mathbb{R}$. Show that the function $f(x) = k$ is integrable on any interval $[a, b]$ and compute $\int_a^b f$.

Solution: Let $P$ be a partition of $[a, b]$. Then
\[ U(f, P) = \sum_{n=1}^{m} k(x_{n+1} - x_n) = k \sum_{n=1}^{m} (x_{n+1} - x_n) = k(b - a) \]
and
\[ L(f, P) = \sum_{n=1}^{m} k(x_{n+1} - x_n) = k \sum_{n=1}^{m} (x_{n+1} - x_n) = k(b - a). \]
Since the infimum and supremum on any interval is $k$. So $U(f) = L(f) = k(b - a)$. This gives that $f$ is integrable and $\int_a^b f = k(b - a)$.

Problem 2: Define $f: [0, 2] \to \mathbb{R}$ by
\[ f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases} \]
Show that $f$ is integrable on $[0, 2]$.

Solution: Fix $\epsilon > 0$. Choose $n$ so that $1/n < \epsilon$. Subdivide $[0, 2]$ into intervals of length $1/n$. Then
\[ L(f, P) = 0 \]
since the infimum of $f$ on any interval is 0.
\[ U(f, P) = \sum_{k=1}^{2n} M_k \frac{1}{n} \]
$M_k$ is 0 except on the interval containing 1. So
\[ U(f, P) = (1) \frac{1}{n} \]
So
\[ U(f, P) - L(f, P) = \frac{1}{n} < \epsilon. \]
This gives that $f$ is integrable and that $\int_0^2 f = L(f) = 0$.

Problem 3: Suppose that $f: [0, 2] \to \mathbb{R}$ is an integrable function and that $g: [0, 2] \to \mathbb{R}$ satisfies
\[ g(x) = f(x) \text{ if } x \neq 1, \quad g(x) \neq f(x) \text{ if } x = 1. \]
Prove that $f$ is integrable and that $\int_0^2 g = \int_0^2 f$.

Solution: Let $h = f - g$. Then $h(x) = 0$ for $x \neq 1$ and $h(1) = g(1) - f(1)$. So
\[ \frac{h(x)}{g(1) - f(1)} \]
is integrable by problem 2 and
\[ \int_0^2 \frac{h(x)}{g(1) - f(1)} = 0 \]
This gives that \( h \) is integrable with 0 integral. Since \( f \) is integrable we get that \( g = -h + f \) is integrable and that \( \int_0^2 f = \int_0^2 g \).

Problem 4: Prove that the function
\[ f(x) = \begin{cases} 
\sin(1/x) & \text{if } x \neq 0 \\
5 & \text{if } x = 0 
\end{cases} \]
is integrable on \([0, 1]\).

Solution: Fix \( \epsilon > 0 \). Subdivide \([0, 1]\) into \([0, \epsilon/12]\) and \([\epsilon/12, 1]\). Since \( \sin(1/x) \) is continuous on \([\epsilon/12, 1]\) we can find a partition of \([\epsilon/12, 1]\) so that \( U(f, P) - L(f, P) < \epsilon/2 \). On \([0, \epsilon/12]\) we have, for any partition \( P' \),
\[ U(f, P') = \sum_{k=1}^{m} M_k(x_{k+1} - x_k) \leq 5 \sum_{k=1}^{m} x_{k+1} - x_k = 5 \frac{\epsilon}{12} \]
and
\[ L(f, P') = \sum_{k=1}^{m} m_k(x_{k+1} - x_k) \geq -1 \sum_{k=1}^{m} x_{k+1} - x_k = -\epsilon/12. \]
So \( U(f, P') - L(f, P') < \epsilon/2 \) and combining the two partitions we get a partition \( P'' \) on \([0, 1]\) so that \( U(f, P'') - L(f, P'') < \epsilon \).

Problem 6: For \( x > 0 \) le
\[ L(x) = \int_1^x \frac{dt}{t} \]
(a) Compute \( L(1) \) and \( L'(x) \).
(b) Prove that \( L(x) \) is strictly increasing for \( x \geq 1 \).
(c) Prove that \( L(xy) = L(x) + L(y) \) for every \( x, y > 0 \).

Solution:
(a) \( L(1) = 0 \) and \( L'(x) = 1/x \) where the second equality is due to the fundamental theorem of calculus.
(b) By (a), \( L'(x) \geq 0 \) so use the mean value theorem to show that \( L(x) \) is strictly increasing.
(c) Fix \( y \). Then
\[ (L(xy))' = L'(xy) y = \frac{y}{xy} = \frac{1}{x} = L'(x) \]
So \( L(xy) = L(x) + C \). Let \( x = 1 \), then \( L(y) = 0 + C \). So \( L(xy) = L(x) + L(y) \).