Representations of the Symmetric Group from Geometry

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Overview

$$\mathrm{PConf}_n(\mathbb{C}) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$$

When viewed as a representation of S_n, there is a notion of "stabilization" of the limit $\lim_{n\to\infty} H^i(\mathrm{PConf}_n(\mathbb{C}))$. We will first understand what is meant by "stabilization", and then investigate what structure this stabilizes to.

Representation Theory of Finite Groups

A *representation* of a finite group G on a finite dimensional $\mathbb{C}$ vector space V is a homomorphism $\rho : G \to GL(V)$.

A representation V of a finite group G can always be uniquely decomposed (up to isomorphism) as a direct sum of irreducible representations of G:

$$V = W_1^{\oplus a_1} \oplus \cdots \oplus W_k^{\oplus a_k}$$

Characters

The *character* χ of a complex representation V is defined to be $\mathrm{Tr} \circ \rho : G \to \mathbb{C}$, where Tr is the trace map.

Two representations V_1, V_2 are isomorphic if and only if their characters χ_1, χ_2 are equal.

Inner Products of Characters

Let V be a G representation which decompose into irreducibles as

$$V = W_1^{\oplus a_1} \oplus \cdots \oplus W_k^{\oplus a_k}$$

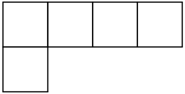
Suppose V has character χ, and W_j has character χ_j. The inner product $\langle\rangle_G$ satisfies the following for all j:

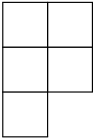
$$\langle \chi, \chi_j \rangle_G = a_j$$

Young Diagrams

Alfred Young described an explicit correspondence between Young Diagrams with n boxes and the irreducible representations of S_n:

 $\iff$ W_1

 $\iff$ W_2

 $\iff$ W_3

$$\vdots$$

Configuration Space

Ordered complex configuration space $\mathrm{PConf}_n(\mathbb{C})$ is defined as:

$$\mathrm{PConf}_n(\mathbb{C}) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$$

S_n acts faithfully on $\mathrm{PConf}_n(\mathbb{C})$ by permuting coordinates.

This induces an action of S_n on $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C})$, which gives $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C})$ the structure of a complex representation of S_n.

Finite Case

Thus, for any i, n there is a unique decomposition of $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C})$ into irreducibles:

$$H^i(\mathrm{PConf}_3(\mathbb{C})) = \ydiagram{3}^{\oplus a_1} \oplus \ydiagram{2,1}^{\oplus a_2} \oplus \ydiagram{1,1,1}^{\oplus a_3}$$

In the 1980's, Lehrer and Solomon described an explicit formula for $H^i(\mathrm{PConf}_n(\mathbb{C}))$ as an S_n representation. However, this does not easily yield the decomposition into irreducibles.

$$H^i(\mathrm{PConf}_n(\mathbb{C})) = \bigoplus_{\mu} \mathrm{Ind}_{Z(c_\mu)}^{S_n}(\xi_\mu)$$

Character Polynomials

Let V_n be the standard representation of S_n. Then, we have:

$$\chi_{V_n}(\sigma) = c_1(\sigma) - 1$$

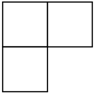
where $c_1(\sigma)$ is the number of fixed points (1-cycles) of σ. Therefore the characters of $V_2, V_3, \ldots$ can be simultaneously expressed as the polynomial $i_1 - 1$,

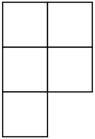
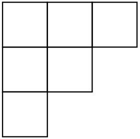
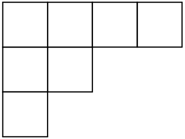
Character Polynomials

Furthermore, the standard representations are given by the following Young Diagrams:

$$V_2 = \ydiagram{1,1}, \qquad V_3 = \ydiagram{2,1}, \qquad V_4 = \ydiagram{3,1}$$

Character Polynomials

Take any Young Diagram $\lambda =$  with L boxes. Let $V_n, V_{n+1}, \ldots$ be the irreducible representations given by adding boxes to the row above λ:

$V_5 =$ , $V_6 =$ , $V_7 =$ , ...

There is a unique polynomial $P \in \mathbb{C}[i_1, i_2, \ldots]$ with $\deg P = L$ which gives the characters of each V_k simultaneously, i.e.:

$$\chi_{V_k}(\sigma) = P(c_1(\sigma), c_2(\sigma), \ldots) \qquad \text{for all } k$$

Representation Stability

Theorem (Church, Farb 2013). For all character polynomials $P \in \mathbb{C}[i_1, i_2, \ldots,]$, the limit

$$\langle P, H^i(\mathrm{PConf}(\mathbb{C}))\rangle_{S_n} := \lim_{n\to\infty} \langle P, H^i(\mathrm{PConf}_n(\mathbb{C}))\rangle_{S_n}$$

exists and furthermore is constant for $n \geq 2i + \deg P$.

Research Question

Question. For character polynomials P representing families of irreducible representations, what is the value of the following limit?

$$\lim_{n\to\infty} \langle P, H^i(\mathrm{PConf}_n(\mathbb{C}))\rangle_{S_n}$$

Polynomial Statistics

For $\mathbb{F}$ a field, define $\mathrm{Conf}_n(\mathbb{F})$ to be the square free polynomials of degree n with coefficients in $\mathbb{F}$

$$\mathrm{Conf}_n(\mathbb{F}) := \{f \in \mathbb{F}[x] \mid \deg(f) = n,\ f \text{ squarefree}\}$$

Given a polynomial $f \in \mathrm{Conf}_n(\mathbb{F}_q)$ which splits as $(x - r_1)\dots(x - r_n)$ for $r_j \in \overline{\mathbb{F}_q}$, applying the Frobenius action to its coefficients will fix $\mathbb{F}_q$. However, the Frobenius action will induce a permutation σ of the roots $r_1, \dots, r_n$. Let σ_f be this permutation, which is unique up to conjugation.

Polynomial Statistics and Representation Stability

Theorem (Farb, Church, 2013). For any character polynomial $P \in \mathbb{C}[i_1, i_2, \ldots]$, the following two limits exist and are equal:

$$\lim_{n\to\infty} \sum_{i=0}^{\infty} (-1)^i \frac{\langle P, H^i(\mathrm{PConf}_n(\mathbb{C}))\rangle_{S_n}}{q^i} = \lim_{n\to\infty} q^{-n} \sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} P(\sigma_f)$$

In particular, both the limit on the left and the series on the right converge, and they converge to the same series in q^{-1},

Example - Standard Representation

Let us consider the standard representations $V_2, V_3, \ldots$ of $S_2, S_3, \ldots$. In this case, their characters are given by the polynomial $P(i_1, i_2, \ldots) = i_1 - 1$. The sum

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} P(\sigma_f)$$

is thus counting the number of linear factors minus 1 of polynomials $f \in \mathrm{Conf}_n(\mathbb{F}_q)$ as $n \to \infty$. This average can be computed directly through a combinatorial argument, yielding:

$$q^{-n} \sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} P(\sigma_f) = -q^{-1} + 2q^{-2} - 2q^{-3} + 2q^{-4} = \ldots$$

Example - Standard Representation

By the previous theorem, we have

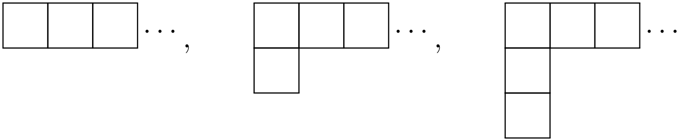
$$\lim_{n\to\infty}\sum_{i=0}^{\infty}(-1)^i\frac{\langle P, H^i(\mathrm{PConf}_n(\mathbb{C}))\rangle_{S_n}}{q^i} = -q^{-1} + 2q^{-2} - 2q^{-3} + 2q^{-4} - \dots$$

Since both sides converge to the same series, we have:

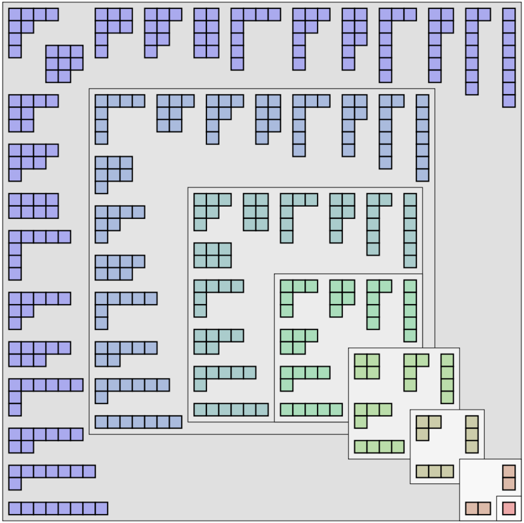
$$\langle P, H^i(\mathrm{PConf}_n(\mathbb{C}))\rangle = \begin{cases} 0 & i = 0 \\ 1 & i = 1 \\ 2 & i \geq 2 \end{cases}$$

Previous Results

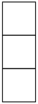
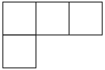
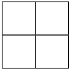
At the start of this project, the limiting power series for a character polynomial P representing a family of irreducible representations had only been explicitly computed for 3 examples:



Current Results



Some Examples of Results

	$-3q^{-3}+9q^{-4}-15q^{-5}+23q^{-6}-34q^{-7}+47q^{-8}-62q^{-9}+78q^{-10}-96q^{-11}+118q^{-12}-141q^{-13}+\dots$
	$q^{-2}-6q^{-3}+19q^{-4}-46q^{-5}+97q^{-6}-178q^{-7}+288q^{-8}-435q^{-9}+634q^{-10}-887q^{-11}+1190q^{-12}-1553q^{-13}+\dots$
	$-2q^{-3}+12q^{-4}-34q^{-5}+66q^{-6}-113q^{-7}+190q^{-8}-298q^{-9}+425q^{-10}-581q^{-11}+790q^{-12}-1047q^{-13}+\dots$

Method

These results are found by computing the following power series for arbitrary character polynomials P:

$$\lim_{n\to\infty} q^{-n} \sum_{f\in\mathrm{Conf}_n(\mathbb{F}_q)} P(f)$$

This can be done directly on the following polynomials

$$\binom{i_j}{k} = \frac{i_j(i_j-1)\dots(i_j-k+1)}{k!} \in \mathbb{C}[i_1, i_2, \dots]$$

and then extended by linearity to arbitrary polynomials.

Conjectures

Suppose that $\lambda = \ydiagram{2,1}$ is a Young Diagram with $k > 0$ boxes. Let the associated power series be of the form

$$a_0 - a_1 q^{-1} + a_2 q^{-2} - \dots$$

Conjectures:

1 The sequence $a_0, a_1, \dots$ is non-decreasing.

2 Suppose that a_j is the first non-zero coefficient. Then

  1.1 $j \leq k$, and $j = k$ if and only if λ is a vertical stack of k boxes.

  2.2 $j \geq k/2$.

Thanks!

Thank you to Dr. Sean Howe for advising me on this project!