High Dimensional Geometry

Geometry is the study of the shape of mathematical objects. There are many intricate connections between geometry and other areas of math, particularly in the study of the complex numbers (\(\mathbb{C}\)) and related algebraic constructions.

Figure 1. The planes \(x = y, y = z,\) and \(x = z\) in \(\mathbb{R}^3\). \(\text{PConf}_n(\mathbb{R})\) is the complement of these planes in \(\mathbb{R}^3\).

We aim to study complex configuration space, denoted as \(\text{PConf}_n(\mathbb{C})\) where \(n\) is the dimension. In recent years, researchers have shown that the geometry of \(\text{PConf}_n(\mathbb{C})\) stabilizes as \(n\) tends toward infinity. However, the limiting geometry of \(\text{PConf}_n(\mathbb{C})\) is still mostly unknown.

Partitions and Young Diagrams

Let \(n\) be a positive integer. A partition of \(n\) is a way of writing \(n\) as a sum of positive integers. For instance, \(n = 4\) has 5 partitions:

- \(4 = 4\)
- \(4 = 3 + 1\)
- \(4 = 2 + 2\)
- \(4 = 2 + 1 + 1\)
- \(4 = 1 + 1 + 1 + 1\)

Young Diagrams are a convenient way of visualising partitions as non-increasing stacks of boxes.

\[\begin{array}{c}
\text{\(4 \quad \downarrow\)}
\end{array}\]

\[\begin{array}{c}
\text{\(4 \quad \downarrow\)}
\end{array}\]

The Symmetric Group

The symmetric group of degree \(n\), denoted \(S_n\), is the group of permutations of the integers \(1, 2, \ldots, n\). The following is an example of an element of \(S_5\):

\[\sigma = (1, 2, 3) \times (1, 2, 4, 5)\]

This notation represents \(\sigma(1) = 2, \sigma(2) = 5, \sigma(3) = 4, \sigma(4) = 3, \sigma(5) = 1\). Any permutation can be written in "cycle form" as the product of disjoint cycles. For instance \(\sigma = (12534)\). The cycle type of a permutation is the length of each cycle, represented as \(\sigma = (a_1, a_2, a_3, \ldots)\). Cycle types describe the conjugacy classes of \(S_n\).

There is a correspondence between Young Diagrams with \(n\) boxes, cycle types in \(S_n\), and partitions of \(n\):

\[\begin{array}{c}
\text{\(5 \quad \downarrow\)}
\end{array}\]

\[\begin{array}{c}
\text{\(5 \quad \downarrow\)}
\end{array}\]

Homology and Cohomology

The \(s\)th degree of homology represents the number and structure of the \(s\)-dimensional "holes" in a space. Cohomology is algebraically dual to homology and expresses the same information. However, cohomology associates a ring rather than a group for each degree \(s\). The \(s\)th degree of cohomology (over the rationals) of a space \(X\) is denoted \(H^s(X; \mathbb{Q})\).

\[\begin{array}{c}
\text{The torus and sphere have isomorphic degree 1 and degree 2 homology but differ in degree 3.}
\end{array}\]

Representation Stability of Configuration Space

Degree \(n\) configuration space over \(\mathbb{F}\) is the space of ordered \(n\)-tuples with pairwise distinct elements in \(\mathbb{F}\):

\[\text{PConf}_n(\mathbb{F}) = \{(x_1, x_2, \ldots, x_n) \mid i \neq j \Rightarrow x_i \neq x_j, x_i \in \mathbb{F}\}\]

\(S_n\) acts on \(\text{PConf}_n(\mathbb{C})\) by permuting the coordinates \(x_1, \ldots, x_n\). This induces an action of \(S_n\) on the homology \(H^s(\text{PConf}_n(\mathbb{C}), \mathbb{Q})\). Since \(H^s(\text{PConf}_n(\mathbb{C}), \mathbb{Q})\) is a complex vector space, this gives this action the structure of a representation of \(S_n\).

For a Young Diagram \(\lambda\) with \(k\) boxes let \(V_\lambda^k\) represent the Young Diagram by adding \(n - k\) boxes above the first row of \(\lambda\):

\[\lambda = \begin{array}{c}
\text{\(5 \quad \downarrow\)}
\end{array}\]

Conjectures

Let \(\lambda\) be a Young Diagram with \(k\) boxes and let \(p_\lambda(q)\) be defined as above.

- The leading non-zero term of \(p_\lambda\) has exponent in the range \([-k, -k/2]\).
- If \(a_{i, j} \neq 0\), then \(a_{i, j} \neq 0\) for \(j > i\).
- \(a_{n-1} = 0\) if and only if \(\lambda\) is a vertical stack of \(k\) boxes.

References

