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Representations of the Symmetric Group from Geometry

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High Dimensional Geometry

Geometry is the study of the shape of mathematical objects. There are many intricate connections between geometry and other areas of math, particularly in the study of the complex numbers ($\mathbb{C}$) and related algebraic constructions.

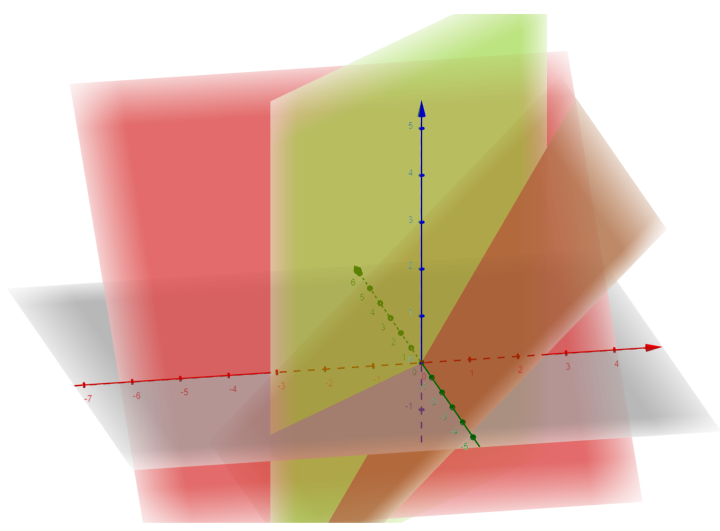


Figure 1. The planes $x = y, y = z$, and $x = z$ in $\mathbb{R}^3$. $\mathrm{PConf}_3(\mathbb{R})$ is the complement of these planes in $\mathbb{R}^3$.

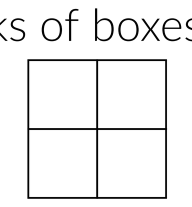
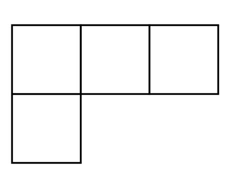
We aim to study *complex configuration space*, denoted as $\mathrm{PConf}_n(\mathbb{C})$ where n is the dimension. In recent years, researchers have shown that the geometry of $\mathrm{PConf}_n(\mathbb{C})$ *stabilizes* as n tends toward infinity. However, the limiting geometry of $\mathrm{PConf}_n(\mathbb{C})$ is still mostly unknown.

Partitions and Young Diagrams

Let n be a positive integer. A *partition* of n is way of writing n as a sum of positive integers. For instance, $n = 4$ has 5 partitions:

$$4 \qquad 3+1 \qquad 2+2 \qquad 2+1+1 \qquad 1+1+1+1$$

Young Diagrams are a convenient way of visualising partitions as non-increasing stacks of boxes.

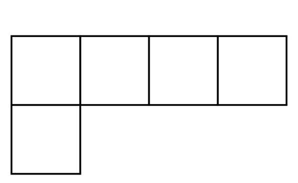
 $\iff \quad 4 = 2+2$  $\iff \quad 4 = 3+1$

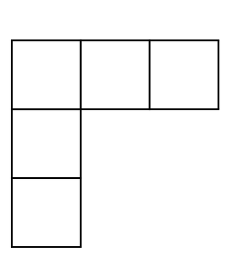
The Symmetric Group

The *symmetric group* of degree n, denoted S_n, is the group of permutations of the integers $1, 2, \ldots, n$. The following is an example of an element of S_5:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$$

This notation represents $\sigma(1) = 2, \sigma(2) = 5, \sigma(3) = 4, \sigma(4) = 3, \sigma(5) = 1$. Any permutation can be written in "cycle form" as the product of disjoint cycles. For instance $\sigma = (125)(34)$. The cycle type of a permutation is the length of each cycle, represented as $\sigma = (abc)(de)$. Cycle types describe the conjugacy classes of S_n. There is a correspondence between Young Diagrams with n boxes, cycle types in S_n, and partitions of n:

 $\iff \quad 5 = 4+1 \quad \iff \quad (abcd)(e) \quad = \quad$ conjugacy class in S_n

 $\iff \quad 5 = 3+1+1 \quad \iff \quad (abc)(d)(e) \quad = \quad$ conjugacy class in S_n

Representations of the Symmetric Group

A *representation* of S_n is a homomorphism $\rho : S_n \to \mathrm{GL}(V)$ where V is a complex vector space.

An *irreducible representation* is a representation which cannot be expressed as a direct sum of other representations. Every representation of S_n can be expressed uniquely as a direct sum of irreducible representations.

There is a correspondence between irreducible representations of S_n and Young Diagrams with n boxes. Therefore, to classify any representation $\rho : S_n \to \mathrm{GL}(V)$, it suffices to find the integer multiplicity corresponding to each Young Diagram with n boxes.

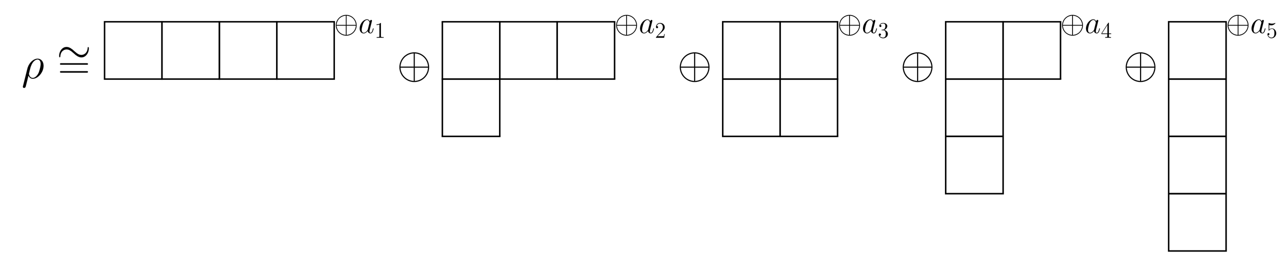


Figure 2. An example of decomposing a representation of S_4 into irreducibles, where the jth Young Diagram corresponds to an irreducible representation ι_j. Then, $a_j = \langle \rho, \iota_j \rangle_{S_n}$.

Homology and Cohomology

The *ith degree of homology* represents the number and structure of the i-dimensional "holes" in a space. *Cohomology* is algebraically dual to homology and expresses the same information. However, cohomology associates a ring rather than a group for each degree i. The ith degree of cohomology (over the rationals) of a space X is denoted $H^i(X; \mathbb{Q})$.

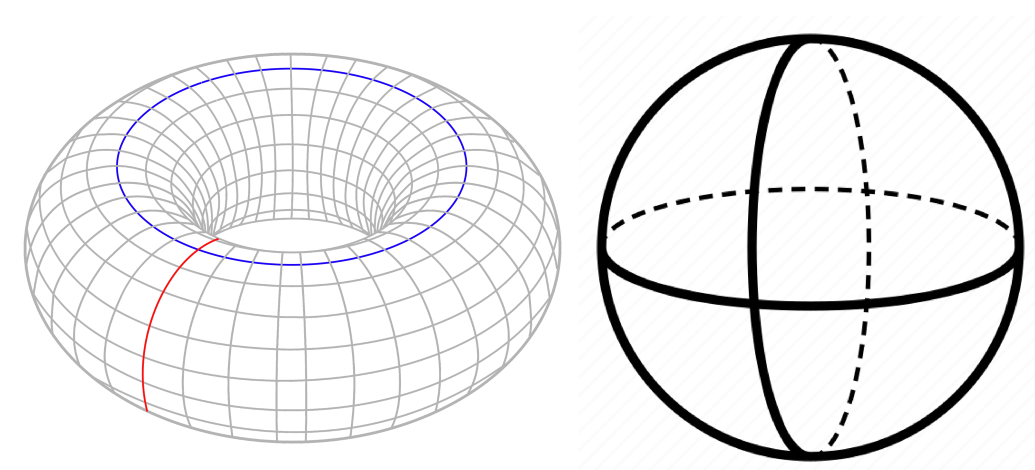


Figure 3. The torus and sphere have isomorphic degree 0 and degree 2 homology but differ in degree 1.

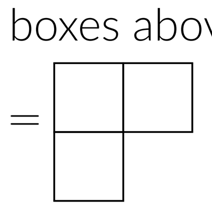
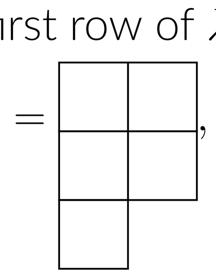
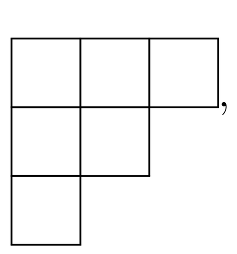
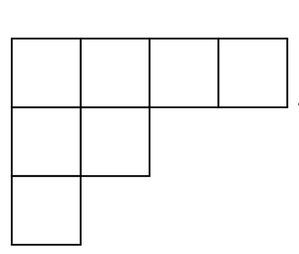
Representation Stability of Configuration Space

Degree n configuration space over $\mathbb{F}$ is the space of ordered n-tuples with pairwise distinct elements in $\mathbb{F}$:

$$\mathrm{PConf}_n(\mathbb{F}) = \{(x_1, x_2, \ldots, x_n) \mid i \neq j \Rightarrow x_i \neq x_j, x_i \in \mathbb{F}\}$$

S_n acts on $\mathrm{PConf}_n(\mathbb{C})$ by permuting the coordinates $x_1, \ldots, x_n$. This induces an action of S_n on the cohomology $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q})$. Since $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q})$ is a complex vector space, this gives this action the structure of a representation of S_n.

For a Young Diagram λ with k boxes let V_n^λ represent the Young Diagram by adding $n-k$ boxes above the first row of λ:

$\lambda =$  $V_5^\lambda =$ , $V_6^\lambda =$ , $V_7^\lambda =$  $\ldots$

Theorem

For any Young Diagram λ and $i \in \mathbb{N}$ the following limit exists:

$$\lim_{n\to\infty} \langle V_n^\lambda, H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q}) \rangle_{S_n}$$

Project Goal

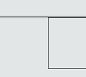
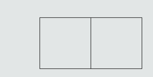
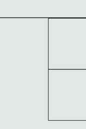
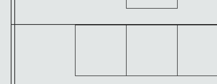
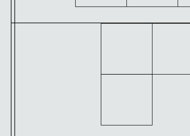
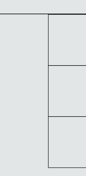
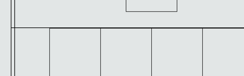
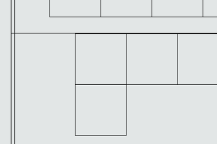
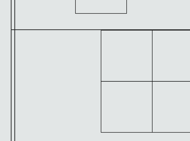
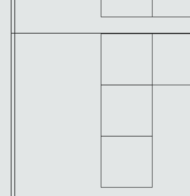
For any Young Diagram λ, determine the value of

$$\lim_{n\to\infty} \langle V_n^\lambda, H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q}) \rangle$$

In the following results, we fix λ and let $a_i = \lim_{n\to\infty} \langle V_n^\lambda, H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q}) \rangle$. Then, we define $p^\lambda(q)$ as a formal power series in q such that

$$p^\lambda(q) = a_0 - a_1 q^{-1} + a_2 q^{-2} - \ldots$$

Results

	$-q^{-1}+2q^{-2}-2q^{-3}+2q^{-4}-2q^{-5}+2q^{-6}-2q^{-7}+2q^{-8}-2q^{-9}+\ldots$
	$-q^{-1}+2q^{-2}-3q^{-3}+6q^{-4}-9q^{-5}+10q^{-6}-11q^{-7}+14q^{-8}-17q^{-9}+\ldots$
	$2q^{-2}-5q^{-3}+6q^{-4}-7q^{-5}+10q^{-6}-13q^{-7}+14q^{-8}-15q^{-9}+\ldots$
	$q^{-2}-4q^{-3}+8q^{-4}-14q^{-5}+24q^{-6}-35q^{-7}+46q^{-8}-61q^{-9}+\ldots$
	$2q^{-2}-7q^{-3}+16q^{-4}-30q^{-5}+47q^{-6}-68q^{-7}+94q^{-8}-123q^{-9}+\ldots$
	$-3q^{-3}+9q^{-4}-15q^{-5}+23q^{-6}-34q^{-7}+47q^{-8}-62q^{-9}+\ldots$
	$-q^{-3}+6q^{-4}-17q^{-5}+33q^{-6}-57q^{-7}+96q^{-8}-149q^{-9}+\ldots$
	$q^{-2}-6q^{-3}+19q^{-4}-46q^{-5}+97q^{-6}-178q^{-7}+288q^{-8}-435q^{-9}+\ldots$
	$-2q^{-3}+12q^{-4}-34q^{-5}+66q^{-6}-113q^{-7}+190q^{-8}-298q^{-9}+\ldots$
	$-4q^{-3}+17q^{-4}-45q^{-5}+97q^{-6}-177q^{-7}+287q^{-8}-435q^{-9}+\ldots$

Conjectures

Let λ be a Young Diagram with k boxes and let $p_\lambda(q)$ be defined as above.

- The leading non-zero term of p_λ has exponent in the range $[-k, -k/2]$.
- If $a_i \neq 0$, then $a_j \neq 0$ for $j > i$.
- $a_{k-1} = 0$ if and only if λ is a vertical stack of k boxes.

References

[1] Thomas Church and Benson Farb. Representation theory and homological stability. *Advances in Mathematics*, 245: 250–314, oct 2013. doi:10.1016/j.aim.2013.06.016. URL `https://doi.org/10.1016%2Fj.aim.2013.06.016`.

[2] Thomas Church, Jordan Ellenberg, and Benson Farb. Representation stability in cohomology and asymptotics for families of varieties over finite fields. *Contemporary Mathematics*, 620:1–54, 2014.

[3] Sean Howe. Motivic random variables and representation stability, i: Configuration spaces. *Algebraic & Geometric Topology*, 20(6):3013–3045, dec 2020. doi:10.2140/agt.2020.20.3013. URL `https://doi.org/10.2140%2Fagt.2020.20.3013`.