REPRESENTATIONS OF THE SYMMETRIC GROUP

FROM GEOMETRY

by

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1. Abstract

Representation stability was introduced to study mathematical structures which stabilize when viewed from a representation theoretic framework. The instance of representation stability studied in this project is that of ordered complex configuration space, denoted $\mathrm{PConf}_n(\mathbb{C})$:

$$\mathrm{PConf}_n(\mathbb{C}) := \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j\}$$

$\mathrm{PConf}_n(\mathbb{C})$ has a natural S_n action by permuting its coordinates which gives the cohomology groups $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q})$ the structure of an S_n representation. The cohomology of $\mathrm{PConf}_n(\mathbb{C})$ *stabilizes* as n tends toward infinity when viewed as a family of S_n representations. From previous work, there is an explicit description for $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q})$ as a direct sum of induced representations for any i, n, but this description does not explain the behavior of families of irreducible representations as $n \to \infty$. We implement an algorithm which, given a Young Tableau, computes the cohomological degrees where the corresponding family of irreducible representations appears stably as $n \to \infty$. Previously, these values were known for only a few Young Tableaus and cohomological degrees. Using this algorithm, results have been found for all Young Tableau with up to 8 boxes and certain Tableau with more, which has led us to conjectures based on the data collected.

2. Introduction

Representation stability is a sub-field of representation theory concerned with the symmetries of certain representations as their degree increases towards infinity. Mathematicians are interested in stabilization phenomena because they describe the part of a problem that remains the same even as the complexity or size of the problem increases. There is a broad family of problems in representation stability that have theoretical guarantees of a structure remaining stable as the problem increases in complexity [3] [7]. However, even in the simplest cases, computing this stable part of the problem is difficult.

While the theory of representations of finite groups is well over a hundred years old, [5, Section 4] the area of representation stability is far more recent, originating in the early 2010's due to Church and Farb [3]. This theory began as a series of conjectures pertaining to the cohomology of configuration spaces [4, Section 3]. *Complex configuration space* is the space of ordered n-tuples of $\mathbb{C}$ [4]. There is a natural action of S_n on complex configuration space by permuting the coordinates, which when considered as a vector space over $\mathbb{C}$ provides a representation of S_n. Church and Farb investigated this representation of S_n as n and the degree of cohomology varied [4] [3]. They were able to find that in terms of the decomposition into irreducible representations, the representation of S_n stabilized as n tended toward infinity under certain conditions [4, Section 3]. This stabilization phenomena was proved using the machinery of étale cohomology, which provides a connection between arithmetic and algebraic geometry. [4, Section 2].

In addition, Farb describes connections between the cohomology of configuration space and the properties of square free polynomial space over a finite field [4, Section 5]. This connection is important to this work, as it means that the coefficients determined here can be interpreted as geometric results in square free polynomial space.

3. Representations of Finite Groups

3.1. Basic Definitions.

Definition 3.1.1. A *representation* of a finite group G on a finite dimensional vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$ from G to the invertible linear transformations of V. V is then said to be a representation of G [5, Section 1.1].

Representation theory is the study of algebraic structures by representing their elements as linear transformations of a vector space. Representations provide a framework to understand a group and its symmetries. Let us now consider some examples of representations.

Example 3.1.2. A relevant representation of the permutation group S_3 is the permutation representation on $\mathbb{C}^3$. The permutation representation is defined by letting $\sigma \in S_3$ act on a vector $v \in \mathbb{C}^3$ by permuting its coordinates:

$$\sigma \in S_3 : (\rho(\sigma))(v_1, v_2, v_3) = (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)})$$

Example 3.1.3. Take some finite group G and a one dimensional vector space V. There exists an identity map $\mathrm{id}_V \in \mathrm{G}L(V)$ such that for all $v \in V$, $\mathrm{id}_V(v) = v$. The *trivial representation* maps all elements of G to the identity map so $\rho(g) = \mathrm{id}_V$. For instance, the trivial representation of the permutation group S_3 on $\mathbb{C}$ is defined as:

$$c \in \mathbb{C}, \sigma \in S_3 : (\rho(\sigma))(c) = c$$

3.2. Subrepresentations.

Definition 3.2.1. Consider a representation V of a finite group G. A *subrepresentation* of V is a subspace W of V which is invariant under G, so that for any $w \in W$, $\rho(g)(w)$ is an element of W [5, Section 1.1].

Example 3.2.2. Consider a representation $V = \mathbb{C}^3$ of S_3 that maps each element of S_3 to the identity automorphism of $\mathbb{C}^3$: $\rho(\sigma)(v_1, v_2, v_3) = (v_1, v_2, v_3)$. There are infinitely many subrepresentations of V - any proper non-zero subspace W of V is a subrepresentation since so $\rho(\sigma)(W) = W$. For instance, the subspace spanned by the vectors $\langle(1,1,0),(0,0,1)\rangle$ is a subrepresentation of V, as well as the subspace spanned by $\langle(1,0,5)\rangle$.

There is a natural way to combine two complementary subrepresentations back into the original representation, by way of direct sum.

Definition 3.2.3. For any two representations V_1, V_2 of G defined by the homomorphisms ρ_1, ρ_2 respectfully, define the direct sum $\rho_1 \oplus \rho_2 : G \to \mathrm{GL}(V_1 \oplus V_2)$ by:

$$(\rho_1 \oplus \rho_2)(g) = \rho_1(g) \oplus \rho_2(g) \qquad v_1 \in V_1, v_2 \in V_2, g \in G$$

$\rho_1 \oplus \rho_2$ is a homomorphism and thus a representation from the properties of the direct sum of matrices in a vector space:

$$(\rho_1(g) \oplus \rho_2(g))(\rho_1(h) \oplus \rho_2(h)) = \rho_1(g)\rho_1(h) \oplus \rho_2(g)\rho_2(h)$$

For a finite group G, it is possible to construct infinitely many representations of G on vector spaces of differing dimensions and structure. A natural question is whether there is some classification of the representations of G so the search for representations is restricted somewhat. We have seen via the direct sum that representations can be built out of other representations, so a natural focus is on the representations that are 'atomic' with respect to this operation, i.e., that cannot be expressed as a direct sum of others [5, Section 1.2]. This provides the motivation for the following definition.

Definition 3.2.4. An *irreducible* representation is one whose only subrepresentations are itself and the zero subspace.

Example 3.2.5. The trivial representation is irreducible, since its vector space has dimension 1, and therefore has no proper nonzero subspaces. Likewise, any representation over a vector space of dimension 1 is irreducible.

Irreducible representations are the building blocks of all representations. Consider example 3.2.2, of $\rho : S_3 \to \mathrm{GL}(V)$ defined by $\rho(g) = \mathrm{id}_V$, and $V = \mathbb{C}^3$. Take the subrepresentations W_1 spanned by $(1, 0, 0)$, W_2 spanned by $(0, 1, 0)$, and W_3 spanned by $(0, 0, 1)$ of V. On each of these subspaces, S_3 acts as the trivial representation. Thus, we may express $V = U \oplus U \oplus U$, where U is the trivial representation of S_3. Since U is trivial and therefore irreducible, this is a decomposition of V into a direct sum of irreducible representations.

Theorem 3.2.6. *For any representation V of a finite group G, there is a decomposition*

$$V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \cdots \oplus V_k^{\oplus a_k}$$

where the V_i are distinct irreducible representations. This decomposition is unique up to reordering and isomorphism [5, Proposition 1.8].

This is a foundational result in representation theory. The proof is omitted for the sake of brevity. It states that all representations can be expressed in terms of the direct sum of irreducible representations, and therefore if the irreducible representations of a group can be found, all of the representations of the group can be described.

Example 3.2.7. Once again, consider the representation $V = \mathbb{C}^3$ of S_3 that maps each element of S_3 to the identity automorphism of $\mathbb{C}^3$: $\rho(\sigma)(v_1, v_2, v_3) = (v_1, v_2, v_3)$. Since there are infinitely many subspaces of V and each subspace is fixed under the identity automorphism, there are infinitely many ways to express V as a sum of irreducible representations. This is why the decomposition in Theorem 3.2.6 is unique *up to isomorphism.* Theorem 3.2.6 states that no matter how V is decomposed into irreducible representations, the decomposition will be isomorphic to $U \oplus U \oplus U = U^{\oplus 3}$, where U is the trivial representation.

Example 3.2.8. Consider the permutation representation V' of S_3. The subspace $W \subset V'$ generated by $(1, 1, 1)$ is a subrepresentation of V', since it is fixed under the action of S_3:

$$\sigma \in S_3 : \rho(\sigma)(v_1, v_2, v_3) = (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)})$$

$$\sigma \in S_3 : \rho(\sigma)(1, 1, 1) = (1, 1, 1)$$

W is the trivial representation since each $\sigma \in S_3$ maps each element of W to itself. Therefore, V' is the direct sum of the trivial representation U with a two dimensional representation V. V is called the *standard representation* of S_3, and is irreducible. Thus, $V' = U \oplus V$ is the decomposition of the permutation representation on S_3 into irreducible representations [5, Section 1.3].

3.3. **Introduction to Character Theory.** We have seen that all the representations of a finite group G can be expressed as the direct sum of irreducible representations. Character theory provides structure that assists in finding the irreducible representations of a group and decomposing a representation into irreducibles.

Definition 3.3.1. A representation ρ is a homomorphism from the elements of a group to linear transformations of a vector space. The linear transformations of a vector space can be interpreted as matrices. Given a representation of a finite group G over a vector space V, the *character* is a function χ_V on G defined to be the trace of the matrix $\rho(g)$:

$$\chi_V(g) = \mathrm{Tr}(\rho(g))$$

An important property of the trace is that the trace of a matrix is invariant under conjugation [5, Section 2.1]. In particular, for $A, B \in \mathrm{GL}(V)$

$$\mathrm{Tr}(BAB^{-1}) = \mathrm{Tr}(A)$$

Therefore, χ_V is a *class function* - it is constant on the conjugacy classes of G:

$$\chi_V(hgh^{-1}) = \mathrm{Tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathrm{Tr}(\rho(g)) = \chi_V(g)$$

As an example, let us determine the character of the permutation representation V' on S_3. First, let us write out the matrices associated with each element of S_3. Since the character is invariant on conjugacy classes, we need only find the character of one representative from each conjugacy class:

$$\chi_{V'}((1)) = \mathrm{Tr}(\rho((1))) = \mathrm{Tr}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 + 1 + 1 = 3$$

$$\chi_{V'}((12)) = \mathrm{Tr}(\rho((12))) = \mathrm{Tr}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0 + 0 + 1 = 1$$

$$\chi_{V'}((123)) = \mathrm{Tr}(\rho((123))) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 0 + 0 + 0 = 0$$

Definition 3.3.2. Consider a group G, and take two class functions $\alpha, \beta : G \to \mathbb{C}$. Then, we define a Hermitian inner product on the class functions of G by

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$$

Lemma 3.3.3. *Given representations W and U over a group G, $\chi_{W \oplus U} = \chi_W + \chi_U$.*

Proof. Take some $g \in G$. Then the linear operator $T = \rho(g)$ of the representation $W \oplus U$ is the direct sum of the linear operators $T_W = \rho_W(g), T_U = \rho_U(g)$ of the representations W and U. The direct sum of linear operators satisfies

$$\mathrm{Tr}(T) = \mathrm{Tr}(T_W \oplus T_U) = \mathrm{Tr}(T_W) + \mathrm{Tr}(T_U)$$

Therefore, $\chi_{W \oplus U}(g) = \chi_W + \chi_U$ as desired. □

Theorem 3.3.4. *For a group G, the characters χ_V of its irreducible representations form an orthonormal basis for the class functions from G to $\mathbb{C}$* [5, Theorem 2.12, Proposition 2.30].

This is an essential theorem that helps explain why character theory is such a valuable tool in representation theory. The proof is omitted for the sake of brevity.

Corollary 3.3.5. *Any representation is determined by its character.*

Proof. Any representation V can be decomposed into a direct sum of irreducible representations in a unique way:

$$V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \cdots \oplus V_k^{\oplus a_k}$$

By lemma 3.3.3,

$$\chi_V = \overbrace{\chi_{V_1} + \chi_{V_1} + \cdots + \chi_{V_1}}^{a_1} + \cdots + \overbrace{\chi_{V_k} + \cdots + \chi_{V_k}}^{a_k} = \sum_{i=1}^{k} a_i \chi_{V_i}$$

Since V_i are distinct irreducible representations, their characters χ_{V_i} form an orthonormal basis by theorem 3.3.4 and are therefore linearly independent of one another. Thus, V is determined by its character. □

This theorem is why character theory is the key to the representation theory of finite groups - instead of working with representations which are generally complicated to construct and visualize, we can instead work with characters, which are simply class functions from G to $\mathbb{C}$.

Corollary 3.3.6. *For a group G, there are exactly as many irreducible representations of G as there are conjugacy classes.*

Proof. Class functions of G are functions on the conjugacy classes $C_1, C_2, \ldots C_k$ of G. Thus, the set of class functions on G has a basis $e_1, e_2, \ldots e_k$ as a vector space over $\mathbb{C}$

defined by:

$$e_i(g) = \begin{cases} 0 & g \notin C_i \\ 1 & g \in C_i \end{cases}$$

Therefore, the vector space of class functions of G has dimension equal to the number of conjugacy classes in G. Therefore, since the characters of the irreducible representations form an orthonormal basis of the class functions of G, there are exactly as many irreducible representations of G as there are conjugacy classes. □

4. Symmetric Polynomials

4.1. **Symmetric Polynomials.**

Definition 4.1.1. A *symmetric polynomial* is a polynomial $p(t_1, t_2, \dots t_n)$ in n variables such that if any of the variables t_i, t_j are interchanged, the polynomial remains the same [1, Section 16.1]. Equivalently, $p(t_1, t_2, \dots t_n)$ is symmetric if:

$$\text{for all } \sigma \in S_n, \quad p(t_1, t_2, \dots t_n) = p(t_{\sigma(1)}, t_{\sigma(2)}, \dots t_{\sigma(n)})$$

Example 4.1.2. Take $n = 3$. Then, p, q, r are symmetric polynomials:

$$p(t_1, t_2, t_3) = t_1 + t_2 + t_3$$

$$q(t_1, t_2, t_3) = t_1t_2 + t_1t_3 + t_2t_3$$

$$r(t_1, t_2, t_3) = (3t_1t_2t_3) + (t_1^2t_2 + t_1^2t_3 + t_2^2t_1 + t_2^2t_3 + t_3^2t_1 + t_3^2t_2)$$

Symmetric polynomials are important in the study of finite representations of the symmetric group. Additionally, they are important for setting up the calculations that are the goal of this research project.

Definition 4.1.3. The *elementary symmetric polynomials* in n variables are defined as:

$$e_1 = t_1 + t_2 + \cdots + t_n$$

$$e_2 = t_1t_2 + t_1t_3 + t_2t_3 + \cdots + t_{n-1}t_n$$

$$\vdots$$

$$e_k = \sum_{1 \leq j_1 < \cdots < j_k \leq n} t_{j_1}t_{j_2}\dots t_{j_k}$$

$$\vdots$$

$$e_n = t_1t_2\dots t_n$$

Theorem 4.1.4. *Every symmetric polynomial can be written in a unique way as a polynomial in the elementary symmetric polynomials* [1, Theorem 16.1.6]

Due to this theorem, elementary symmetric polynomials are fundamental to the study of all symmetric polynomials. The remaining symmetric polynomials defined in this section are pertinent to the calculations performed on the representations of S_n which stabilize as n tends toward infinity.

Definition 4.1.5. The *power sum* symmetric polynomials are defined as:

$$p_1 = t_1 + t_2 + \cdots + t_n$$

$$p_2 = t_1^2 + t_2^2 + \cdots + t_n^2$$

$$\vdots$$

$$p_k = t_1^k + t_2^k + \cdots + t_n^k$$

$$\vdots$$

$$p_n = t_1^n + t_2^n + \cdots + t_n^n$$

Definition 4.1.6. Let p_i' be a symmetric polynomial in n variables defined as:

$$p_i' = \frac{1}{i}\sum_{d|i} \mu(i/d)p_d$$

where $\mu(r)$ is the *mobius function*, and p_d is the dth power sum polynomial in n variables.

Definition 4.1.7. The *mobius function* μ is a function from the positive integers to $\{-1, 0, 1\}$:

$$\mu(d) = \begin{cases} 0 & \text{if } d \text{ is not square free} \\ 1 & \text{if } d \text{ is square free with an even number of prime factors} \\ -1 & \text{if } d \text{ is square free with an odd number of prime factors} \end{cases}$$

Definition 4.1.8. Take some symmetric polynomial g and non-negative integer j. Let the binomial symmetric polynomial $\binom{g}{j}$ be defined as:

$$\binom{g}{j} = \frac{g(g-1)\dots(g-j+1)}{j!}$$

5. Representations of S_n

The representations of S_n have a rich theory which will be key to understand the results of this thesis. The two simplest representations of S_n are the trivial representation sending each element of S_n to 1 and the sign representation given by $\rho(\sigma) = \text{sgn}(\sigma)$. These are the only one dimensional representations of S_n, which the reader can verify as an exercise. The next obvious place to look for representations of S_n is the action of permuting the coordinates of $\mathbb{C}^n$.

5.1. Standard Representation.

Definition 5.1.1. For any S_n, there is an associated permutation representation $\rho : S_n \to \mathbb{C}^n$:

$$\sigma \in S_n : \rho(\sigma)(v_1, v_2, \dots, v_n) = (v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n)})$$

The permutation representation of S_n is not irreducible, since the subspace $U \subset \mathbb{C}^n$ spanned by the vector $(1, 1, \dots, 1)$ is invariant under S_n, and is therefore the trivial representation. Therefore, the permutation representation can be decomposed as $V \oplus U$ for some $n-1$ dimension representation V and the trivial representation U. Denote V the *standard representation*.

Lemma 5.1.2. *The standard representation V of S_n is irreducible.*

Proof. The subspace V is complementary to the subspace of $\mathbb{C}^n$ generated by $(1, 1, \ldots, 1)$. Thus, V can be taken to be the subspace:

$$V = \{(v_1, v_2, \ldots, v_n) \in \mathbb{C}^n : v_1 + v_2 + \cdots + v_n = 0\}$$

To show that V is irreducible, we must show that there is no proper nonzero subspace of V that is invariant under the permutation representation. Take some non-zero subspace $W \subset V$ that is invariant under S_n. Let us show that $W = V$, which will imply V is irreducible.

Take a non-zero vector $u = (u_1, u_2, \ldots u_n) \in W$ that has a maximal number of 0 coordinates among the non-zero vectors of W. Since $u_1 + u_2 + \cdots + u_n = 0$ and not all the terms are zero, there is some u_i, u_j such that $u_i \neq u_j$ and u_i, u_j are non-zero. Since W is invariant by the action of S_n, W contains all permutations of the coordinates of u. Thus, W contains a vector

$$v = (v_1, v_2, \ldots v_n)$$

such that $v_1 \neq v_2$, v_1 and v_2 are non-zero, and v has a maximal number of 0 coordinates. Then, W must contain the vector $\rho((12))(v)$. Since W is closed under linear combinations, W contains:

$$\begin{aligned} v' = v - \frac{v_2}{v_1}\rho((12))(v) &= (v_1, v_2, \ldots v_n) - \frac{v_2}{v_1}\Big(v_2, v_1, v_3, \ldots v_n\Big) \\ &= \left(v_1 - \frac{v_2^2}{v_1}, 0, \frac{v_3v_1 - v_3v_2}{v_1}, \ldots, \frac{v_nv_1 - v_nv_2}{v_1}\right) \\ &= \left(v_1 - \frac{v_2^2}{v_1}, 0, v_3\frac{v_1 - v_2}{v_1}, \ldots, v_n\frac{v_1 - v_2}{v_1}\right) \\ &= \frac{v_1 - v_2}{v_1}\Big(v_2, 0, v_3, \ldots, v_n\Big) \end{aligned}$$

For $i \geq 3$, the ith coordinate of v' is 0 if and only if v_i is zero since $v_1 \neq v_2$. Therefore, v' has at least 1 more zero coordinate than v, since v_1, v_2 are non-zero by assumption and the second coordinate of v' is zero. Since v was taken to be a non-zero vector in W with a maximal number of zero coordinates, v' must be the zero vector. Therefore, $v_3, v_4, \ldots v_n$ must be equal to 0, so v is of the form:

$$v = (v_1, -v_1, 0, \ldots 0)$$

Therefore, since W is invariant by the permutation action of S_n, W contains the vectors:

$$e_1 = (v_1, -v_1, 0, \ldots 0)$$

$$e_2 = (v_1, 0, -v_1, \ldots 0)$$

$$\vdots$$

$$e_{n-1} = (v_1, 0, \ldots, -v_1)$$

The vectors $e_1, \ldots e_{n-1}$ form a basis for V, so $W = V$ as desired. □

5.2. **Wedge Powers of Standard Representation.**

Lemma 5.2.1. *For all S_n and k, the representation $\wedge^k V$ (where V is the standard representation) is irreducible.*

The proof of this statement is excluded for brevity, but can be found in Fulton & Harris [5, Section 3.2]. The wedge powers of V are an important source of irreducible representations of S_n. Now let us describe the character of $\wedge^k V$. Let us find the character for the wedge powers of the permutation representation V' of S_n.

Lemma 5.2.2. *The character of $\wedge^k V'$ where V' is the permutation representation is equal to the kth symmetric polynomial e_k evaluated on the eigenvalues of $\rho_{V'}(\sigma)$.*

Once again, the proof is excluded for the sake of brevity and to avoid dealing hands on with the technicalities of the alternating powers.

Lemma 5.2.3. *The character of $\wedge^k V$, where V is the standard representation of S_n, is equal to the symmetric polynomial $e_k - e_{k-1} + \cdots + (-1)^{k-1} e_1$ evaluated on the eigenvalues of the linear transformation associated with V'.*

Proof. Let V' be the permutation representation of S_n. For any $\sigma \in S_n$, the character $\chi_{\wedge^k V'}$ is equal to the kth symmetric polynomial e_k evaluated on the eigenvalues of $\rho_{V'}(\sigma)$ by the lemma above. The proof of this statement is excluded for brevity. Recall that V' reduces into irreducible representations as $V' = V \oplus U$, where V is the standard representation and U is the trivial representation. Then,

$$\wedge^k V' = \wedge^k (V \oplus U) = \oplus_{a=0}^{k} \wedge^a V \otimes \wedge^{n-a} U$$

Since U is one dimensional, $\wedge^b U$ is empty for all $b > 1$. Therefore, we have:

$$\wedge^k V' = \wedge^k V \oplus \wedge^{k-1} V \otimes U = \wedge^k V \oplus \wedge^{k-1} V$$

Take the character of both sides:

$$e_k = \chi_{\wedge^k V} = \chi_{\wedge^k V'} + \chi_{\wedge^{k-1} V'}$$

Then, by substitution and telescoping sums, the character of $\wedge^k V$ is equal to:

$$\chi_{\wedge^k V} = \chi_{\wedge^k V'} - \chi_{\wedge^{k-1} V'} + \cdots + \chi_{(-1)^{k-1} V'} = e_k - e_{k-1} + \cdots + (-1)^{k-1} e_1$$

with the symmetric polynomials e_i evaluated on the eigenvalues of $\rho_{V'}(\sigma)$ for a given $\sigma \in S_n$.

□

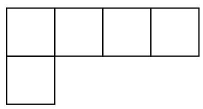
5.3. **Young Tableaux.** In general, it is a difficult problem to determine all of the irreducible representations of a finite group. However, this problem has been solved for the symmetric group with the use of *Young Tableaux*.

Definition 5.3.1. A *partition* of a positive integer n is an unordered set of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ such that $n = \lambda_1 + \cdots + \lambda_r$.

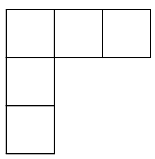
The partitions of a positive integer n are in correspondence with the conjugacy classes of S_n by cycle type.

Definition 5.3.2. To each partition of a positive integer n is an associated *Young Diagram*, which is a stack of boxes ordered by the number of boxes in each row.

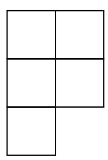
Example 5.3.3. The correspondence between Young Diagrams, cycle types (conjugacy classes of S_n), and partitions is demonstrated in the following examples with $n = 5$:



$5 = 4 + 1$ $\quad (abcd)(e)$



$5 = 3 + 1 + 1$ $\quad (abc)(d)(e)$



$5 = 2 + 2 + 1$ $\quad (ab)(cd)(e)$

Definition 5.3.4. A *Young Tableau* (plural, Tableaux) is a Young Diagram with a specific numbering, as in the following:

1	2	3
4	5	

Young Tableaux with this specific numbering, left to right, top to bottom, provide a framework for a description of all irreducible representations of S_n. Take any partition λ of n. Then, let λ also denote the Young Tableaux with numbers 1 through n written in the boxes left to right, top to bottom. Then, define the following two subgroups of S_n:

$$P = \{g \in S_n \mid g \text{ preserves each row of } \lambda\}$$

$$Q = \{g \in S_n \mid g \text{ preserves each column of } \lambda\}$$

Then in the group algebra $\mathbb{C}S_n$ (a $\mathbb{C}$ vector space with basis vectors v_σ for each $\sigma \in S_n$, and multiplication defined by linearity from multiplication in S_n), introduce two elements corresponding to the subgroups P, Q:

$$a = \sum_{g \in P} g \qquad b = \sum_{g \in Q} \operatorname{sgn}(g) g$$

Finally, define the *Young Symmetrizer* as

$$c = ab \in \mathbb{C}S_n$$

Then, the image of c on $\mathbb{C}S_n$ is an irreducible representation (with representation induced from the group algebra), and every irreducible representation of S_n can be obtained uniquely in this way [5, Section 4.1]. As an example, the wedge power $\wedge^k V$ of S_n is given by the following Young tableau

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & \multicolumn{1}{c}{\ldots} & \multicolumn{1}{|c|}{\scriptstyle n-k} \\
\hline
\multicolumn{1}{c}{\vdots} \\
\hline
\scriptstyle n-1 \\
\hline
n \\
\hline
\end{array}$$

5.4. **Frobenius' Character Formula.** Now that we have an explicit description of the irreducible representations of S_n, the next natural question is whether a similar description exists for the characters. This description does exist and is given by *Frobenius' Character Formula.* There is considerable set up to describe the formula.

Take any partition λ of n and let V_λ be the associated representation described in the previous section. Let C_i denote the conjugacy class in S_n determined by the sequence

$$(i_1, i_2, \ldots, i_n)$$

where $\sum_j j \cdot i_j = n$, and C_i represents the conjugacy class of S_n with elements having i_1 1-cycles, i_2 2-cycles, $\ldots$, i_n n-cycles. Introduce formal variables $x_1, x_2, \ldots, x_k$, where k is at least as large as the number of rows in λ. Define the *discriminant* of the x_i by the symmetric polynomial

$$\Delta(x) = \prod_{i<j} (x_i - x_j)$$

And define the power sum symmetric polynomials:

$$p_1 = x_1 + x_2 + \cdots + x_n$$

$$p_2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\vdots$$

$$p_n = x_1^n + x_2^n + \cdots + x_n^n$$

If λ has rows of length $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$, then define

$$l_1 = \lambda_1 + k - 1,\ l_2 = \lambda_2 + k - 2, \ldots,\ l_k = \lambda_k$$

Theorem 5.4.1. ***(Frobenius' Character Formula)*** *The character of the representation V_λ evaluated on the conjugacy class C_i is given by the coefficient of $x_1^{l_1} x_2^{l_2} \ldots x_3^{l_k}$ in the following symmetric polynomial:*

$$\chi_\lambda(C_i) = \left[\Delta(x) \cdot \prod_j p_j(x)^{i_j} \right]_{\textit{coefficient of } x_1^{l_1} x_2^{l_2} \ldots x_k^{l_k}}$$

This theorem is essential in the study of the representations of the symmetric group. This result is the primary tool used in the computational results of this research.

6. Representation Stability

6.1. **Topology of Configuration Space.**

Definition 6.1.1. $\text{Conf}_n(\mathbb{F})$ is the space of all monic square free polynomials of degree n with coefficients in a field $\mathbb{F}$. $\text{PConf}_n(\mathbb{F})$ is the subspace of $\mathbb{F}^n$ such that all coordinates are distinct:

$$\mathrm{PConf}_n(\mathbb{F}) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n \mid x_i \neq x_j\}$$

Notice that $\mathrm{PConf}_n(\mathbb{F})$ has a natural S_n action permuting the coordinates. When $\mathbb{F}$ is algebraically closed, the quotient of this action is $\mathrm{Conf}_n(\mathbb{F})$. Let us see why this is the case. Consider the continuous map $\pi : \mathrm{PConf}_n(\mathbb{F}) \to \mathrm{Conf}_n(\mathbb{F})$ defined by the following:

$$\pi((x_1, x_2, \ldots, x_n)) \to (t - x_1)(t - x_2) \ldots (t - x_n)$$

Notice that π is invariant on the permutation action of S_n, so it factors through $\mathrm{PConf}_n(\mathbb{F})/S_n$. If $\mathbb{F}$ is an algebraically closed field, then every square free polynomial can be factored into distinct linear factors. Therefore, π is surjective, so the induced map from $\mathrm{PConf}_n(\mathbb{F})/S_n$ to $\mathrm{Conf}_n(\mathbb{F})$ is a homeomorphism, and $\mathrm{PConf}_n(\mathbb{F})$ is thus an S_n Galois cover of $\mathrm{Conf}_n(\mathbb{F})$.

The action of S_n on $\mathrm{PConf}_n(\mathbb{C})$ induces an action of S_n on its cohomology $H^i(\mathrm{PConf}_n(\mathbb{C}); R)$ for any choice of coefficients R. If R is a field, this gives the cohomology $H^i(\mathrm{PConf}_n(\mathbb{C}); R)$ the structure of a representation of S_n.

In the 1980's, Lehrer and Solomon provided an explicit description for $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C})$ as a representation of S_n.

Theorem 6.1.2. *(Lehrer and Solomon, 1986)* [8] *As a representation of S_n,*

$$H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C}) \cong \bigoplus_{\mu} \mathrm{Ind}_{Z(c_\mu)}^{S_n} \xi_\mu$$

summed over conjugacy classes of μ with $n - i$ cycles in their cycle type.

In this formula, c_μ is a representative of the conjugacy class μ, and $Z(c_\mu)$ is the centralizer of c_μ in S_n. ξ_μ is a one-dimensional representation of $Z(c_\mu)$ given by the following. Suppose that μ is the set of elements with cycle type of i_1 one cycles,

i_2 two cycles, and so on, so $\sum i_j = n$. Then,

$$Z(c_\mu) = \prod_{j=1}^{n} (\mathbb{Z}/j)^{i_j} \rtimes S_{i_j}$$

where S_{i_j} acts on $(\mathbb{Z}/j)^{i_j}$ by permutation. Then, we define ξ_μ on each term $(\mathbb{Z}/j)^{i_j} \rtimes S_{i_j}$ by

$$\xi_\mu(k, \sigma) = e^{2\pi i k/j}(-1)^{(j+1)\operatorname{sgn}\sigma}$$

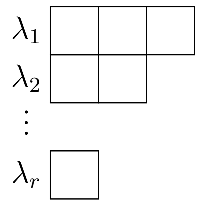
so ξ_μ acts on $\mathbb{Z}/j$ by sending 1 to the jth root of unity, and acts on S_{i_j} trivially if j is odd and as the sign representation if j is even.

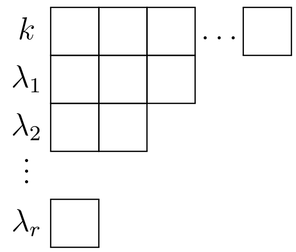
There is not currently an explicit way of converting this formula to a decomposition of $H^i(\mathrm{PConf}_n(\mathbb{C}))$ into irreducibles. However, there are useful properties of this formula which will become apparent in fugure sections.

6.2. **Character Polynomials and Families of Representations.** The main result of representation stability on $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q})$ is that in some sense, the limit $\lim_{n\to\infty} H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q})$ converges when viewed representation of $\lim_{n\to\infty} S_n$. The immediate problem with this statement is there is not an immediate way to recognize a limit of representations of S_n as $n \to \infty$ as "converging". There is a natural way to extend a representation of S_n to S_{n+1} which will allow us to define a notion of "stability" of S_n representations.

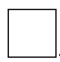
Suppose λ is a Young Diagram representing an irreducible representation of S_n. Then, to extend λ to a representation of S_{n+1}, we will add a box to the first row of λ. This results in a family of Young Tableau of different groups $\{S_n, S_{n+1}, \dots\}$. The characters of each of these representations is given simultaneously by a *character polynomial*, which is the reason why adding boxes to the first row is a natural way to extend representations of S_n to higher groups.

Definition 6.2.1. Let λ be a Young Tableau with λ_i boxes in the ith row. For $k \geq \lambda_1$, let λ_k be the Young Tableau with k boxes in the first row and λ_i boxes in the $i+1$th row. Then, let V_{λ_k} be the irreducible representation given by λ_k:

$\lambda =$ 

V_{λ_k} irreducible representation given by 

Then, we say $\mathscr{V}_\lambda := \{V_k^\lambda \mid k \geq \lambda_1\}$ is the *family of irreducible representations given by the Young Tableau* λ.

Example 6.2.2. Let $\lambda =$ . Then, $\mathscr{V}_\lambda$ is the set of standard representations of S_n for $n > 1$. In particular, $V_n^\lambda \in \mathscr{V}_\lambda$ is the standard representation of S_{n+1}. Let χ_n denote the character χ_n, and let σ be an element of S_{n+1} with μ_1 one cycles. Since $V_n \oplus U_{n+1} = P_{n+1}$ where U_{n+1} is the trivial representation and P_{n+1} is the permutation representation, we have that

$$\chi_n(\sigma) = \mu_1 - 1$$

Notice that the formula for χ_n does not involve n: it is written solely in terms of the number of 1 cycles in σ. In this way, the elements of $\mathscr{V}$ can be identified together by their characters. Even though each V_n^λ is a representation of a different symmetric group, they all have the same character in terms of the number of 1 cycles of σ. This is the motivating example for *character polynomials*, which extend this idea to all such families of irreducible representations of S_n.

Definition 6.2.3. For an element $\sigma \in S_n$, define

$$c_i(\sigma) := \text{number of } i \text{ cycles in } \sigma$$

Definition 6.2.4. Let $P \in \mathbb{Q}[x_1, x_2, \ldots,]$ be a polynomial of finitely many terms. For all n, define $P : S_n \to \mathbb{Q}$ by

$$P(\sigma) := P(c_1(\sigma), c_2(\sigma), \ldots, c_n(\sigma))$$

With respect to the function $P : S_n \to \mathbb{Q}$, P is called a *character polynomial*. The degree of a character polynomial similar to the usual polynomial degree, except $\deg x_k = k$, and is extended to remaining polynomials by linearity. For instance, treating $P(x_1, x_2, \ldots) = x_1 x_2 x_4$ as a character polynomial,

$$\deg P = \deg x_1 x_2 x_4 = \deg x_1 + \deg x_2 + \deg x_4 = 7$$

Notice that a character polynomial P is a class function on S_n since cycle type is independent of conjugation. In the case of the standard representation where λ has a single row of length 1, all of the standard representations $V_{\lambda_k} \in \mathscr{V}_\lambda$ are given simultaneously by the character polynomial $P = x_1 - 1$. We might hope that this holds for all families of representations $\mathscr{V}$ - it turns out that this is the case, which is a corollary of the Frobenius Character Formula.

Theorem 6.2.5. *Suppose $\mathscr{V}_\lambda$ is a family of irreducible representation of S_n given by a Young Tableau λ with rows of length $\lambda_1, \ldots, \lambda_r$. Then there is a unique polynomial $P \in \mathbb{C}[x_1, x_2, \ldots]$ such that for all representations $V_{\lambda_k} \in \mathscr{V}_\lambda$, the character of V_{λ_k} is given by the character polynomial P.*

Proof. Choose arbitrary $k \geq \lambda_1$, and let us consider the character χ_{λ_k} of V_{λ_k} given by the Frobenius formula. Take a conjugacy class $C_i = (i_1, i_2, \ldots, i_r)$ with i_j j-cycles. Let $\Delta(x) = \prod_{a<b} x_a - x_b$ for formal variables $x_0, x_1, \ldots$. Furthermore, define nonnegative integers $l_0, l_1, \ldots, l_r$ by $l_0 = k + r$ and $l_j = \lambda_j + r - j$ for $j \geq 1$. Then by the Frobenius formula, we have:

$$\chi_{\lambda_k}(C_i) = \left[\Delta(x) \cdot \prod_j p_j(x)^{i_j}\right]_{\text{coefficient of } x_0^{l_0} x_1^{l_1} \ldots x_r^{l_r}}$$

We aim to show that the expression above for $\chi_\lambda(C_i)$ is a polynomial in $i_1, \ldots, i_n$. Notice that this expression is a homogeneous symmetric polynomial in $x_0, x_1, \ldots, x_n$, so any term of the form $x_0^m x_1^{l_1} \ldots x_r^{l_r}$ must have $m = l_0$. Thus, it suffices to solely consider the expression in terms of the exponents of $x_1, \ldots, x_k$. Let us compute an example to understand the general case. Let us compute the coefficient of the term $x_0^{k+r-m} x_1^m$:

$$\prod_j p_j(x)^{i_j} = \prod_j (x_0^j + x_1^j + x_2^j + \cdots + x_r^j)^{i_j}$$

The coefficient of $x_0^{k+r-m} x_1^m$ is equal to the number of ways to choose x_1^j terms in the above expansion so the sum of their exponents is m. The remaining terms must all be chosen to be the x_0 term, and thus are determined by the choices of x_1 terms. Therefore, the coefficient of $x_0^{k+r-m} x_1^m$ is equal to

$$\binom{i_1}{m} + \binom{i_1}{m-2}\binom{i_2}{1} + \cdots + \binom{i_m}{1}$$

which is a character polynomial in $i_1, \ldots, i_m$. For instance, the $\binom{i_1}{m}$ term corresponds to choosing m x_1 terms, and the $\binom{i_m}{1}$ term corresponds to choosing a single x_1^m term. More generally, the coefficient of $x_0^m x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}$ with $m = k + r - \sum m_i$ is equal to the sum of all ways to choose a total of l_1 x_1 terms, l_2 x_2 terms, and so on in the product $\prod_j p_j(x)^{i_j}$. By similar reasoning, this can be expressed as a sum of products of binomials corresponding to the number of ways to choose each sum of x_j terms. Therefore, the coefficient of $x_0^m x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}$ for all choices of $m_1, \ldots, m_r$ is a finite degree character polynomial in $i_1, i_2 \ldots$. Furthermore, the discriminant $\Delta(x)$ is constant with respect to k. After expanding $\Delta(x)$, we obtain a sum of monomials $x_1^{a_0} x_1^{a_1} \ldots x_r^{a_r}$. Then, the coefficient of $x_0^{l_0} \ldots x_r^{l_r}$ in $\Delta(x) \cdot \prod_j p_j(x)^{i_j}$ is equal to the sum of the coefficients of $x_0^{l_0 - a_0} \ldots x_r^{l_r - a_k}$ for each monomial term of the determinant. Since each of these coefficients is a character polynomial in $i_1, \ldots, i_k$, their sum is a polynomial in $i_1, \ldots, i_k$, so the coefficient of $x_0^{l_0} \ldots x_r^{l_r}$ in $\Delta(x) \cdot \prod_j p_j(x)^{i_j}$ is a polynomial in $i_1, \ldots, i_k$ as desired. $\square$

This explains why $\mathscr{V}_\lambda$ gives a natural definition of a family of irreducible representations - all their characters are given simultaneously by a single character polynomial χ_λ.

6.3. **Decomposition of Cohomology into Irreducible Families.** A character polynomial P is a class function on S_n for all n, so the S_n inner product $\langle P, Q\rangle_{S_n}$ for a class function Q on S_n is defined in the usual way:

$$\langle P, Q\rangle_{S_n} = \frac{1}{n!}\sum_{\sigma\in S_n} P(\sigma)\overline{Q(\sigma)}$$

Theorem 6.3.1. *(Church, Farb, Ellenburg, 2013* [2]*) For all polynomials* $P \in \mathbb{C}[x_1, x_2, \ldots,]$, *the limit*

$$\lim_{n\to\infty} \langle P, H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q})\rangle_{S_n}$$

exists and is constant for $n \geq 2i + \deg P$.

This theorem implies that the inner product of a family $\mathscr{V}_\lambda$ of irreducible representations and $H^i(\mathrm{PConf}(\mathbb{C}); \mathbb{Q})$ stabilizes as $n \to \infty$. This is the canonical example of representation stability. Benson and Farb expand on this result to more general topological spaces with S_n actions that resemble $\mathrm{PConf}(\mathbb{C})$, which they term FI-CHA for "FI-complement of hyperplane arrangement" [2] [3] [4].

6.4. **Research Question.** When studying a representation V of S_n, one of the best ways to understand V is to decompose it into irreducibles. The goal of this project is to determine for each family $\mathscr{V}_\lambda$ of irreducible representations with character polynomial P the limit $\langle P, H^i(\mathrm{PConf}(\mathbb{C}); \mathbb{Q})\rangle$ for each degree of cohomology i.

$$\langle P, H^i(\mathrm{PConf}(\mathbb{C}); \mathbb{Q})\rangle := \lim_{n\to\infty} \langle P, H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q})\rangle$$

This is the "stable" version of decomposing $H^i(\mathrm{PConf}(\mathbb{C}); \mathbb{Q})$ into irreducibles.

7. Polynomial Statistics

The tools of étale cohomology provide a connection between cohomology and polynomial statistics over a finite field, which allow for the computation of cohomology of $H^i(\mathrm{PConf}(\mathbb{C});\mathbb{Q})$ through arithmetic.

7.1. Polynomial Statistics.

For p a power of a prime, let $\mathbb{F}_p$ denote the field with p elements. Recall that $\mathrm{Conf}_n(\mathbb{F}_p)$ denotes the set of degree n square free polynomials with coefficients in $\mathbb{F}_p$. For P a character polynomial, let us describe a function $P : \mathrm{Conf}_n(\mathbb{F}_p) \to \mathbb{Q}$.

Take $f \in \mathrm{Conf}_n(\mathbb{F}_p)$, and fix a splitting field extension $\mathbb{F}_q$ of f. Recall that the p Frobenius action on $\mathbb{F}_q$ acts by $\mathrm{Frob}_p(a) = a^q$ and the stabilizer of Frob_p is $\mathbb{F}_p \subset \mathbb{F}_q$. Considered as a polynomial in $\mathbb{F}_p[x]$, f splits into n terms:

$$f(x) = (x - r_1)(x - r_2) \dots (x - r_n)$$

Since Frob_p is a field automorphism,

$$\mathrm{Frob}_p(f) = (x - \mathrm{Frob}_p(r_1))(x - \mathrm{Frob}_p(r_2)) \dots (x - \mathrm{Frob}(r_n))$$

Furthermore, $\mathrm{Frob}_p(f) = f$ since f has coefficients in $\mathbb{F}_p$. Therefore, Frob_p induces a permutation $\sigma_f \in S_n$ of the roots $r_1, \dots, r_n$. While σ_f is dependent on the labeling of the roots $r_1, \dots, r_n$, it is unique up to conjugation. Therefore, for a class function χ of S_n, $\chi(\sigma_f)$ is well defined.

Definition 7.1.1. Let $f \in \mathrm{Conf}_n(\mathbb{F}_p)$ and $P \in \mathbb{Q}[x_1, x_2, \dots]$ a character polynomial. Let σ_f be the permutation of the roots of f by the Frobenius automorphism Frob_p, determined up to conjugation. Then, define

$$P(f) = P(\sigma_f) = P(c_1(\sigma_f), c_2(\sigma_f), \dots)$$

Theorem 7.1.2. *(Church, Farb, and Ellenburg, 2013)* [2] *For any character polynomial $P \in \mathbb{Q}[x_1, x_2, \dots]$, the following two limits exist and are equal:*

$$(1) \qquad \lim_{n\to\infty} \sum_{i=0}^{\infty} (-1)^i \frac{\langle P, H^i(\mathrm{PConf}_n(\mathbb{C}))\rangle_{S_n}}{q^i} = \lim_{n\to\infty} q^{-n} \sum_{f\in \mathrm{Conf}_n(\mathbb{F}_q)} P(\sigma_f)$$

In particular, both the limit on the left and the series on the right converge to a power series in q^{-1} with the same coefficients.

Example 7.1.3. Consider the character polynomial $P = 1$, which is the character polynomial of the family of trivial representations. The action of P on a square free polynomial f is therefore just $P(f) = 1$. Therefore, the right side of equation (1) becomes $\lim_{n\to\infty} q^{-n}|\mathrm{Conf}_n(\mathbb{F}_q)|$. By the well known formula for $|\mathrm{Conf}_n(\mathbb{F}_q) = q^n - q^{n-1}|$, this expression converges (and is constant for $n \geq 2$) to the power series $1 - q^{-1}$. Therefore,

$$\lim_{n\to\infty} \sum_{i=0}^{\infty} (-1)^i \frac{\langle P, H^i(\mathrm{PConf}_n(\mathbb{C}))\rangle_{S_n}}{q^i} = 1 - q^{-1}$$

Equating like terms, this implies that

$$\lim_{n\to\infty} \langle P, H^0(\mathrm{PConf}_n(\mathbb{C}))\rangle = 1$$

$$\lim_{n\to\infty} \langle P, H^1(\mathrm{PConf}_n(\mathbb{C}))\rangle = 1$$

$$\lim_{n\to\infty} \langle P, H^i(\mathrm{PConf}_n(\mathbb{C}))\rangle = 0 \qquad \text{for all } i \geq 2$$

Example 7.1.4. Consider the character polynomial $P = x_1 - 1$, which is the character polynomial of the family of standard representations. The action of P on a square free polynomial f is equal to $c_1(\sigma_f) - 1$. Recall that $c_1(\sigma_f)$ counts the number of fixed points of the permutation of σ_f, and a fixed point of σ_f corresponds to a root of f in Frob_p. Therefore, the right side of equation (1) becomes

$$\lim_{n\to\infty} q^{-n} \sum_{f\in \mathrm{Conf}_n(\mathbb{F}_q)} (\text{number of roots of } f \text{ in } \mathbb{F}_q) - 1$$

The right side can be explicitly computed through a combinatorial argument by stating the number of irreducible polynomials over $\mathbb{F}_q$ of a given degree as a generating function. This results in the following power series in q:

$$-q^{-1}+2q^{-2}-2q^{-3}+2q^{-4}+\cdots=\lim_{n\to\infty} q^{-n}\sum_{f\in \mathrm{Conf}_n(\mathbb{F}_q)} (\text{number of roots of } f \text{ in } \mathbb{F}_q)-1$$

Therefore,

$$\lim_{n\to\infty}\sum_{i=0}^{\infty}(-1)^i\frac{\langle P, H^i(\mathrm{PConf}_n(\mathbb{C}))\rangle_{S_n}}{q^i}=-q^{-1}+2q^{-2}-2q^{-3}+2q^{-4}+\dots$$

Equating like terms in the power series, this implies that

$$\lim_{n\to\infty}\langle P, H^0(\mathrm{PConf}_n(\mathbb{C}))\rangle=0$$

$$\lim_{n\to\infty}\langle P, H^1(\mathrm{PConf}_n(\mathbb{C}))\rangle=1$$

$$\lim_{n\to\infty}\langle P, H^i(\mathrm{PConf}_n(\mathbb{C}))\rangle=2 \qquad \text{for all } i\geq 2$$

8. Computations of Limiting Multiplicities

Using combinatorial identities for polynomial statistics of $\mathrm{Conf}_n(\mathbb{F}_q)$, it is possible to solve explicitly for the right hand side of equation (1) for a given character polynomial. Furthermore, for a specific Young Tableau λ, the Frobenius formula gives a way of determining the character polynomial for $\mathscr{V}_\lambda$. Combining these two steps, we obtain an algorithm to compute the limiting multiplicities

$$\lim_{n\to\infty}\langle \mathscr{V}_\lambda, H^i(\mathrm{PConf}(\mathbb{C}))\rangle$$

as the coefficients of a power series in q^{-1} for all choices of Young Diagrams λ.

8.1. **Algorithm Description.** The algorithm utilized was derived by Dr. Sean Howe [7], [6] and myself, which sends the representation V_λ to the desired power series.

First, we express the character $\mathscr{V}_\lambda$ as a character polynomial $P \in \mathbb{Q}[i_1, i_2, \dots]$ in the following basis

$$\mathscr{C} := \left\{ \binom{i_1}{j_1} \dots \binom{i_r}{j_r} \mid j_i \in \mathbb{N} \right\}$$

for all character polynomials. Notice that the natural way of writing the Frobenius formula is in this basis, so working from the Frobenius formula, the polynomial P is already in the correct basis. For a sketch of how to compute a character polynomial from the Frobenius formula, reference the proof of lemma 6.2.5.

Along with being the natural form of the Frobenius formula, the basis $\mathscr{C}$ is a natural basis for class functions on S_n since for a permutation σ of S_{kj}, we have:

$$\binom{c_k}{j}(\sigma) = \begin{cases} 1 & \text{if the cycle type of } \sigma \text{ is exactly } j \text{ } k\text{-cycles} \\ 0 & \text{else} \end{cases}$$

Once the character of $\mathscr{V}_\lambda$ is expressed as a character polynomial P written as a sum of terms in the basis $\mathscr{C}$, we send each term $\binom{c_k}{j}$ to a power series in q^{-1} by a function denoted as CTP_q here, for "character to power series"

$$\mathrm{CTP}_q\left(\binom{i_k}{j}\right) := \binom{\frac{1}{k}\sum_{d|k} q^d \mu(k/d)}{j} (q^{-k} - q^{-2k} + \dots)^j$$

where μ is the Möbius function. This equation represents the fact that

$$\lim_{n\to\infty} q^{-n} \sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} \binom{i_k}{j}(\sigma_f) = \binom{\frac{1}{k}\sum_{d|k} q^d \mu(k/d)}{j} (q^{-k} - q^{-2k} + \dots)^j$$

which can be proven through existing combinatorial techniques for polynomial statistics. The notation "CTP_q" is solely to reduce notation. CTP_q is extended to all basis elements of $\mathscr{C}$ by multiplying term by term, and extended by linearity to remaining terms:

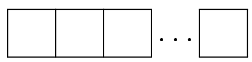
$$\mathrm{CTP}_q\left(\binom{i_1}{j_1} \dots \binom{i_r}{j_r}\right) = \mathrm{CTP}_q\left(\binom{i_1}{j_1}\right) \dots \mathrm{CTP}_q\left(\binom{i_r}{j_r}\right)$$

CTP_q can be extended in this way because the sum of polynomial statistics over $\mathrm{Conf}_n(\mathbb{F}_q)$ of different length cycles are independent as $n \to \infty$. Finally, we multiply this final power series by $1 - q^{-1}$ (normalizing by the number of elements in $\mathrm{PConf}_n(\mathbb{F}_q)$, which yields the desired limit

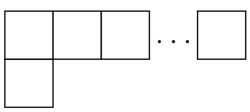
$$\lim_{n\to\infty} \sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} P(\sigma_f)$$

as a power series in q^{-1}

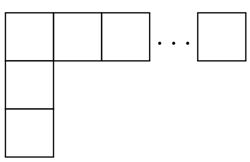
8.2. **Previous Results.** Benson and Farb compute $\langle \mathscr{V}_\lambda, H^i(\mathrm{PConf}(\mathbb{C}); \mathbb{Q})$ for the following Young Tableaus.



$$\longrightarrow \quad 1 - q^{-1}$$



$$\longrightarrow \quad -q^{-1} + 2q^{-2} - 2q^{-3} + 2q^{-4} - 2q^{-5} + 2q^{-6} - 2q^{-7} + \dots$$



$$\longrightarrow \quad 2q^{-2} - 5q^{-3} + 6q^{-4} - 7q^{-5} + 10q^{-6} - 13q^{-7} + \dots$$

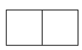
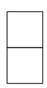
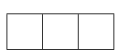
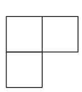
8.3. **Table of Complete Computational Results.** The appendix contains a table of computational results. For each row, there is a Young Tableau λ in the first column and a power series $p(q^{-1})$ in q^{-1} with alternating sign integer entries in the second column. The power series p represents the stable multiplicity

$$\langle \mathscr{V}_\lambda, H^i(\mathrm{PConf}(\mathbb{C}); \mathbb{Q}) \rangle$$

for all i. In particular, the coefficient of q^{-i} in $p(q^{-1})$ is equal to

$$\lim_{n\to\infty} (-1)^i \langle \mathscr{V}_\lambda, H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{Q}) \rangle_{S_n}$$

Here, we include a single page of the computational results.

	$-q^{-1}+2q^{-2}-2q^{-3}+2q^{-4}-2q^{-5}+2q^{-6}-2q^{-7}+2q^{-8}-2q^{-9}+2q^{-10}-2q^{-11}+2q^{-12}-2q^{-13}+2q^{-14}-2q^{-15}+2q^{-16}-2q^{-17}+2q^{-18}-2q^{-19}+2q^{-20}-2q^{-21}+2q^{-22}-2q^{-23}+2q^{-24}-2q^{-25}+2q^{-26}-2q^{-27}+2q^{-28}-2q^{-29}+2q^{-30}+\dots$
	$-q^{-1}+2q^{-2}-3q^{-3}+6q^{-4}-9q^{-5}+10q^{-6}-11q^{-7}+14q^{-8}-17q^{-9}+18q^{-10}-19q^{-11}+22q^{-12}-25q^{-13}+26q^{-14}-27q^{-15}+30q^{-16}-33q^{-17}+34q^{-18}-35q^{-19}+38q^{-20}-41q^{-21}+42q^{-22}-43q^{-23}+46q^{-24}-49q^{-25}+50q^{-26}-51q^{-27}+54q^{-28}-57q^{-29}+58q^{-30}+\dots$
	$2q^{-2}-5q^{-3}+6q^{-4}-7q^{-5}+10q^{-6}-13q^{-7}+14q^{-8}-15q^{-9}+18q^{-10}-21q^{-11}+22q^{-12}-23q^{-13}+26q^{-14}-29q^{-15}+30q^{-16}-31q^{-17}+34q^{-18}-37q^{-19}+38q^{-20}-39q^{-21}+42q^{-22}-45q^{-23}+46q^{-24}-47q^{-25}+50q^{-26}-53q^{-27}+54q^{-28}-55q^{-29}+58q^{-30}+\dots$
	$q^{-2}-4q^{-3}+8q^{-4}-14q^{-5}+24q^{-6}-35q^{-7}+46q^{-8}-61q^{-9}+79q^{-10}-97q^{-11}+117q^{-12}-140q^{-13}+165q^{-14}-192q^{-15}+220q^{-16}-250q^{-17}+284q^{-18}-319q^{-19}+354q^{-20}-393q^{-21}+435q^{-22}-477q^{-23}+521q^{-24}-568q^{-25}+617q^{-26}-668q^{-27}+720q^{-28}-774q^{-29}+832q^{-30}+\dots$
	$2q^{-2}-7q^{-3}+16q^{-4}-30q^{-5}+47q^{-6}-68q^{-7}+94q^{-8}-123q^{-9}+156q^{-10}-194q^{-11}+235q^{-12}-280q^{-13}+330q^{-14}-383q^{-15}+440q^{-16}-502q^{-17}+567q^{-18}-636q^{-19}+710q^{-20}-787q^{-21}+868q^{-22}-954q^{-23}+1043q^{-24}-1136q^{-25}+1234q^{-26}-1335q^{-27}+1440q^{-28}-1550q^{-29}+1663q^{-30}+\dots$

8.4. **Algorithm Correctness.** The code used to generate these results has been tested extensively for correctness at every stage of the algorithm. Furthermore, the algorithm agrees with the existing results for the trivial representation, $\lambda = \square$, and $\lambda = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. Another nice feature of the program which points to its correctness is that for a general character polynomial P, even with integer coefficients, $\mathrm{CTP}_q(P)$ is not a power series with alternating integer coefficients. The coefficients can be rational with potentially large denominator, and the sign can fluctuate between positive, negative, and zero in any periodic fashion. Therefore, because the algorithm returns power series with alternating integer coefficients, it is unlikely there is a small computational error in the algorithm.

9. Observations and Conjectures

Using the data found on the stable multiplicities of irreducible representations in $H^i(\mathrm{PConf}(\mathbb{C}), \mathbb{Q})$, we conjecture bounds for the leading degree of the power series $\mathrm{CTP}_q(\chi_\lambda)$ (i.e., the first term with non-zero coefficient). Only the upper bound has currently been proven.

9.1. Upper Bound on Leading Degree.

Lemma 9.1.1. *Let λ be a Young-diagram with k boxes and $\mathscr{V}_\lambda$ its associated family of irreducible representations with character polynomial χ_λ. Let $CTP_q(\chi_\lambda) = a_0 - a_1 q^{-1} + a_2 q^{-2} - \ldots$. Then, the first non-zero coefficient a_r has $r \geq k/2$.*

This lemma is proven using the Lehrer-Solomon description of $H^i(\mathrm{PConf}_n(\mathbb{C}); C)$ as an S_n representation [8]. We will prove the lemma using the fact that if V_λ is a subrepresentation of $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C})$, $\dim V_\lambda \leq \dim H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C})$. First we prove that the dimension of $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C})$ is a polynomial of degree $2i$ in n. Then we prove that the dimension of V_λ for a tableau λ with k boxes is a degree k polynomial in n. Therefore, for V_λ to be a subrepresentation of $H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C})$, we must have $k \leq 2i$, and so the first non-zero coefficient a_r of $\mathrm{CTP}_q(\chi_\lambda)$ is at least $a_{k/2}$.

Proof. Let us consider the dimension of $H^i(\mathrm{PConf}_n(\mathbb{C}); C)$ in terms of n. Recall that by Lehrer-Solomon,

$$H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C}) = \bigoplus_\mu \mathrm{Ind}_{Z(c_\mu)}^{S_n}(\xi_\mu)$$

Therefore,

$$\dim(H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C})) = \sum_\mu \dim \mathrm{Ind}_{Z(c_\mu)}^{S_n}(\xi_\mu) = \sum_\mu [S_n : Z(c_\mu)]$$

Suppose c_μ has μ_j j-cycles for each j, such that $\sum_j \mu_j = n - i$ and $\sum_j j\mu_j = n$. Recall that:

$$Z(c_\mu) = S_{\mu_1} \rtimes \left((\mathbb{Z}/2)^{\mu_2} \times S_{\mu_2}\right) \ldots \left((\mathbb{Z}/r)^{\mu_r} \rtimes S_{\mu_r}\right)$$

Therefore,

$$|Z(c_\mu)| = \prod_j \mu_j! j^{\mu_j}$$

Since $\sum_j \mu_j = n - i$ and $\sum_j j\mu_j = n$, we have $\mu_1 \geq n - 2i$ by pigeonhole principle. Furthermore, $\mu_1 = n - 2i$ is achieved when $\mu_3 = \cdots = \mu_r = 0$. Therefore, in this case, we have:

$$[S_n : Z(c_\mu)] = \frac{n!}{(n-2i)! 2^{i+1}}$$

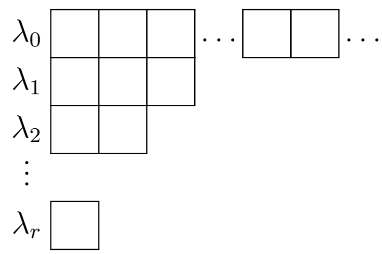
which is a polynomial of degree $2i$ in n. Furthermore, for any other choice of c_μ, we will have

$$[S_n : Z(c_\mu)] = \frac{n!}{\mu_1! \ldots \mu_r! r^{\mu_r}}$$

Since $\mu_1 \geq n - 2i$ and we fix each other μ_j as n tends to infinity, $[S_n : Z(c_\mu)]$ is a polynomial of degree $2i$ or less in P. Furthermore, this polynomial has a positive leading coefficient. Thus, $[S_n : Z(c_\mu)]$ is the sum of degree $2i$ or less polynomials with positive leading degree, and at least one polynomial of degree exactly $2i$ with positive leading coefficient. Therefore, $[S_n : Z(c_\mu)]$ is a polynomial of degree exactly $2i$ in n

as desired.

Furthermore, the dimension of a representation λ of the form:



can be computed by the hook formula [5]. As an example, let us compute the dimension of a representation given by a Young Diagram. In each box, write the number of boxes in that row and column, the *hook length*. For instance:

$V \Longleftrightarrow$

7	5	2	1
3	2		
2	1		
1			

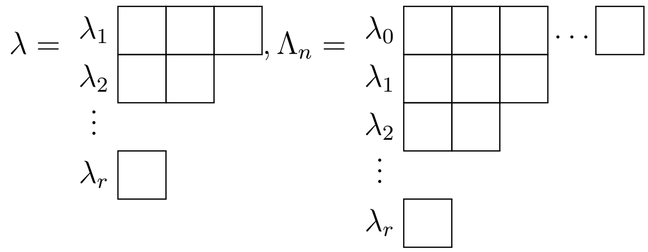
Then, the dimension of the irreducible representation V given by the specified Young Tableaux is equal to

$$\frac{n!}{\prod \text{hook lengths}}$$

so in this case,

$$\dim V = \frac{9!}{7 * 5 * 3 * 2 * 2 * 2} = 432$$

Now let us return to the situation of a family of irreducible representations. Let λ be a Young Diagram with k boxes, and let $\mathscr{V}_\lambda$ be the family of irreducible representations formed by adding boxes to a row above the first of λ. Let Λ_n be one of the elements of $\mathscr{V}_\lambda$ by adding $n-k$ boxes above λ.



After writing the hook lengths in each box of Λ_n, the hook length of the boxes along the top row will be at least $1, 2, \ldots, n-k$ reading right to left, where λ (below the first row) has k boxes. Therefore,

$$\prod_{\Lambda_n} \text{hook lengths} \geq (n-k)!$$

Therefore,

$$\dim \Lambda_n \geq \frac{n!}{(n-k)!}$$

so Λ_n has dimension bounded below by a monic degree k polynomial in n. Therefore, for any Young Diagram λ with k boxes and $i < k/2$, there is some n such that $\dim H^i(\mathrm{PConf}_n(\mathbb{C}); \mathbb{C}) < \dim \Lambda_n$. Therefore, it is impossible for $H^i(\mathrm{PConf}(\mathbb{C}); \mathbb{C})$ to stably contain the irreducible family $\mathscr{V}_\lambda$, so $\langle \mathscr{V}_\lambda H^i(\mathrm{PConf}(\mathbb{C}))\rangle = 0$ for $i < k/2$. Therefore, the coefficients a_l are zero for $l < k/2$ as desired. □

9.2. **Conjectures.** Using the algorithm described, many limiting multiplicities of irreducible representations were computed. We describe some conjectures from the data about behavior of the limiting multiplicities. Let λ be a Young-diagram with k boxes and $\mathscr{V}_\lambda$ its associated family of irreducible representations with character polynomial P. Let $\mathrm{CTP}_q(P) = a_0 - a_1 q^{-1} + a_2 q^{-2} - \ldots$. Then, the first non-zero coefficient a_r has $r \geq k/2$.

Conjecture 9.2.1. *So long as λ is nonempty ($\mathscr{V}_\lambda$ is not the family of trivial representations), the sequence $a_0, a_1, \ldots$ is non-decreasing.*

Remark. We have shown by approximating the function CTP_q that the sequence $a_0, a_1, \ldots$ in this setting is *eventually* non-decreasing for any nonempty diagram λ. However, we have still not provided any bounds on when it will become non-decreasing.

Conjecture 9.2.2. *The first non-zero term a_l of $a_0, a_1, \ldots$ satisfies $l \leq k$.*

The bound of $l = k$ is achieved if and only if λ is a vertical stack of k boxes, so $\mathscr{V}_\lambda$ is the family of kth wedge powers of the standard representation.

10. Conclusion

10.1. **Analysis.** This work was effective in computing explicit values in representation stability. The example of $\mathrm{PConf}_n(\mathbb{C})$ is the simplest example known of representation stability, so these computations provide thorough data to inform future discoveries and insights relating to representation stability.

10.2. **Future Research.** Completing the remainder of the conjectures described in section 9.2 are a natural next step . I am currently working on proving the remainder of these conjectures, and plan to publish my results in the future.

Representation stability has also been shown to apply to other topological spaces, and can be stated in even more generality. In this more general setting, there may be an analog of the computations performed here. One obstacle to this direction of future research is that $\mathrm{PConf}_n(\mathbb{C})$ yields the simplest example of representation stability and has a powerful connection between its cohomology and polynomial statistics. While other spaces with representation stability share a similar connection between cohomology and arithmetic, the exact statistics are generally more complicated than in the example of $\mathrm{PConf}_n(\mathbb{C})$. Therefore, it may be infeasible to compute limiting cohomology through polynomial statistics in a more general setting, although more research would be required to investigate the possibility of such an algorithm.

References

[1] Michael Artin. *Algebra*. Birkhäuser, 1998.

[2] Thomas Church, Jordan Ellenberg, and Benson Farb. Representation stability in cohomology and asymptotics for families of varieties over finite fields. *Contemporary Mathematics*, 620:1–54, 2014.

[3] Thomas Church and Benson Farb. Representation theory and homological stability. *Advances in Mathematics*, 245:250–314, oct 2013.

[4] Benson Farb. Representation stability, 2014.

[5] William Fulton and Joe Harris. *Representation theory: a first course*, volume 129. Springer Science & Business Media, 2013.

[6] Sean Howe. Motivic random variables and representation stability ii: Hypersurface sections. *Algebraic & Geometric Topology*, 2016.

[7] Sean Howe. Motivic random variables and representation stability, i: Configuration spaces. *Algebraic & Geometric Topology*, 20(6):3013–3045, dec 2020.

[8] G.I Lehrer and Louis Solomon. On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes. *Journal of Algebra*, 104(2):410–424, 1986.