Math 33a Practice first hour exam. **This is a closed book exam.**

1(30). Find the following limits (include calculations!):

a) $\lim_{x\to 0} \frac{x^2}{x+\sqrt{x}} = \lim_{x\to 0} \frac{\frac{1}{x}x^2}{\frac{1}{x}(x+\sqrt{x})} = \lim_{x\to 0} \frac{x}{1+\frac{1}{\sqrt{x}}} = \frac{0}{1+\frac{1}{\infty}} = \frac{0}{1} = 0$

(or use l'Hopital)

b) $\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} \overset{\left(\frac{-\infty}{\infty}\right)}{=} \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} \frac{x}{-1} = \frac{0}{-1} = 0$

c) $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x-\sin x}{x\sin x} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} \overset{\left(\frac{0}{0}\right)}{}$

$= \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2-0} = 0$

e) $\lim_{n\to\infty} \frac{n!}{2^n}$

$\frac{n!}{2^n} = \left(\frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}\cdot\overbrace{\frac{4}{2}\cdot\frac{5}{2}\cdot\ldots\cdot\frac{n}{2}}^{n-3 \text{ terms}}\right)$

$> \left(\frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}\right)(\overbrace{2\cdot 2\ldots\cdot 2}^{n-3}) \longrightarrow \infty$

so answer is ∞

f) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$

$L = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$

$\ln L = \lim_{x\to 0} \frac{1}{x}\ln(1+x) \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$

$\Rightarrow L = e^{\ln L} = e^1$

2(25). Determine which of the following sums converge. Carefully explain which test you are using.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$ Use limit comparison test with $\sum \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{\frac{1}{n^2-n+1}}{\frac{1}{n^2}} = \lim \frac{n^2}{n^2-n+1} = 1 < \infty$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ it follows that $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1} < \infty$.

↳ p test

b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+k}} \le \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} < \infty$ (p-test, $p = 3/2 > 1$.)

(comparison test)

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test since $1 \ge \frac{1}{2} \ge \frac{1}{3} \ge \cdots$ and $\frac{1}{n} \to 0$

d) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ Converges: compare with $\sum \frac{n^{\frac{1}{2}}}{n^2} = \sum \frac{1}{n^{3/2}} < \infty$ (limit comparison test)

Suffices to show $\lim \frac{a_n}{b_n} < \infty$

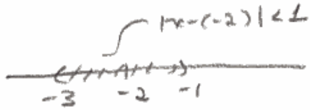
$$\frac{\frac{\ln n}{n^2}}{\frac{n^{1/2}}{n^2}} = \frac{\ln n}{n^{1/2}} \qquad \lim_{n\to\infty} \frac{\ln n}{n^{1/2}} = \lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2x^{1/2}}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0 \ (< \infty)$$

e) Determine where the series $\sum \frac{(x+2)^n}{n}$ converges (include a careful picture of the points x for which it converges).

$$\rho = \lim \frac{\frac{|x+2|^{n+1}}{n+1}}{\frac{|x+2|^n}{n}} = |x+2| \lim \frac{n}{n+1} = |x+2|$$

$\rho = |x+2| < 1$ convergence



$\rho = |x+2| > 1$ divergence

$x=-1$ $\sum^{\infty} \frac{1}{n}$ diverges (p-test, $p = 1$)

$x = 1$ $\sum_1^{\infty} \frac{(-1)^n}{n}$ converges (alt. series test)

3a) **Define** $\lim_{n\to\infty} x_n = L$.

for all $\varepsilon > 0$, there exists an N such that $n \ge N \Rightarrow |x_n - L| < \varepsilon$

b) **Prove** that if a sequence x_n converges, then there exists a constant M such that $x_n \le M$ for all n.

Letting $\varepsilon = 1$, we may choose N such that $n \ge N \Rightarrow |x_n - L| < 1$.

From the picture $n \ge N \Rightarrow x_n < L+1$.

Let $M = \max\{x_1, \dots, x_{N-1}, L+1\}$. Then

$x_n \le M$ for all n.

c) **Define**: the series $\sum_{n=1}^{\infty} a_n$ converges.

This means that the ~~sequence~~ of partial sums $S_N = \sum_{n=1}^{N} a_n$ converges.

d) **Prove** that if $0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

Since $a_n \geq 0$,

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$

is an increasing sequence. By the "axiom of continuity" $S_1, S_2, \ldots$ will converge if it is bounded above.

But we have

$$S_N = \sum_{n=1}^{N} a_n \le \sum_{n=1}^{N} b_n \le \sum_{n=1}^{\infty} b_n < \infty$$

Hence $S_1, S_2, \ldots \le M = \sum_{n=1}^{\infty} b_n$. It follows that $S_1, S_2, \ldots$ converges.