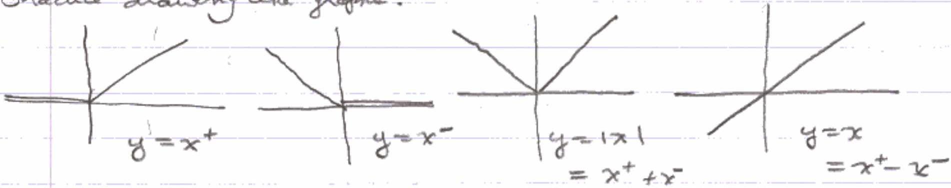
Handout 6 More convergence tests

We define $x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ and $x^- = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Practice drawing the graphs!



<u>Absolute convergence test</u>: If $\sum_{n=0}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$

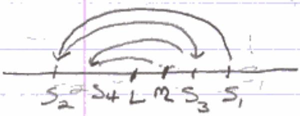
<u>Proof</u>: Suppose $\sum_{n=0}^{\infty} |a_n|$ converges. Since $0 \leq a_n^+ \leq |a_n|$ and $0 \leq a_n^- \leq |a_n|$

it follows that $\sum_0^\infty a_n^+$ and $\sum_0^\infty a_n^-$ converge (comparison test). Thus

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (a_n^+ - a_n^-) = \sum_{n=0}^{\infty} a_n^+ - \sum_{n=0}^{\infty} a_n^- \text{ converges.}$$

<u>Alternating series test</u>: If $a_1 \geq a_2 \geq \ldots$ and $a_n \to 0$, then $\sum_{n=0}^{\infty} (-1)^{n+1} a_n$ converges.

<u>Proof</u>: $S_1 = a_1$ $S_2 = a_1 - a_2$ $S_3 = a_1 - a_2 + a_3$ $S_4 = a_1 - a_2 + a_3 - a_4$



We have $\underbrace{S_1 \geq S_3} \geq \ldots \geq \underbrace{S_4 \geq S_2}$

$S_3 = S_1 - \underbrace{(a_2 - a_3)}_{\geq 0}$ $S_4 = S_2 + \underbrace{(a_3 - a_4)}_{\geq 0}$

Since $S_2 \leq S_4 \leq S_6 \leq \ldots \leq S_1$, the "axiom of continuity" shows that $S_{2n} \to L$ for some L. Similarly since $S_1 \geq S_3 \geq S_5 \geq \ldots \geq S_2$, $S_{2n+1} \to M$. We have

$$M - L = \lim_{n\to\infty} S_{2n+1} - S_{2n} = \lim_{n\to\infty} a_{2n+1} = 0 \quad \text{(BY ASSUMPTION)}$$

hence $L = M$ and $\lim S_n = L$.

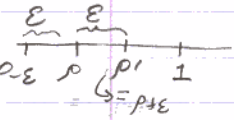
RATIO TEST: Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$.

a) If $\rho < 1$ then $\sum a_n$ converges

b) If $\rho > 1$ then $\sum a_n$ diverges

c) If $\rho = 1$ no information.

Proof: a) Choose ρ' with $\rho < \rho' < 1$ and let $\varepsilon = \rho' - \rho$.



Since $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow \rho$ we may choose N such that

$n \geq N \Rightarrow \left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon$. From the picture

it follows that $n \geq N \Rightarrow \frac{|a_{n+1}|}{|a_n|} < \rho' \Rightarrow |a_{n+1}| < \rho' |a_n|$

$$S = |a_0| + |a_1| + \dots + |a_{N-1}| + |a_N| + |a_{N+1}| + |a_{N+2}| + \dots$$

We have $|a_{N+1}| < \rho' |a_N|$, $|a_{N+2}| < \rho' |a_{N+1}| < \rho'^2 |a_N|$, $|a_{N+3}| < \rho'^3 |a_N|$ etc.

so $S \leq |a_0| + |a_1| + \dots + |a_{N-1}| + \underbrace{|a_N| + |a_N|\rho' + |a_N|(\rho')^2 + |a_N|(\rho')^3 + \dots}_{\text{CONVERGING GEOMETRIC SERIES SINCE } \rho' < 1}$

$< \infty$

b) Since $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow \rho$, letting $\varepsilon = \rho - 1$, there exists an N such

that $n \geq N \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow |a_{n+1}| > |a_n|$.

It follows that

$|a_n| > |a_{n-1}| > \dots > |a_N|$

hence a_n does not converge to zero, and $\sum a_n$ does not converge.
(SEE HANDOUT 2).

c) Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.