Handout 3 : LIMIT TESTS

COMPLETENESS AXIOM : If $a_1 \le a_2 \le \ldots \le b$, then a_n converges.

CONVERGENCE TESTS FOR $\sum_{n=1}^{\infty} a_n$

$\boxed{\text{(a) SERIES WITH } a_n \ge 0}$

Th: Suppose $a_n \ge 0$. Then $\sum_{n=1}^{\infty} a_n$ converges $\iff$ the partial sums $S_N = \sum_{n=1}^{N} a_n$ are bounded above.

Proof: By definition $\sum_{n=1}^{\infty} a_n$ converges $\iff$ S_N converges. Since $a_n \ge 0$, $S_1 \le S_2 \le \ldots$ If S_N are bounded above, i.e., $S_1 \le S_2 \le \ldots \le b$ for some b, then by completeness axiom, S_N converges. Conversely if S_N converges, we have from previous handout that S_N is bounded.

"∞-convention" If $a_n \ge 0$, we write $\sum_{n=1}^{\infty} a_n < \infty$ if the series converges and $\sum_{n=1}^{\infty} a_n = \infty$ if it does not converge.

COMPARISON TEST

Th: If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\sum_{n=1}^{\infty} a_n < \infty$

Pf: We have that

$$S_N = \sum_{n=1}^{N} a_n \le \sum_{n=1}^{N} b_n \le \sum_{n=1}^{\infty} b_n < \infty$$

hence by previous theorem $\sum_{n=1}^{\infty} a_n < \infty$.

Here is a "shorthand proof":

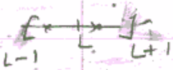
$$\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n < \infty \implies \sum_{n=1}^{\infty} a_n < \infty.$$

Cor: If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} a_n = \infty$, then $\sum_{n=1}^{\infty} b_n = \infty$

Pf: If $\sum b_n < \infty$, then $\sum a_n \le \sum b_n < \infty$ (as above), contradicts $\sum a_n = \infty$

LIMIT COMPARISON TEST (Stronger Version) Say that $0 \le a_n$ and $0 \le b_n$. If $\lim \frac{a_n}{b_n} = L < \infty$ and $\sum b_n < \infty$, then $\sum a_n < \infty$.

Proof: Say $\lim \frac{a_n}{b_n} = L < \infty$. Choose N so that $n \ge N \implies$



$n \ge N \implies \left| \frac{a_n}{b_n} - L \right| < 1$. Then $n \ge N \implies \frac{a_n}{b_n} \le L+1$

$\implies a_n < (L+1) b_n$. Then $\sum_N^{\infty} a_n \le \sum_N^{\infty} (L+1) b_n \le (L+1) \sum_1^{\infty} b_n < \infty$

$\implies \sum_1^{\infty} a_n = \left(\sum_1^{N-1} a_n\right) + \sum_N^{\infty} a_n < \infty.$

→ LIMIT COMPARISON TEST (AS IN STEWART)

→ Cor: Say $\lim \frac{a_n}{b_n} = L \neq 0, 1$. Then

$$\sum a_n < \infty \iff \sum b_n < \infty$$

Pf: $\Leftarrow$] above

$\Rightarrow$] Since $L \neq 0$, $\lim \frac{b_n}{a_n} = \frac{1}{L} \neq \infty$ so $\sum a_n < \infty \Rightarrow \sum b_n < \infty$

EXAMPLE: $\sum_{n=1}^{\infty} \frac{\log n}{n\sqrt{n}}$

COMPARE THIS WITH $\sum \frac{n^{1/4}}{n\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \left(\frac{\log n}{n\sqrt{n}}\right) \Big/ \frac{n^{1/4}}{n\sqrt{n}} = \lim_{n\to\infty} \frac{\log n}{n^{1/4}}$$

$$= \lim_{x\to\infty} \frac{\log x}{x^{1/4}} \overset{(\frac{\infty}{\infty})}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{4}x^{-3/4}} \quad \text{(L'HOPITAL)}$$

$$= 4 \lim_{x\to\infty} \frac{x^{3/4}}{x} = 4 \lim_{x\to\infty} x^{-1/4} = 4 \lim_{x\to\infty} \frac{1}{\sqrt[4]{x}} = 0$$

We have

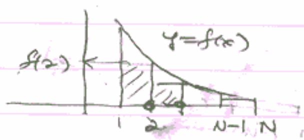
$$\sum \frac{n^{1/4}}{n\sqrt{n}} = \sum \frac{1}{n^{3/2 - 1/4}} = \sum \frac{1}{n^{5/4}} < \infty \quad (p = \tfrac{5}{4} > 1)$$

Def: $\int_1^\infty f(x)\,dx = \lim_{R\to\infty} \int_1^R f(x)\,dx$.

Integral Test: Say that $f(x)$ is decreasing, $\lim_{x\to\infty} f(x) = 0$. Then

$$\sum_{n=1}^{\infty} f(n) < \infty \iff \int_1^\infty f(x)\,dx < \infty$$

Pf:



$$\overbrace{\int_1^N f(x)\,dx}^{\text{area under curve } (1 \le x \le N)} \geq \text{sum of areas of rectangles under curve}$$

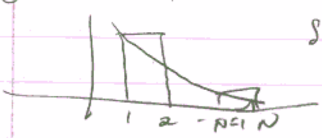
$$= f(2)\cdot 1 + \dots + f(N)\cdot 1$$

USE RIGHT POINT TO DETERMINE HEIGHT OF RECTANGLE | area of first rect

$$\sum_1^N f(n) = f(1) + \sum_2^N f(n) \le f(1) + \int_1^N f(x)\,dx$$

If $\int_1^\infty f(x)\,dx < \infty$, then $\sum_1^N f(n) \le f(1) + \int_1^\infty f(x)\,dx$ proves $\sum_1^\infty f(n) < \infty$. Next say $\sum_1^\infty f(n) < \infty$. Then

$$\int_1^N f(x)\,dx \le \sum_{n=1}^{N-1} f(n)\cdot 1 \le \sum_{n=1}^{\infty} f(n) < \infty$$



hence $\int_1^\infty f(x)\,dx < \infty$.